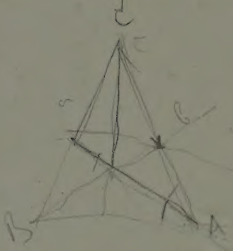

$\cos\frac{A}{2} = \sqrt{\frac{1-\cos 2A}{2}}$

$1 - \frac{b^2+c^2-a^2}{2bc}$

$=$

$\frac{2\,bc\cos\frac{B}{2}}{c+a} \cdot \frac{bc}{2bc-b^2-c^2+a^2}$

$=$

$\frac{a\cos\frac{B}{2}}{c+a}$

$\frac{1}{2}R(b+c)\sin\frac{\alpha}{2}$

$= \frac{\frac{1}{2}R\,b\cos\frac{\alpha}{2}}{b+c}$

$=$

$\frac{b\cos\frac{\alpha}{2}}{b+c} =$

$(a+b+c)$

$2\frac{b}{b+c}\sqrt{\frac{(s-a)(s-c)}{ac}}$

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ELEMENTARY COURSE
OF
INFINITESIMAL CALCULUS.

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AN

ELEMENTARY COURSE

OF

INFINITESIMAL CALCULUS

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PREFACE TO THE SECOND EDITION.

THIS book attempts to teach those portions of the Calculus which are of primary importance in the application to such subjects as Physics and Engineering. Purely analytical developments, and processes, however ingenious, which are seldom useful in practice, are for the most part omitted.

Although the author has had in view the needs of a special class of students, he has not scrupled to discuss questions of theoretical interest when these appeared to him to be at once simple and relevant. He trusts therefore that the book may be acceptable also to mathematical students who desire to take a general survey of the subject before involving themselves deeply in any of its more elaborate developments.

The arrangement is substantially that which has been adopted for a long series of years in lectures at the Owens College. As far as possible, priority is given to the simpler and more generally useful parts of the subject; in particular, the Integral Calculus is introduced at an early period. An attempt is made, as each stage in the subject is reached, to bring in applications of the theory to questions of interest. Thus, after the rules for differentiation come the applications of the derived function to problems of Maxima and Minima, and to Geometry; the rules for integration are followed by the application to areas, volumes, &c.; and so on. Moreover, in the exposition of the theory, dynamical and physical, as well as geometrical, illustrations have been freely employed.

The demands made on the previous knowledge of the student are not extensive. He is of course assumed to be familiar with the elements of Geometry, Algebra, and Trigonometry, and with the fundamental ideas of Analytical Geometry, but not necessarily with such things as the theory of Infinite Series, or the detailed theory of the Conic Sections[1]. Imaginary quantities are nowhere employed in the book. The introductory Chapter will supply what is required outside the above range[2]; but its main object is to establish carefully the fundamental notions of continuity, and of limiting values, on the basis of the geometrical conception of magnitude.

In this edition the book has been carefully revised, and a number of errors have been corrected, principally in the Examples. A few paragraphs in the latter portion of the book, relating to Infinite Series, have been amplified.

The author is much indebted to various friends, and in particular to Professor W. F. Osgood, who have kindly favoured him with criticisms and corrections.

---

[1] Except, to some extent, in the latter part of the book, where the curvature and related properties of special curves are investigated.

[2] In particular, the hyperbolic functions, direct and inverse, whose systematic employment in the Calculus Prof. Greenhill has done so much to promote, are here introduced and studied.

September, 1902.

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CHAPTER I.

CONTINUITY.

1. Continuous Variation.

In every problem of the Infinitesimal Calculus we have to deal with a number of magnitudes, or quantities, some of which may be constant, whilst others are regarded as variable, and (moreover) as admitting of *continuous* variation.

Thus in the applications to Geometry, the magnitudes in question may be lengths, angles, areas, volumes, &c.; in Dynamics they may be masses, times, velocities, forces, &c.

Algebraically, any such magnitude is represented by a letter, such as a or x, denoting the *ratio* which it bears to some standard or 'unit' magnitude of its own kind. This ratio may be integral, or fractional, or it may be 'incommensurable,' *i.e.* it may not admit of being exactly represented by any fraction whose numerator and denominator are finite integers. Its symbol will in any case be subject to the ordinary rules of Algebra.

A 'constant' magnitude, in any given process, is one which does not change its value. A magnitude to which, in the course of any given process, different values are assigned, is said to be 'variable.' The earlier letters $a, b, c, \ldots$ of the alphabet are generally used to denote constant, and the later letters $\ldots u, v, w, x, y, z$ to denote variable magnitudes.

Some kinds of magnitude, as for instance lengths, masses, densities, do not admit of variety of sign. Others, such as altitudes, rotations, velocities, may be either positive or negative.

When we wish to designate the 'absolute' value of a magnitude of this latter class, without reference to sign, we enclose the representative symbol between two short vertical lines, thus

$$|x|, \qquad |\sin x|, \qquad |\log x|.$$

It is important to notice that, if a and b have the same sign,

$$|a+b| = |a| + |b| \text{ .... } (1),$$

whilst, if they have opposite signs,

$$|a+b| < |a| + |b| \text{ .... } (2).$$

A geometrical representation of any class of magnitudes is obtained by taking an unlimited straight line $X'X$, and in it a fixed origin O, and by measuring lengths OM proportional on any convenient scale to the various magnitudes considered. In the case of sign-less magnitudes (such as masses), these lengths are to be measured on one side only of O; in cases where there is a variety of sign, OM must be drawn to the right or left of O according as the magnitude to be represented is positive or negative. To each magnitude of the kind in question will then correspond a definite point M in the line $X'X$.

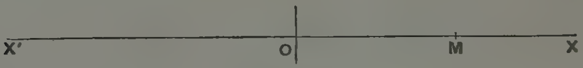

Fig. 1.

When we say that a magnitude admits of 'continuous' variation, we mean that the point M may occupy any position whatever in the line $X'X$ within (it may be) a certain range.

It will be observed that two things are postulated with respect to the magnitudes of the particular kind under consideration, viz. that every possible magnitude of the kind is represented by some point or other of the line $X'X$, and (conversely) that to every point on the line, within a certain range, there corresponds some magnitude of the kind. These conditions are fulfilled by all the kinds of magnitude with which we meet, either in Geometry, or in Mathematical Physics. It will be found on examination that these all involve in their specification a reference, direct or indirect, to linear magnitude.

2. Upper or Lower Limit of a Sequence.

Suppose we have an endless sequence of magnitudes of the same kind

$$x_1,\quad x_2,\quad x_3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

each greater than the preceding, so that the differences

$$x_2 - x_1,\quad x_3 - x_2,\quad x_4 - x_3,\ldots$$

are all positive. Suppose, further, that the magnitudes (1) are all less than some finite quantity a. The sequence will in this case have an 'upper limit,' *i.e.* there will exist a certain quantity μ, greater than any one of the magnitudes (1), but such that if we proceed far enough in the sequence its members will ultimately exceed any assigned magnitude which is less than μ.

In the geometrical representation the magnitudes (1) are represented by a sequence of points

$$M_1,\quad M_2,\quad M_3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(2),$$

each to the right of the preceding, but all lying to the left of some fixed point A. Hence every point on the line $X'X$, without exception, belongs to one or other of two mutually exclusive categories. Either it has points of the sequence (2) to the right of it, or it has not. Moreover, every point in the former category lies to the left of every point of the latter. Hence there must be some point M, say, such that all points on the left of M belong to the former category and all points on the right of it to the latter. Hence if we put $\mu = OM$, μ fulfils the definition of an 'upper limit' above given.

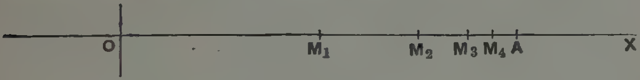

Fig. 2.

In a similar manner we can shew that if we have an endless sequence of magnitudes

$$x_1,\quad x_2,\quad x_3,\ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

each less than the preceding, so that the differences

$$x_1 - x_2,\quad x_2 - x_3,\quad x_3 - x_4,\ldots$$

are all positive, whilst the magnitudes all exceed some finite quantity b, there will be a lower limit ν, such that every magnitude in the sequence is greater than ν, whilst the members of the sequence ultimately become less than any assigned magnitude which is greater than ν.

3. Application to Infinite Series. Series with positive terms.

The above has been called the fundamental theorem of the Calculus. An important illustration is furnished by the theory of infinite series whose terms are all of the same sign. In strictness, there is no such thing as the 'sum' of an infinite series of terms, since the operations indicated could never be completed, but under a certain condition the series may be taken as defining a particular magnitude.

Consider a series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots \qquad \ldots\ldots\ldots\ldots(1)$$

whose terms are all positive, and let

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \ldots\ldots\; s_n = u_1 + u_2 + \ldots + u_n, \ldots \quad \ldots(2).$$

If the sequence

$$s_1, \; s_2, \ldots\ldots\; s_n, \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

has an upper limit S, the series (1) is said to be 'convergent,' and the quantity S is, by convention, called its 'sum.'

Ex. 1. The series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \ldots$$

If in Fig. 2 we make $OM_1 = 1$, $OA = 2$, and bisect M_1A in M_2, M_2A in M_3, and so on, the points M_1, M_2, M_3, ... will represent the magnitudes s_1, s_2, s_3, ... And since these points all lie to the left of A, whilst $M_nA = 1/2^{n-1}$ and can therefore be made as small as we please by taking n large enough, it appears that the sequence has the upper limit OA, $=2$.

Ex. 2. The series

$$\frac{1}{1.2} + \frac{1}{2.3} + \ldots + \frac{1}{n(n+1)} + \ldots$$

If we write this in the form

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \ldots,$$

we see that $$s_n = 1 - \frac{1}{n+1},$$
which has evidently the upper limit 1.

Ex. 3. Further illustrations are supplied by every arithmetical process in which the digits of a non-terminating decimal are obtained in succession. For example, the ordinary process of extracting the square root of 2 gives the series

$$1{\cdot}414213\ldots$$

or $$1 + \frac{4}{10} + \frac{1}{10^2} + \frac{4}{10^3} + \frac{2}{10^4} + \frac{1}{10^5} + \frac{3}{10^6} + \ldots$$

Since s_n is always less than 1·5, there is an upper limit.

Again, if (1) be a convergent series of positive terms, and if

$$u_1' + u_2' + u_3' + \ldots + u_n' + \ldots \qquad \ldots\ldots\ldots\ldots(4),$$

be a series of positive terms which are respectively less than the corresponding terms in (1), *i.e.* $u_n' < u_n$ for all values of n, then (4) is also convergent. For if s_n' be the sum of the first n terms in (4), we have $s_n' < s_n$, and since the magnitudes s_n have by hypothesis an upper limit, the magnitudes s_n will have one *à fortiori.*

Thus the terms of the series

$$1 + 1 + \frac{1}{2\,!} + \frac{1}{3\,!} + \ldots + \frac{1}{n\,!} + \ldots$$

are (after the first three) respectively less than those of the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} + \ldots$$

The latter series is convergent and has the sum 3. Hence the former is also convergent, and its sum is less than 3.

4. Perimeter of a Circle.

Another important example occurs in the definition of the 'perimeter' of a circle.

If we have any number of points on the circumference of a given circle, then by joining them in order we obtain an inscribed polygon, and by drawing the tangents at the points we obtain a circumscribed polygon. Let Π_1 be the perimeter of the former polygon, and Π_1' that of the latter.

It is plain that $\Pi_1 < \Pi_1'$. By taking additional (intervening) points on the several arcs we can form another inscribed polygon of perimeter Π_2 (say), and a corresponding circumscribed polygon of perimeter Π_2'; and it is evident that $\Pi_2 > \Pi_1$, whilst $\Pi_2' < \Pi_1'$. The process can be continued to any extent, and if we imagine it to be followed out according to some definite law, we obtain an ascending sequence of magnitudes

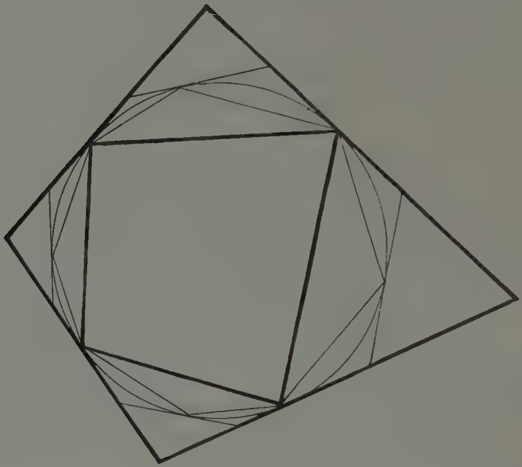

Fig. 3.

$$\Pi_1, \quad \Pi_2, \quad \Pi_3, \ldots,$$

and a descending sequence

$$\Pi_1', \quad \Pi_2', \quad \Pi_3', \ldots.$$

Also, every member of the former sequence is less than every member of the second. Hence the former sequence will have an upper limit Π, and the latter a lower limit Π', and we can assert that $\Pi \not> \Pi'$.

We can further shew that if the law of construction of the successive polygons be such that the angle subtended at the centre by any two successive points on the circumference is ultimately less than any assignable magnitude, the limits Π and Π' are identical. For, let PQ be a side of the inscribed polygon, PT and QT the tangents at P and Q. If PQ meets OT in N, we have

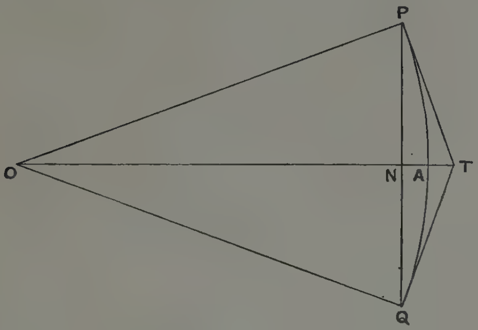

Fig. 4.

$$\frac{PQ}{TP+TQ}=\frac{PN}{PT}=\frac{ON}{OP}.$$

Now if Σ be a symbol of summation extending round the polygons, we have

$$\Pi_s=\Sigma\,(PQ);\quad \Pi_s'=\Sigma\,(TP+TQ).$$

Hence, by a known theorem, the ratio

$$\frac{\Pi_s}{\Pi'_s},=\frac{\Sigma\,(PQ)}{\Sigma\,(TP+TQ)},$$

will be intermediate in value between the greatest and least of the ratios

$$\frac{ON}{OP}.$$

But in the limit, when the angles POQ are indefinitely diminished, each of the ratios ON/OP becomes equal to unity. Hence the limit of Π_s is identical with that of Π_s', or $\Pi=\Pi'$.

Finally, whatever be the law of construction of the successive polygons, subject to the above-mentioned condition, the limit obtained is the same. For suppose, for a moment, that in one way we obtain the limit Π, and in another the limit P. Regarding Π as the limit of an inscribed, and P as that of a circumscribed polygon, it is plain that $\Pi \ngtr P$. In like manner, regarding P as the limit of an inscribed, and Π as that of a circumscribed polygon, it appears that $P \ngtr \Pi$. Hence $\Pi = P$.

The definite limit to which, as we have here proved, the perimeter of an inscribed (or circumscribed) polygon tends, when each side is ultimately less than any assignable magnitude, is adopted by definition as the 'perimeter' of the circle. The proof that the ratio (π) which this limit bears to the diameter is the same for all circles is given in most books on Trigonometry.

In a similar manner we may define the length of *any* arc of a circle less than the whole perimeter, and shew that the length so defined is unique.

5. Limiting Value in a Sequence.

Suppose that we have an endless sequence of magnitudes

$$x_1,\ x_2,\ x_3, \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

arranged in any definite order, and that a point in the sequence can always be found beyond which every member of it differs from a certain quantity μ by a quantity less (in absolute value) than σ, where σ may be any assigned magnitude, however small. The sequence is then said to have the 'limiting value' μ.

We have had particular cases of this relation in the upper and lower limits discussed in Art. 2, but in the present wider definition it is not implied that the members of the sequence are arranged in order of magnitude, or that they are all greater or all less than the limiting value μ.

The point in the sequence after which the difference of each member from μ is less (in absolute value) than σ will in general vary with the value of σ, but it is implied in the definition that, however small σ be taken, such a point exists.

Ex. Let the sequence be that obtained by putting $n=0, 1, 2, 3, \ldots$ in the formula

$$\frac{1}{2n+1} \sin \frac{2n+1}{2} \pi.$$

The sine is alternately equal to ± 1, whilst the first factor diminishes indefinitely. The limiting value is therefore zero.

6. **Application to Infinite Series.**

If in the infinite series

$$u_1 + u_2 + \ldots + u_n + \ldots \quad \ldots\ldots\ldots\ldots\ldots(1),$$

whose terms are no longer restricted to be all of the same sign, we write

$$s_1 = u_1, \quad s_2 = u_1 + u_2, \ \ldots\ldots \ s_n = u_1 + u_2 + \ldots + u_n, \ldots \ \ldots(2),$$

and if the sequence

$$s_1, \ s_2, \ \ldots\ldots \ s_n, \ \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

has a limiting value S, the series is said to be 'convergent,' and S is called its 'sum.'

The most important theorem in this connection is that if the series

$$|u_1| + |u_2| + \ldots + |u_n| + \ldots \quad \ldots\ldots\ldots\ldots(4),$$

formed by taking the absolute values of the several terms of (1), be convergent, the series (1) will be convergent. If (4) be convergent, the positive terms of (1) must *à fortiori* form a convergent series, and so also must the negative terms. Let the sum of the positive terms be p and that of the negative terms be $-q$. Also, let s_{m+n}, the sum of the first $m+n$ terms of (1), consist of m positive terms whose sum is p_m, and n negative terms whose sum is $-q_n$.

We have, then,

$$\begin{aligned}(p-q) - s_{m+n} &= (p-q) - (p_m - q_n) \\ &= (p - p_m) - (q - q_n) \ldots\ldots\ldots(5).\end{aligned}$$

If $m+n$ be sufficiently great, $p - p_m$ and $q - q_n$ will both be less than σ, where σ is any assigned magnitude, however small; and the difference of these positive quantities will be *à fortiori* less than σ in absolute value. Hence s_{m+n} has the limiting value $p - q$.

When the series (4), composed of the absolute values of the several terms of (1), is convergent, the series (1) is said to be 'absolutely,' or 'essentially,' or 'unconditionally' convergent.

It is possible, however, for a series to be convergent, whilst the series formed by taking the absolute values of the terms has no upper limit. In this case, the convergence of the given series is said to be 'accidental,' or 'conditional.'

The following very useful theorem holds whether the series considered be essentially or only accidentally convergent:

If the terms of a series are alternately positive and negative, and continually diminish in absolute value, and tend ultimately to the limit zero, the series is convergent, and its sum is intermediate between the sum of any finite odd number of terms and that of any finite even number, counting in each case from the beginning.

Let the series be

$$\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4 + \dots \qquad \dots\dots\dots\dots\dots(6),$$

where, by hypothesis,

$$\alpha_1 > \alpha_2 > \alpha_3 > \dots.$$

In the figure, let

$$OM_1 = \alpha_1, \quad M_1M_2 = \alpha_2, \quad M_2M_3 = \alpha_3, \quad M_3M_4 = \alpha_4, \dots.$$

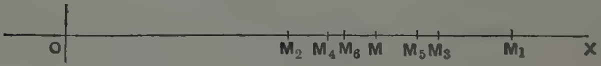

Fig. 5.

It is plain that the points M_1, M_3, M_5, ... form a descending sequence, and that the points M_2, M_4, M_6, ... form an ascending sequence. Also that every point of the former sequence lies to the right of every point of the latter. Hence each sequence has a limiting point, and since

$$M_{2n}M_{2n+1} = \alpha_{2n+1},$$

and therefore is ultimately less than any assignable magnitude, these two limiting points must coincide, say in M. Then OM represents the sum of the given series (6).

Ex. The series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots$$

converges to a limit between 1 and $1 - \frac{1}{2}$.

This series belongs to the 'accidentally' convergent class. It is shewn in books on Algebra that the sum of n terms of the series

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \dots$$

can be made as great as we please by taking n great enough.

It cannot be too carefully remembered that the word 'sum' as applied to an infinite series is used in a purely conventional sense, and that we are not at liberty to assume, without examination, that we may deal with such a series as if it were an expression consisting of a finite number of terms. For example, we may not assume that the sum is unaltered by any rearrangement of the terms. In the case of an essentially convergent series this assumption can be justified, but an accidentally convergent series can be made to converge to any limit we please by a suitable adjustment of the order in which the terms succeed one another. For the proofs of these theorems we must refer to books on Algebra; they are hardly required in the present treatise.

We shall, however, occasionally require the theorem that if

$$u_1 + u_2 + \dots + u_n + \dots \qquad \dots\dots\dots\dots\dots(7)$$

and

$$u_1' + u_2' + \dots + u_n' + \dots \qquad \dots\dots\dots\dots\dots(8)$$

be two convergent series whose sums are S and S', respectively, the series

$$(u_1 \pm u_1') + (u_2 \pm u_2') + \dots + (u_n \pm u_n') + \dots \quad \dots(9),$$

composed of the sums, or the differences, of corresponding terms in (7) and (8), will converge to the sum $S \pm S'$. This is easily proved. If s_n, s_n' denote the sums of the first n terms of (7) and (8) respectively, the sum of the first n terms of (9) will be $s_n \pm s_n'$. Now

$$(S \pm S') - (s_n \pm s_n') = (S - s_n) \pm (S' - s_n') \quad \dots(10).$$

By hypothesis, if σ be any assigned magnitude, however small, we can find a value of n such that for this and for all higher values we shall have

$$|S - s_n| < \tfrac{1}{2}\sigma, \quad \text{and} \quad |S' - s_n'| < \tfrac{1}{2}\sigma \quad \dots\dots(11),$$

and therefore $$|(S \pm S') - (s_n \pm s_n')| < \sigma \quad \text{............(12)},$$
which is the condition that $s_n \pm s_n'$ should have the limiting value $S \pm S'$.

7. General Definition of a Function.

One variable quantity is said to be a 'function' of another when, other things remaining the same, if the value of the latter be fixed that of the former becomes determinate.

The two quantities thus related are distinguished as the 'dependent' and the 'independent' variable respectively.

The notion of a function of a variable quantity is one which presents itself in various branches of Mathematics. Thus, in Arithmetic, the number of permutations of n objects is a function of n; the number of balls in a square or a triangular pile of shot is a function of the number contained in each side of the base; the sum (s_n) of the first n terms of any given series is a function of the number n; and so on. In some of these cases there are definite mathematical formulæ for the functions in question, but it is to be noticed that the idea of functionality does not necessarily require this; for example, the sum of the first n terms of the series

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

is a definite function of n, although no exact mathematical expression exists for it. So, again, the number of primes not exceeding a given integer n is a definite function of n, although it cannot be represented by a formula.

In these examples, the independent variable, from its nature, can only change by finite steps. The Infinitesimal Calculus, on the other hand, deals specially with cases where the independent variable is continuous, in the sense of Art. 1. For instance, in Geometry the area of a circle, or the volume of a sphere, is a function of the radius; in theoretical Physics the altitude, or the velocity, of a falling particle is regarded as a function of the time; the period of oscillation of a given pendulum as a function of the amplitude; the pressure of a given gas at a given temperature as a function of the density; the pressure of saturated steam as a function of the temperature; and so on. Here, again, the existence or non-existence of a mathematical expression for the function is not material; all that is necessary to establish a functional relation between two variables is that, when other things are unaltered, the value of one shall determine that of the other.

In general investigations it is usual to denote the independent variable by x, and the dependent variable by y. The relation between them is often expressed in such a form as

$$y = \phi(x), \quad \text{or} \quad y = f(x), \ \&c.,$$

the symbol $\phi(x)$, for instance, meaning 'some particular function of x.'

When a quantity varies from one value to another, the amount (positive or negative) by which the new value exceeds the former value is called the 'increment' of the quantity. This increment is often denoted by prefixing δ or Δ (regarded as a symbol of *operation*) to the symbol which represents the variable magnitude. Thus we speak of the independent variable changing from x to $x + \delta x$, and of the dependent variable consequently changing from y to $y + \delta y$.

Hence if $$y = \phi(x), \quad \text{......................(1)},$$
we must have $$y + \delta y = \phi(x + \delta x), \quad \text{.................(2)},$$
and therefore $$\delta y = \phi(x + \delta x) - \phi(x) \quad \text{.........(3)}.$$

At present there is no implication that δx or δy is small; the increments may have any values subject to the relation (2).

Ex. If $y = x^3$, then if $x = 100$, $\delta x = 1$, we have

$$\delta y = (101)^3 - (100)^3 = 30301.$$

8. Geometrical Representation of Functions.

We may construct a graphical representation of the relation between two variables x, y, one of which is a function of the other, by taking rectangular coordinate axes OX, OY. If we measure OM along OX, to represent any particular value of the independent variable x, and ON along OY to represent the corresponding value of the function y, and if we complete the rectangle $OMPN$, the position of the point P will indicate the values of both the associated variables.

Since, by hypothesis, M may occupy any position on OX, between (it may be) certain fixed termini, we obtain

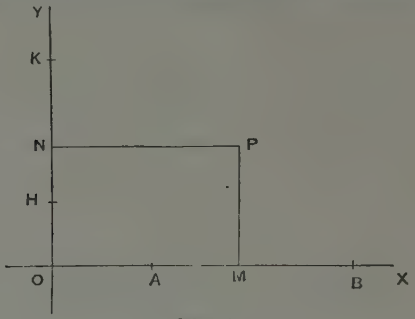

Fig. 6.

in this way an infinite assemblage of points P. A question arises as to the nature of this assemblage; whether, or in what sense, the points constituting it can be regarded as lying on a curve.

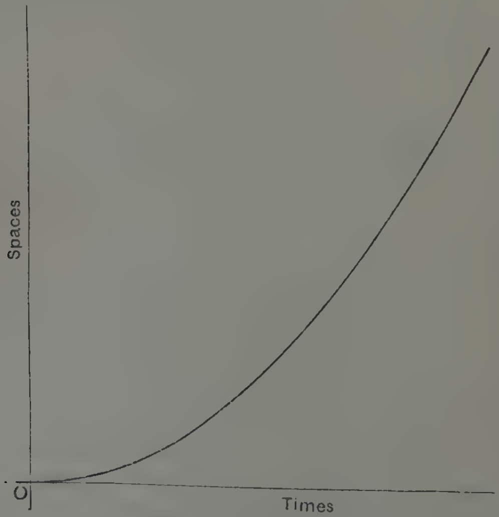

Fig. 7.

In many cases, of course, there is no difficulty about the answer. For example, if, to represent the relation between the area of a circle and its radius, we make OM proportional to the radius, and PM proportional to the area, then $PM \propto OM^2$, and the points P lie on a parabola. The same curve will represent the relation between the space (s) described by a falling body and the time (t) from rest, since $s \propto t^2$. See Fig. 7.

The general question must, however, be answered in the negative. The definition of a function given at the beginning of Art. 7 stipulates that for each value of x there shall be a definite value of y; but there is no necessary relation between the values of y corresponding to different values of x, however close together these may be.

Without some further qualification the definition referred to is indeed far too wide for our present purposes, the functions ordinarily contemplated in the Calculus being subject to certain very important restrictions.

The first of these restrictions is that of 'continuity.' This implies that, as M ranges over any finite portion AB of the line OX, N ranges over a finite portion HK of the line OY, *i.e.* N occupies once at least every position between H and K. Further, that if the range AB be continually contracted, the range HK will also contract, and can be made as small as we please by taking AB small enough.

Since, as we shall see in Art. 10, the second of the above properties includes the first, it is adopted as the basis of the formal definition of a 'continuous function,' to which we now proceed.

9. Definition of a Continuous Function.

Let x and y be corresponding values of the independent variable and of the function. Let δx be any admissible increment of x, and δy the corresponding increment of y. Then if, σ being any positive quantity different from zero, we can always find a positive quantity ϵ, different from zero, such that for all admissible values of δx which are less (in absolute value) than ϵ the value of δy will be less in absolute value than σ, the function is said to be 'continuous' for the particular value x of the independent variable.

Otherwise, if $\phi(x)$ be the function, the definition requires that it shall be possible to find a quantity ϵ such that

$$|\phi(x+h)-\phi(x)|<\sigma \qquad (1),$$

for all admissible values of h such that $|h|<\epsilon$. The value of ϵ will in general be limited by that of σ, but it is implied that the condition can always be satisfied by *some* value of ϵ, however small σ may be. The restriction to 'admissible' values of δx (or h) means that $x+\delta x$ must be within the range of values of the independent variable.

The above definition is sometimes (imperfectly) summed up in the statement that an infinitely small change in the independent variable produces an infinitely small change in the function.

It follows from the definition that if $x_1, x_2, x_3, \ldots$ be any sequence of admissible values of the independent variable having x for its limit (Art. 5), the sequence $y_1, y_2, y_3, \ldots$ of the corresponding values of the function will have the limit y. Conversely, if this hold for *every* admissible sequence $x_1, x_2, x_3, \ldots$ having x for its limit, the function will be continuous at x.

Ex. 1. To shew that the function

$$\phi(x)=x^2 \qquad (2)$$

is continuous, according to the above definition, for all values of x.

We have

$$(x+h)^2-x^2=2xh+h^2 \qquad (3).$$

Now, x being fixed, and σ being any assigned positive quantity, however small, we can always find a positive quantity ϵ such that

$$2\epsilon|x|+\epsilon^2=\sigma \qquad (4).$$

This is in fact a quadratic equation to find ϵ, and it is easily seen that one root is positive. And it is evident that if $|h|$ be less than the value of ϵ thus found, we shall have

$$|\phi(x+h)-\phi(x)|<\sigma.$$

A general investigation of the continuity of rational algebraic functions will be given later (Arts. 14, 15).

Ex. 2. Let

$$\phi(x) = \sqrt{(1-x)} \qquad\qquad (5),$$

a function only defined (so long as we recognize only 'real' quantities) for $x \ngtr 1$. We have

$$\phi(1-h) = \sqrt{(h)},$$

which can be made less than any assigned quantity σ by making $h < \sigma^2$. The function is therefore continuous for $x = 1$.

10. **Property of a Continuous Function.**

If $\phi(x)$ be a function which is continuous from $x = a$ to $x = b$ inclusive, and if $\phi(a)$ and $\phi(b)$ have opposite signs, there must be some value of x *between* a and b for which $\phi(x) = 0$.

For definiteness we will suppose that $\phi(a)$ is positive and $\phi(b)$ negative. Let A, B be the points of the line $X'X$ for which $x = a$, $x = b$, respectively; and let HA, BK represent the corresponding values of the function. In virtue of the continuity of $\phi(x)$ there is a certain range extending to the right of A for every point of which the function differs (in absolute value) from HA by less than HA, and is therefore necessarily positive. Since this range does not extend as far as B, it will terminate at some point C between A and B. Moreover the value of the function at the point C itself must be zero. For if it were positive, then (in virtue of the

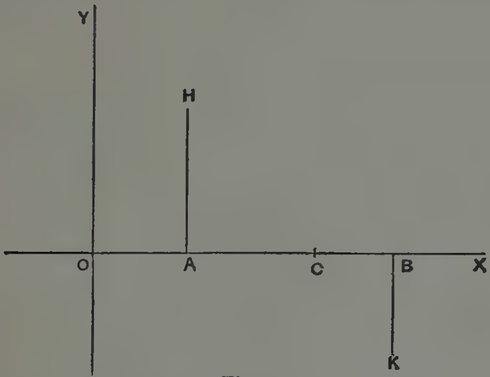

Fig. 8.

continuity) there would be a certain range extending to the right of C for every point of which the function would be positive; and if it were negative there would be a certain range extending to the left of C for every point of which the function would be negative. Either of these suppositions is inconsistent with the above determination of C.

We may express this theorem shortly by saying that a continuous function cannot change sign except by passing through the value zero.

It follows that if $\phi(x)$ be a function which is continuous from $x=a$ to $x=b$ inclusive, and if $\phi(a)$, $\phi(b)$ be unequal, there must be some value of x *between* a and b, such that $\phi(x)=\beta$, where β may be any quantity intermediate in value to $\phi(a)$ and $\phi(b)$. For, let

$$f(x)=\phi(x)-\beta;$$

since β is a constant, $f(x)$ also will be continuous. By hypothesis,

$$\phi(a)-\beta \quad \text{and} \quad \phi(b)-\beta$$

have opposite signs, and therefore $f(a)$ and $f(b)$ have opposite signs. Hence, by the above theorem, there is some value of x between a and b for which $f(x)=0$, or $\phi(x)=\beta$.

In other words, a continuous function cannot pass from one value to another without assuming once (at least) every intermediate value.

11. Graph of a Continuous Function.

It follows from what precedes that the assemblage of points which represents, in the manner explained in Art. 8, any *continuous* function is a 'connected' assemblage. By this it is meant that a line cannot be drawn across the assemblage without passing through some point of it.

For, denoting the function by $\phi(x)$, and the ordinate of any line by $f(x)$, then if $\phi(x)$ and $f(x)$ are both continuous the difference

$$\phi(x)-f(x)$$

will be continuous (Art. 13), and therefore cannot change sign without passing through the value zero.

The question whether any connected assemblage of points is to be regarded as lying on a curve is to some extent a verbal one, the answer depending upon what properties are held to be connoted by the term 'curve.' We shall have occasion (in Art. 32) to return to this question; but in the meantime it is obvious that a good representation of the general course or 'march' of any given continuous function can be obtained by actually plotting on paper the positions of a sufficient number of points belonging to the assemblage, and drawing a line through them with a free hand. A figure constructed in this way is called a 'graph' of the function.

The figure shews the method of constructing a graph of the function $y = x^2$; the series of corresponding values of x and y employed for the purpose being as follows:

$$\left\{\begin{array}{llllll} x = 0, & \pm\cdot 5\,, & \pm 1, & \pm 1\cdot 5\,, & \pm 2, & \pm 2\cdot 5\,, \\ y = 0, & \cdot 25, & 1, & 2\cdot 25, & 4, & 6\cdot 25. \end{array}\right.$$

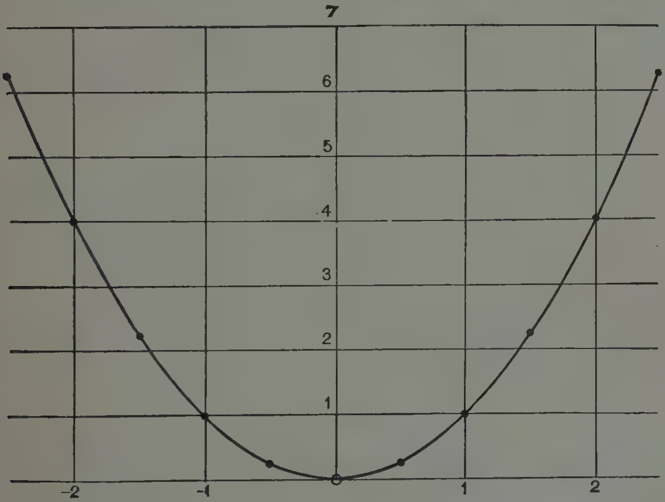

Fig. 9.

It will be noticed that different scales have been adopted for x and y, respectively. This is often convenient; indeed in the physical applications of the method the scales of x and y have to be fixed independently.

The method of graphical representation is often used in practice when the mathematical form of the function is unknown; a certain number of corresponding values of the dependent and independent variables being found by observation. An example is furnished by the annexed diagram, which represents the pressure of saturated steam as a function of the temperature (Centigrade).

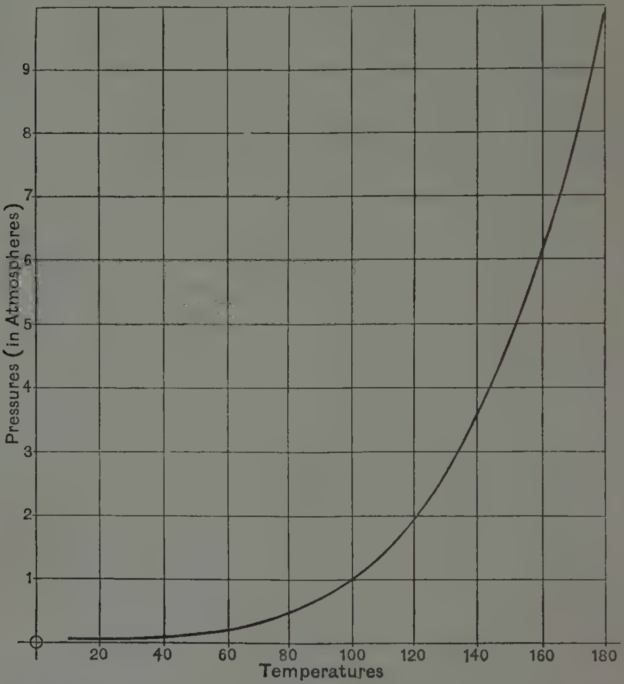

Fig. 10.

The reader may also be reminded of the meteorological charts which exhibit the height of the barometer or thermometer as a function of the time.

The substratum of fact underlying a graph constructed in the above manner is of course no more than is contained in a numerical table giving a series of pairs of corresponding values of x and y; but the graphical form appeals far more effectively to the mind, by helping us to supply, in imagination, the intermediate values of the function.

The graphical method will be freely used in this book (as in other elementary treatises on the subject) by way of illustration. It is necessary, however, to point out that, as applied to *mathematical* functions, it has certain limitations. In the first place, it is obvious that no finite number of isolated values can determine the function completely; and, indeed, unless some judgment is exercised in the choice of the values of x for which the function shall be calculated the result may be seriously misleading. Again, the streak of ink, or graphite, by which we represent the course of the function, has (unlike the ideal mathematical line) a certain breadth, and the same is true of the streak which represents the axis of x; the distance between these streaks is therefore affected by a certain amount of vagueness. For the same reason, we cannot reproduce details of more than a certain degree of minuteness; the method is therefore intrinsically inadequate to the representation of functions (such as can be proved to exist) in which new details reveal themselves *ad infinitum* as the scale is magnified. Functions of this latter class are not, however, encountered in the ordinary applications of the Calculus.

In the representation of *physical* functions, as determined experimentally, the vagueness due to the breadth of the lines is usually no more serious than that due to the imperfection of our senses, errors of observation, and the like.

12. Discontinuity.

A function which for any particular value (x_1) of the independent variable fails in any way to satisfy the condition stated at the beginning of Art. 9 is said to be 'discontinuous' for that value of x.

Functions exist (and can be mathematically defined) which are discontinuous for *every* value of x within a certain range. But ordinarily, in the applications of the Calculus, we have to deal with functions which are discontinuous (if at all) only for certain isolated values of x.

This latter kind of discontinuity, again, may occur in various ways. In the first place, the function may become 'infinite' for some particular value (x_1) of x. The meaning of this is that by taking x sufficiently nearly equal to x_1 the value of the function can be made to exceed (in absolute value) any assigned magnitude, however great.

Examples of this are furnished by the functions $1/x$, which becomes infinite for $x = 0$, and $\tan x$, which becomes infinite for $x = \frac{1}{2}\pi$, &c. See Fig. 19, p. 34.

Again, the time of oscillation of a given pendulum, regarded as a function of the amplitude (α), becomes infinite for $\alpha = \pi$. A graph of this function is appended.

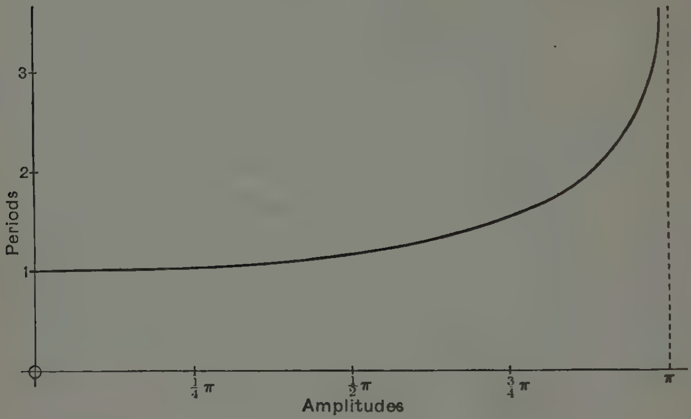

Fig. 11.

Again, it may happen that as x approaches x_1, the function tends to a definite limiting value λ (see Art. 26), whilst the value actually assigned to the function for $x = x_1$ is different from λ.

Consider, for example, the function defined as equal to 0 for $x = 0$ and equal to 1 for all other values of x.

Moreover if x_1 lie within the range of the independent variable, it may happen that the function tends to different limiting values as x approaches x_1 from the right or left, respectively. In this case the value of the function for

$x = x_1$ (if assigned) cannot be equal to *both* these limiting values, and there is necessarily a discontinuity.

An illustration from theoretical dynamics is furnished by the velocity of a particle which at a given instant receives a sudden impulse in the direction of motion. In this case the velocity at the instant of the impulse is undefined.

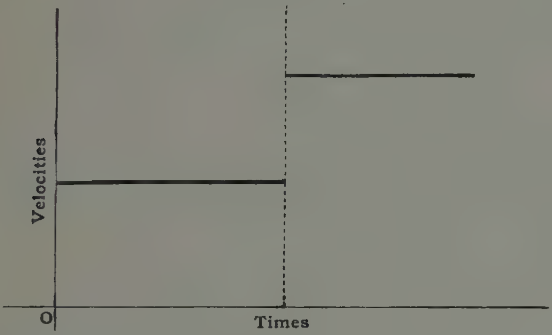

Fig. 12.

Other more general varieties of isolated discontinuity are imaginable, but are not met with in the ordinary applications of the subject.

A sufficient example is afforded by the function

$$\sin\frac{1}{x}.$$

This is undefined for $x = 0$, but whatever value we supply to complete the definition, the point $x = 0$ will be a point of discontinuity. For the function oscillates between the values ± 1 an infinite number of times within any interval to the right or left of $x = 0$, however short, since the angle $1/x$ increases indefinitely.

13. Theorems relating to Continuous Functions.

We may now proceed to investigate the continuity, or otherwise, of various functions which have an explicit mathematical definition, and to examine the character of their graphical representations.

For this purpose the following preliminary theorems will be useful:

1°. The *sum* of any finite number of continuous functions is itself a continuous function.

First suppose we have two functions u, v of the independent variable x. Then

$$\begin{aligned}\delta(u+v) &= (u+\delta u+v+\delta v)-(u+v)\\ &= \delta u+\delta v.\end{aligned}$$

From the definition of continuity it follows that, whatever the value of σ, we can find a quantity ϵ such that for $|\delta x| < \epsilon$ we shall have $|\delta u| < \frac{1}{2}\sigma$ and $|\delta v| < \frac{1}{2}\sigma$, and therefore (Art. 1)

$$|\delta u+\delta v| < \sigma.$$

Hence the function $u+v$ is continuous.

Next, if we have three continuous functions u, v, w, then $u+v$ is continuous, as we have just seen, and consequently $(u+v)+w$ is continuous. In this way the theorem may be extended, step by step, to the case of any finite number of functions.

2°. The *product* of any finite number of continuous functions is itself a continuous function.

First, take the case of two functions u, v. We have

$$\begin{aligned}\delta(uv) &= (u+\delta u)(v+\delta v)-uv\\ &= v\delta u+u\delta v+\delta u\delta v.\end{aligned}$$

By hypothesis we can, by taking $|\delta x|$ small enough, make $|\delta u|$ and $|\delta v|$ less than any assigned quantity, however small. Hence, since u and v are finite, $|v\delta u|$ and $|u\delta v|$ can be made less than any assigned quantity, however small. The same is evidently true of $|\delta u\delta v|$. Hence, also, the value of

$$|v\delta u+u\delta v+\delta u\delta v|$$

can be made less than any assigned quantity, however small. That is, uv is a continuous function.

Next, suppose we have three continuous functions u, v, w. We have seen that uv is continuous; hence also $(uv)w$ is continuous. And so on for the product of any finite number of continuous functions.

3°. The quotient of two continuous functions is a continuous function, except for those values (if any) of the independent variable for which the divisor vanishes.

We have
$$\delta\left(\frac{u}{v}\right) = \frac{u+\delta u}{v+\delta v} - \frac{u}{v}$$
$$= \frac{v\delta u - u\delta v}{v(v+\delta v)}.$$

By hypothesis $v \neq 0$; there is therefore a lower limit M, different from zero, to the absolute magnitude of $v(v+\delta v)$. This makes
$$\left|\delta\left(\frac{u}{v}\right)\right| < \left|\frac{v}{M}\delta u - \frac{u}{M}\delta v\right|.$$

Since v/M and u/M are finite, we can, by taking δx small enough, make $|v/M \,.\, \delta u|$ and $|u/M \,.\, \delta v|$ less than any assigned magnitude, however small. The same will therefore be true of $|\delta(u/v)|$; *i.e.* the quotient u/v is continuous.

4°. If y be a continuous function of u, where u is a continuous function of x, then y is a continuous function of x.

For let δx be any increment of x, δu the consequent increment of u, and δy the consequent increment of y. Since y is a continuous function of u, we can find a quantity ϵ', such that if
$$|\delta u| < \epsilon', \text{ then } |\delta y| < \sigma,$$
where σ may be any assigned quantity, however small. And since u is a continuous function of x, we can find a quantity ϵ, such that if
$$|\delta x| < \epsilon, \text{ then } |\delta u| < \epsilon'.$$
Hence if $\quad |\delta x| < \epsilon$, we have $|\delta y| < \sigma$,

which is the condition of continuity of y, considered as a function of x.

14. Algebraic Functions. Rational Integral Functions.

An 'algebraic' function is one which is obtained by performing with the variable and known constants any finite number of operations of addition, subtraction, multiplication, division, and extraction of integral roots.

All other functions are classed as 'transcendental'; they involve, in one form or another, the notion of a 'limiting value' (Arts. 5, 26).

A 'rational' algebraic function is one which is formed in like manner by operations of addition, subtraction, multiplication, and division, only. Any such function can be reduced to the form

$$\frac{F(x)}{f(x)}$$

where the numerator and denominator are rational 'integral' functions; *i.e.* each of them is made up of a finite number of terms of the type

$$Ax^m,$$

where m is a positive integer, and A is a constant.

A rational integral function is finite and continuous for all finite values of the variable. For x^m, being the product of a finite number (m) of continuous functions (each equal to x), is finite and continuous. Hence also Ax^m is finite and continuous; and the sum of any finite number of such terms is finite and continuous (Art. 13).

A rational integral function becomes infinite for $x = \pm\infty$. Let the function be

$$A_n x^n + A_{n-1} x^{n-1} + A_{n-2} x^{n-2} + \ldots + A_1 x + A_0.$$

Writing this in the form

$$x^n \left(A_n + \frac{A_{n-1}}{x} + \frac{A_{n-2}}{x^2} + \ldots + \frac{A_1}{x^{n-1}} + \frac{A_0}{x^n} \right),$$

we see that by taking x great enough (in absolute value) we can make the first factor (x^n) as great as we please, whilst the second factor can be made as nearly equal to A_n as we please. Hence the product can be made as great as we please. Moreover, if x be positive the sign of the product will be the same as that of A_n, whilst if x be negative the sign will be the same as that of A_n, or the reverse, according as n is even or odd.

It follows that in the graphical representation of a rational integral function $y=f(x)$ the curve is everywhere at a finite distance from the axis of x, but recedes from it without limit as x is continually increased, whether positively or negatively. In actually constructing the curve, it is convenient if possible to solve the equation $f(x)=0$, as this gives the intersections of the curve with the axis of x.

Ex. 1. The graph of the function

$$y=b$$

is a straight line parallel to the axis of x at a distance b from it.

Ex. 2. The graph of

$$y=mx+b$$

is a straight line. For this reason a rational integral function (such as $mx+b$) of the first degree is often called a 'linear' function.

The line cuts the axis of y at the point $(0, b)$, and the axis of x at the point $(-b/m, 0)$.

Ex. 3. To trace the curve

$$y=x(1-x).$$

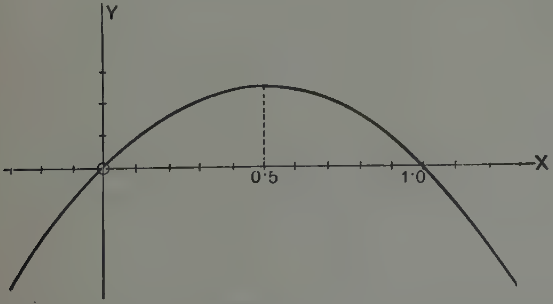

Fig. 13.

We note that $y=0$ for $x=0$ and for $x=1$. Again, for

$$x=-\infty,\quad -2,\quad -1,\quad 0,\quad 1,\quad 2,\quad 3,\quad +\infty,$$

we find $y=-\infty,\quad -6,\quad -2,\quad 0,\quad \cdot 0,\quad -2,\quad -6,\quad -\infty.$

This is sufficient for shewing the general course of the curve except in the interval between $x=0$ and $x=1$. Inserting a few additional values we find

$$\left\{\begin{array}{lllllllll} x=\cdot 1, & \cdot 2, & \cdot 3, & \cdot 4, & \cdot 5, & \cdot 6, & \cdot 7, & \cdot 8, & \cdot 9, \\ y=\cdot 09, & \cdot 16, & \cdot 21, & \cdot 24, & \cdot 25, & \cdot 24, & \cdot 21, & \cdot 16, & \cdot 09. \end{array}\right.$$

The figure shews the resulting curve.

The function might also have been studied under the form

$$y=\tfrac{1}{4}-(x-\tfrac{1}{2})^2,$$

which shews that the curve is symmetrical with respect to the line $x=\frac{1}{2}$; and also that the greatest value of y is $\frac{1}{4}$. Any other quadratic function may be treated in a similar manner (cf. Art. 52).

15. **Rational Fractions.**

A function
$$y=\frac{F(x)}{f(x)} \quad \text{.......................(1)}$$

which is rational but not integral, is finite and continuous for all finite values of x except those which make $f(x)=0$[*]. For the rational integral functions $F(x)$ and $f(x)$ have been proved to be finite and continuous; and it follows, by Art. 13, that the quotient will be finite and continuous except when the denominator vanishes.

The curve represented by (1) will cut the axis of x in the points (if any) for which $F(x)=0$. It will have asymptotes parallel to y wherever $f(x)=0$, whilst for all other finite values of x, y will be finite and continuous. The values of y for $x=\pm\infty$ will depend on the relative order of magnitude of $F(x)$ and $f(x)$. If $F(x)$ be of higher degree than $f(x)$ the ordinates become infinite; if of lower degree the ordinates diminish indefinitely, the axis of x being an asymptote; if the degrees are the same, there is an asymptote parallel to x.

---

[*] It is assumed that the fraction has been reduced to its lowest terms.

In cases where the degree of the numerator is not less than that of the denominator, it is convenient to perform the division indicated until the remainder is of lower degree than the divisor, and so express y as the sum of an integral function and a 'proper' fraction.

Ex. 1. $$y = \frac{1-x}{2x} = -\tfrac{1}{2} + \frac{1}{2x}$$

This makes $y=0$ for $x=1$, and $y=\pm\infty$ for $x=0$. Also y is positive for $1>x>0$, and negative outside this interval. From the second form of y it appears that for $x=\pm\infty$ we have $y=-\tfrac{1}{2}$. We further find, as corresponding values of x and y:

$$\begin{cases} x = -\infty, & -3, & -2, & -1, & -\cdot5, & 0, & \cdot5, & 1, & 2, & 3, & +\infty, \\ y = -\cdot5, & -\cdot67, & -\cdot75, & -1, & -1\cdot5, & \mp\infty, & \cdot5, & 0, & -\cdot25, & -\cdot33, & -\cdot5. \end{cases}$$

The figure shews the curve.

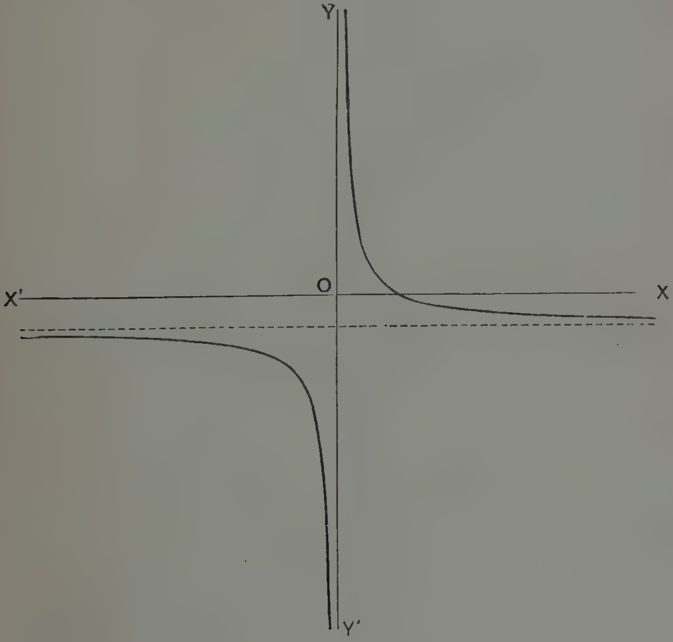

fig. 14.

Ex. 2. $$y=\frac{x(1-x)}{1+x}=-x+2-\frac{2}{1+x}.$$

Here $y=0$ for $x=0$ and $x=1$, and $y=\pm\infty$ for $x=-1$. Also y changes sign as x passes through each of these values. For numerically large values of x, whether positive or negative, the curve approximates to the straight line

$$y=-x+2,$$

lying beneath this line for $x=+\infty$, and above it for $x=-\infty$. The figure shews the curve.

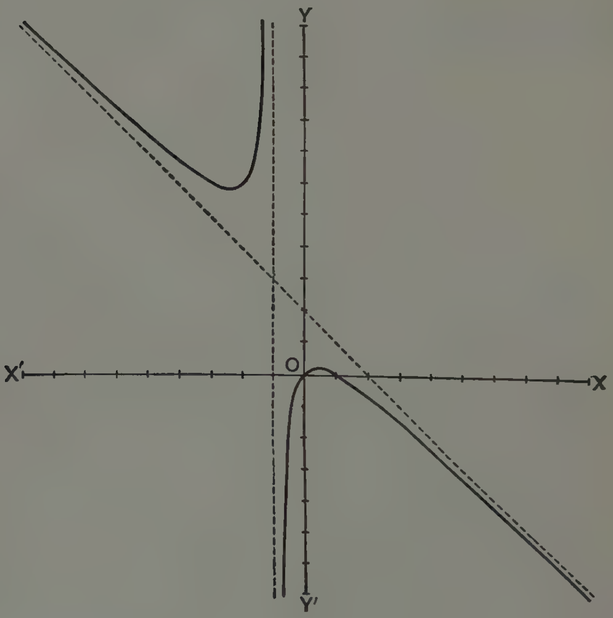

fig. 15.

Ex. 3. $$y = \frac{2x}{1-x^2}.$$

Here y vanishes for $x = 0$, and for $x = \pm\infty$, and becomes infinite for $x = \pm 1$. Again, y is positive for $1 > x > 0$ and negative for $x > 1$. Also, y changes sign with x.

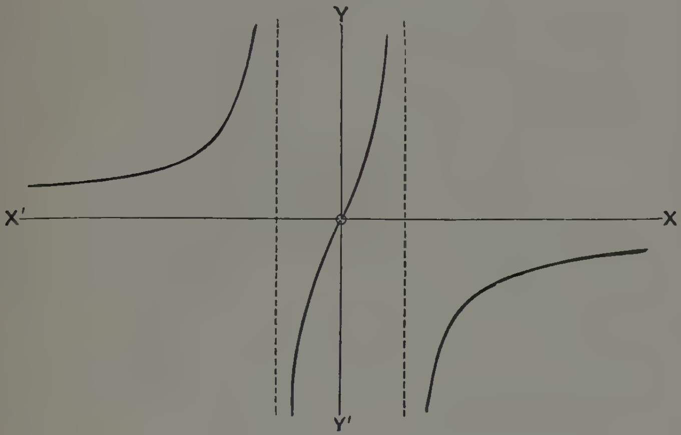

Fig. 16.

Ex. 4. $$y = \frac{2x}{1+x^2}.$$

As in the preceding Ex. y vanishes for $x = 0$ and $x = \pm\infty$, and changes sign with x. But the denominator does not vanish for any real value of x, so that y is always finite.

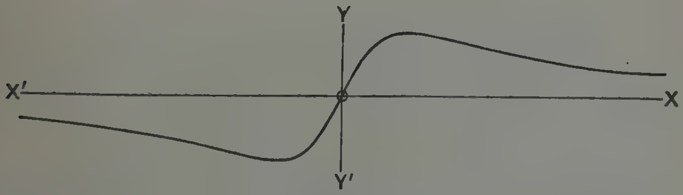

Fig. 17.

EXAMPLES. I.

1. Draw graphs of the following functions*:

(1) $1-x^2, \quad x^3.$

(2) $(x-1)(x-2), \quad x^2-x+1.$

(3) $x(1-x)^2, \quad x^2(1-x)^2.$

(4) $\frac{1}{x}, \quad x+\frac{1}{x}, \quad x-\frac{1}{x}.$

(5) $\sqrt{x}, \quad \frac{1}{\sqrt{x}}.$

(6) $\sqrt{(1-x^2)}, \quad \frac{1}{\sqrt{(1-x^2)}}.$

(7) $\frac{(x-1)(x-2)}{x}, \quad \frac{(x-1)(x-3)}{x-2}.$

(8) $\frac{x^2}{1-x^2}, \quad \frac{x^3}{1+x^2}.$

(9) $\frac{1-x+x^2}{1+x+x^2}.$

2. $\frac{x}{1-x}, \quad \frac{x}{(1-x)^2}.$

3. Prove that the equation

$$2x^3+5x^2-5x-3=0$$

has a root between $-\infty$ and -1, another between -1 and 0, and a third between 1 and 2.

4. Prove that the equation

$$2x^3+7x^2+3x-5=0$$

has three real roots, and find roughly their situations.

5. Find roughly the situations of the roots of

$$2x^3-3x^2-36x+10=0.$$

6. Prove that every algebraic equation of odd degree has at least one real root; and that every equation of even degree, whose first and last coefficients have opposite signs, has at least two real roots, one positive and one negative.

---

* The curves should be drawn carefully to scale; for this purpose paper ruled into small squares is useful. The numerical tables of squares, square-roots, and reciprocals, given in the Appendix (Tables A, B, C), will occasionally help to shorten the arithmetical work.

16. Transcendental Functions. The Circular Functions.

The first place among transcendental functions is claimed by the 'circular' functions

$$\sin x, \quad \cos x, \quad \tan x, \text{ \&c.},$$

whose definitions and properties are given in books on Trigonometry.

The function $\sin x$ is continuous for all values of x. For

$$\begin{aligned}\delta(\sin x) &= \sin(x+\delta x) - \sin x\\ &= 2\sin\tfrac{1}{2}\delta x \,.\, \cos(x+\tfrac{1}{2}\delta x).\end{aligned}$$

The last factor is always finite, and the product of the remaining factors can be made as small as we please by taking δx small enough.

In the same way we may shew that $\cos x$ is continuous. This result is, however, included in the former, since

$$\cos x = \sin(x + \tfrac{1}{2}\pi).$$

Again, since $$\tan x = \frac{\sin x}{\cos x},$$

the continuity of $\sin x$ and $\cos x$ involves (Art. 13) that of $\tan x$, except for those values of x which make $\cos x = 0$. These are given by $x = (n + \frac{1}{2})\pi$, where n is integral.

In the same way we might treat the cases of $\sec x$, $\operatorname{cosec} x$, $\cot x$.

The figures on p. 34 shew the graphs of $\sin x$ and $\tan x$. The reader should observe how immediately such relations as

$$\sin(-x) = -\sin x, \quad \sin(\pi - x) = \sin x,$$
$$\sin(x+\pi) = -\sin x, \quad \tan(x+\pi) = \tan x$$

can be read off from the symmetries of the curves.

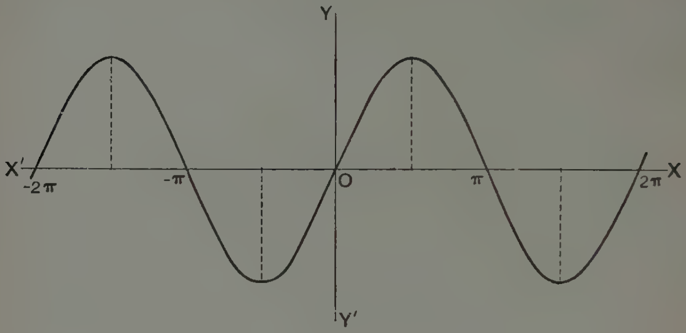

Fig. 18.

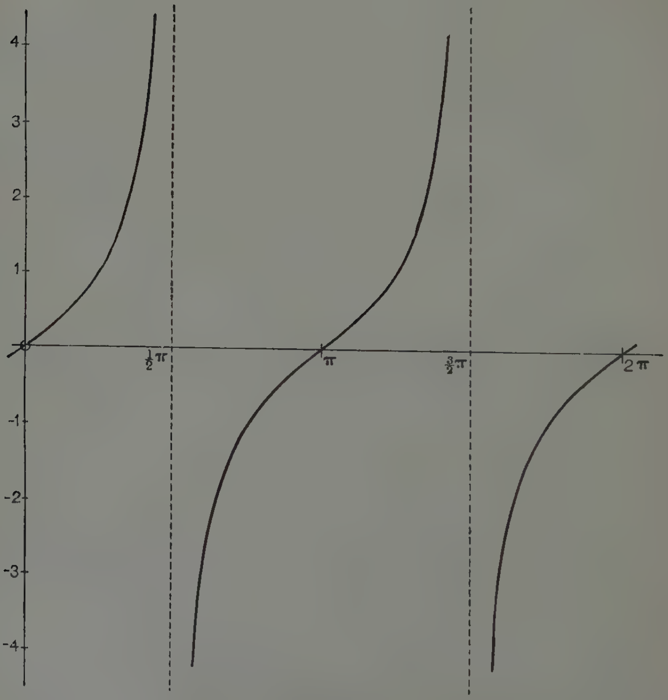

Fig. 19.

17. The Exponential Function.

We consider next the 'exponential' function. This may be defined in various ways; perhaps the simplest, for our purpose, is to define it as the sum of the infinite series

$$1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\dots \qquad \text{.................(1).}$$

To see that this series is convergent, and has therefore a definite 'sum,' for any given value of x, we notice that the ratio of the $(m+1)$th term to the mth is x/m. This ratio can be made as small as we please (in absolute value) by taking m great enough. Hence a point in the series can always be found after which the successive terms will diminish more rapidly than those of any given geometrical progression whatever. The series is therefore convergent, by Art. 6. It is, moreover, 'absolutely' convergent.

If we denote the sum of the series (1) by $E(x)$, it may be shewn that

$$E(x)\times E(y)=E(x+y) \qquad \text{...............(2).}$$

The proof involves the rule for multiplication of absolutely convergent series. Let

and

$$\left.\begin{array}{l} u_0+u_1+u_2+\dots+u_n+\dots \\ v_0+v_1+v_2+\dots+v_n+\dots \end{array}\right\} \text{..................(3)}$$

be two such series, whose sums are U and V respectively; and consider the series

$$w_0+w_1+w_2+\dots+w_n+\dots \qquad \text{...............(4),}$$

where w_n consists of all the products of 'weight'* n which can be formed by multiplying a term of one of the given series (3) by a term of the other; viz.

$$w_0=u_0v_0, \quad w_1=u_0v_1+u_1v_0, \;\dots\dots \quad w_n=u_0v_n+u_1v_{n-1}+u_2v_{n-2}+\dots$$
$$+u_{n-1}v_1+u_nv_0 \text{........(5).}$$

We will first shew that the series (4) will be absolutely convergent, and that its sum will be UV, the product of the sums of the two former series.

Let us write

$$\left.\begin{array}{l} U_n=u_0+u_1+u_2+\dots+u_n, \\ V_n=v_0+v_1+v_2+\dots+v_n, \\ W_n=w_0+w_1+w_2+\dots+w_n \end{array}\right\} \text{...............(5).}$$

---

* The 'weight' is the sum of the suffixes.

Then it is evident, from the actual multiplication, that the product $U_n V_n$ includes all the terms of W_n, and others; whilst W_{2n} includes all the terms of the product $U_n V_n$, and others. Hence in the case where the given series (3) consist entirely of positive terms, we shall have

$$W_n < U_n V_n < UV \qquad \text{(6)}.$$

The ascending sequence of magnitudes W_0, W_1, W_2, ... will therefore (Art. 2) have an upper limit not exceeding UV. Again, since

$$W_{2n} > U_n V_n \qquad \text{(7)},$$

it is evident that the upper limit in question cannot be less than UV. Hence the series (4) is convergent, and its sum is UV. It also follows that if we write

$$U_n V_n = W_n + R_n \qquad \text{(8)},$$

where $$R_n = (u_1 v_n + u_2 v_{n-1} + \ldots + u_n v_1) + \ldots + u_n v_n \qquad \text{(9)},$$

then R_n can be made as small as we please by taking n great enough.

We have still to examine the case where the series (3), one or both, contain negative as well as positive terms. With the above definition of R_n it appears that if both series be absolutely convergent, the limiting value of R_n will be zero when all the letters are made positive. That is, the sum of the absolute values of the several terms of R_n can be made less than any assigned magnitude by taking n great enough. It will be *à fortiori* true that the absolute value of the actual quantity R_n, which includes both positive and negative terms, can be made as small as we please by taking n great enough. It follows from (8) that the limiting value of W_n will be equal to that of $U_n V_n$, *i.e.* it will be equal to UV.

In the application to the exponential series, we put

$$u_n = \frac{x^n}{n\,!}, \qquad v_n = \frac{y^n}{n\,!},$$

with the convention that $u_0 = 1$, $v_0 = 1$, so that

$$U = E(x), \qquad V = E(y).$$

Hence $w_0 = 1$, and (for $n > 0$)

$$w_n = \frac{x^n}{n!} \cdot 1 + \frac{x^{n-1}}{(n-1)!} \cdot \frac{y}{1} + \frac{x^{n-2}}{(n-2)!} \cdot \frac{y^2}{2!} + \ldots + \frac{x}{1} \cdot \frac{y^{n-1}}{(n-1)!} + 1 \cdot \frac{y^n}{n!}$$

$$= \frac{1}{n!} \left\{ x^n + \frac{n}{1} x^{n-1} y + \frac{n(n-1)}{1 \cdot 2} x^{n-2} y^2 + \ldots + \frac{n}{1} x y^{n-1} + y^n \right\}$$

$$= \frac{(x+y)^n}{n!},$$

by the Binomial Theorem. The several terms of the series (4) coincide therefore with those of the series for $E(x+y)$. Hence the result (2).

It follows from (2) that

$$E(x) \times E(y) \times E(z) = E(x+y) \times E(z) = E(x+y+z) \qquad \text{......(10)},$$

and so on for any number of such factors.

Also, since $$E(x) \times E(-x) = E(0) = 1 \qquad \text{............(11)},$$

we have $$E(-x) = \frac{1}{E(x)} \qquad \text{..................(12)}.$$

The function $E(x)$ is continuous for all finite values of x. For, writing h for δx, we have

$$E(x+h) - E(x) = E(x) \cdot E(h) - E(x)$$

$$= E(x) \{E(h) - 1\}.$$

Now $$E(h) - 1 = h \left(1 + \frac{h}{2!} + \frac{h^2}{3!} + \ldots\right).$$

It is easily seen that the series in brackets is absolutely convergent, and that its sum approaches the limit 1 as h is indefinitely diminished; whilst the factor h can be made as small as we please. Hence $E(x+h) - E(x)$ can be made as small as we please by taking h small enough.

When x is positive, every term of the series (1) continually increases with x, and becomes infinite for $x = +\infty$. The same holds *à fortiori* for the sum $E(x)$. Also, in virtue of (12), it appears that if x be positive $E(-x)$ is positive and continually diminishes in absolute value as x increases, and vanishes for $x = \infty$. Hence as x increases from $-\infty$ to $+\infty$, the function $E(x)$ continually increases from 0 to $+\infty$, and assumes once, and only once, every intermediate value.

The accompanying figure shews the curve

$$y = E(x).$$

A column of numerical values of the function $E(x)$ is given in Table E at the end of the book.

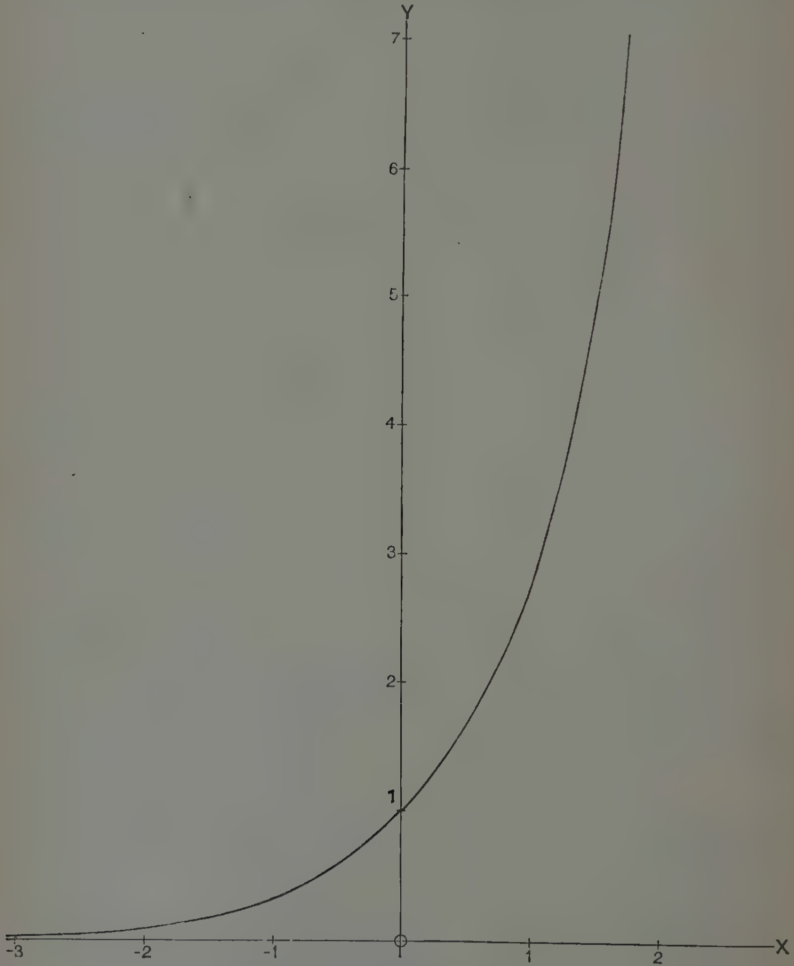

Fig. 20.

18. **The number** *e*.

If we form the product of n factors, each equal to $E(1)$, we have, as in Art. 17, (10),

$$\{E(1)\}^n = E(1 + 1 + \dots \text{ to } n \text{ terms}) = E(n) \quad \dots(1).$$

It is usual to denote the quantity $E(1)$, or

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots \quad \dots\dots\dots\dots\dots\dots(2),$$

by the symbol e. Its value to seven places of decimals is

$$e = 2{\cdot}7182818.$$

With this notation, we have, if n be a positive integer,

$$E(n) = e^n \quad \dots\dots\dots\dots\dots\dots\dots(3).$$

Again, if m/n be an arithmetical fraction (in its lowest terms), we have

$$\left\{E\left(\frac{m}{n}\right)\right\}^n = E\left(\frac{m}{n} + \frac{m}{n} + \dots \text{ to } n \text{ terms}\right) = E(m) = e^m,$$

and therefore

$$E\left(\frac{m}{n}\right) = e^{\frac{m}{n}} \quad \dots\dots\dots\dots\dots\dots(4).$$

Hence, if x be any positive rational quantity, integral or fractional, we have

$$E(x) = e^x \dots\dots\dots\dots\dots\dots\dots(5).$$

It follows, from Art. 17 (12), that

$$E(-x) = \frac{1}{e^x} = e^{-x},$$

so that the formula (5) holds for *all* rational values of x, whether positive or negative*.

---

* The investigations of Arts. 17, 18 are due, substantially, to Cauchy, *Analyse Algébrique* (1821).

The actual calculation of e is very simple. The first 13 terms of the series (2) are as follows:

$$1 + 1 = 2$$

$$\frac{1}{2!} = {\cdot}5$$

$$\frac{1}{3!} = {\cdot}166\ 666\ 667$$

$$\frac{1}{4!} = {\cdot}041\ 666\ 667$$

$$\frac{1}{5!} = {\cdot}008\ 333\ 333$$

$$\frac{1}{6!} = {\cdot}001\ 388\ 889$$

$$\frac{1}{7!} = {\cdot}000\ 198\ 413$$

$$\frac{1}{8!} = {\cdot}000\ 024\ 802$$

$$\frac{1}{9!} = {\cdot}000\ 002\ 756$$

$$\frac{1}{10!} = {\cdot}000\ 000\ 276$$

$$\frac{1}{11!} = {\cdot}000\ 000\ 025$$

$$\frac{1}{12!} = {\cdot}000\ 000\ 002$$

The sum of these numbers is 2·718281830. The error involved in neglecting the remaining terms is

$$\frac{1}{13!} + \frac{1}{14!} + \frac{1}{15!} + \dots,$$

which is less than

$$\frac{1}{13!}\left(1 + \frac{1}{13} + \frac{1}{13^2} + \frac{1}{13^3} + \dots\right), \text{ or } \frac{1}{12 \,.\, 12!},$$

and therefore does not affect the ninth place of decimals. Hence, allowing for the errors of the last figures in the above table, we may say with confidence that the result just found represents the value of e correctly to seven decimal places.

In this book we shall have no need to recognize irrational indices, and the symbol e^x, when x is irrational, is therefore to be considered as (so far) undefined. We may now define it as merely another symbol for the sum of the series $E(x)$. The advantage of this definition is that the notation serves to remind us of the algebraical laws to which the function is subject. Thus we have

$$e^x \,.\, e^y = E(x) \times E(y) = E(x + y) = e^{x+y},$$

whether x and y be rational or irrational.

19. The Hyperbolic Functions.

There are certain combinations of exponential functions whose properties have a close formal analogy with those of the ordinary trigonometrical functions. They are called the hyperbolic sine, cosine, tangent, &c.*, and are defined and denoted as follows:

$$\left.\begin{aligned} \sinh x &= \tfrac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots, \\ \cosh x &= \tfrac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots \end{aligned}\right\} \ldots (1)\dagger,$$

$$\left.\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x}, & \operatorname{sech} x &= \frac{1}{\cosh x}, \\ \coth x &= \frac{\cosh x}{\sinh x}, & \operatorname{cosech} x &= \frac{1}{\sinh x} \end{aligned}\right\} \ldots\ldots\ldots (2).$$

We notice that $\cosh x$, like $\cos x$, is an 'even' function of x; *i.e.* it is unaltered by writing $-x$ for x; whilst $\sinh x$, like $\sin x$, is an 'odd' function, *i.e.* the function is unaltered in absolute value but reversed in sign by the same substitution of $-x$ for x.

The continuity of $\cosh x$ and $\sinh x$ follows from that of e^x and e^{-x} by a theorem of Art. 13. The figure on the next page shews the curves

$$y = e^x, \quad y = e^{-x},$$

together with the curves

$$y = \cosh x, \quad y = \sinh x,$$

which are derived from them by taking half the sum, and half the difference, of the ordinates, respectively‡.

---

* They have in some respects the same relation to the rectangular hyperbola that the circular functions have to the circle. See Art. 98, Ex. 4.

† The series are added according to the rule proved at the end of Art. 6.

‡ The curve $y = \cosh x$ is known in Statics as the 'catenary,' from its being the form assumed by a chain of uniform density hanging freely under gravity.

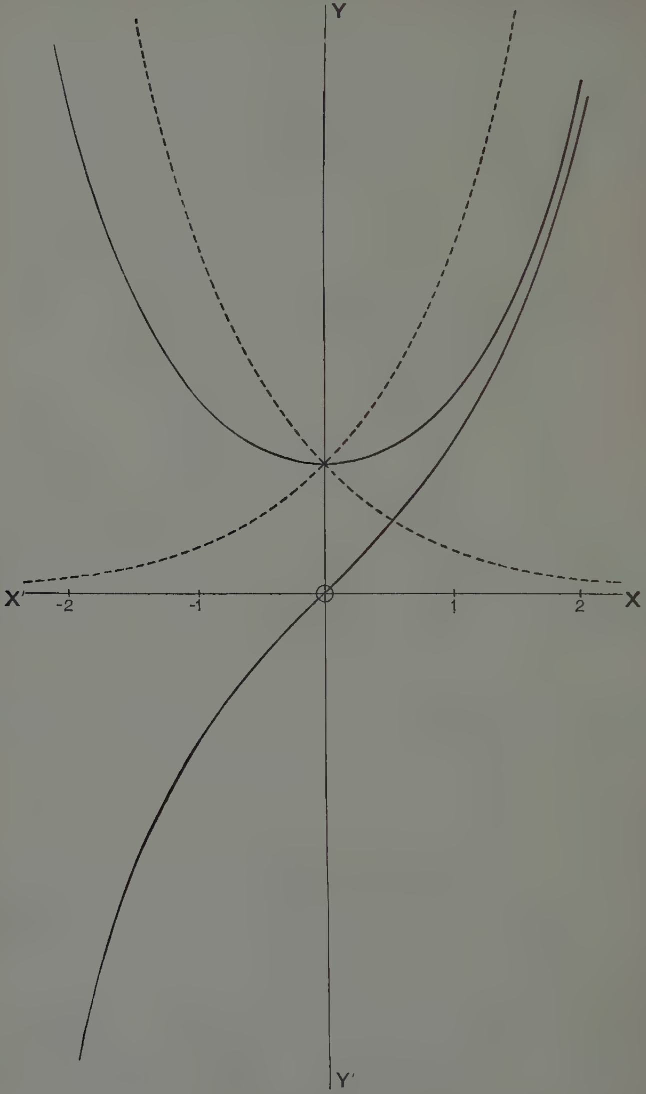

Fig. 21.

Since $\sinh x$ and $\cosh x$ are continuous, whilst $\cosh x$ never vanishes, it follows that $\tanh x$ is continuous for all values of x. Fig. 22 shews the curve

$$y = \tanh x.$$

This has the lines $y = \pm 1$ as asymptotes*.

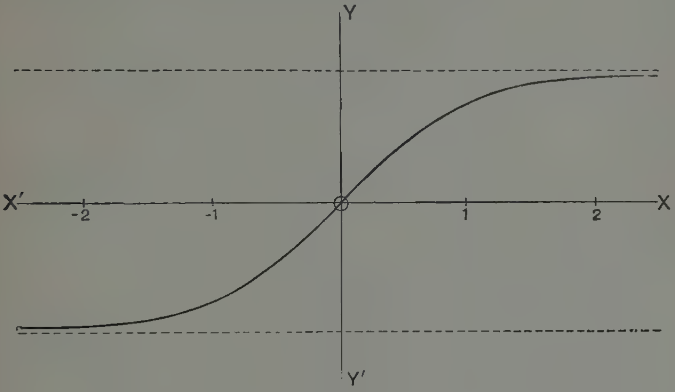

Fig. 22.

It is evident from the definitions (1) that

$$\cosh x + \sinh x = e^x, \quad \cosh x - \sinh x = e^{-x} \quad \dots (3),$$

whence, by multiplication,

$$\cosh^2 x - \sinh^2 x = 1 \quad \dots\dots\dots\dots\dots\dots (4).$$

From this we derive, dividing by $\cosh^2 x$ and $\sinh^2 x$, respectively,

$$\left.\begin{aligned} \operatorname{sech}^2 x &= 1 - \tanh^2 x, \\ \operatorname{cosech}^2 x &= \coth^2 x - 1 \end{aligned}\right\} \quad \dots\dots\dots\dots\dots\dots (5).$$

The formulæ (4) and (5) correspond to the trigonometrical formulæ

$$\cos^2 x + \sin^2 x = 1 \dots\dots\dots\dots\dots\dots\dots\dots (6),$$

$$\left.\begin{aligned} \sec^2 x &= 1 + \tan^2 x, \\ \operatorname{cosec}^2 x &= \cot^2 x + 1 \end{aligned}\right\} \dots\dots\dots\dots\dots\dots\dots (7).$$

---

* The numerical values (to three places) of the functions $\cosh x$, $\sinh x$, $\tanh x$, for values of x ranging from 0 to 2·5 at intervals of 0·1, are given in the Appendix, Table E.

Again, from (1) and (3) we easily find

$$\left.\begin{aligned}\sinh(x \pm y) &= \sinh x \cosh y \pm \cosh x \sinh y,\\ \cosh(x \pm y) &= \cosh x \cosh y \pm \sinh x \sinh y,\end{aligned}\right\}\dots(8),$$

whence, as particular cases,

$$\left.\begin{aligned}\sinh 2x &= 2 \sinh x \cosh x,\\ \cosh 2x &= \cosh^2 x + \sinh^2 x,\end{aligned}\right\}\dots\dots\dots\dots(9).$$

These are the analogues of the trigonometrical formulæ

$$\left.\begin{aligned}\sin(x \pm y) &= \sin x \cos y \pm \cos x \sin y,\\ \cos(x \pm y) &= \cos x \cos y \mp \sin x \sin y,\end{aligned}\right\}\dots\dots\dots(10),$$

and

$$\left.\begin{aligned}\sin 2x &= 2 \sin x \cos x,\\ \cos 2x &= \cos^2 x - \sin^2 x\end{aligned}\right\}\dots\dots\dots\dots\dots(11),$$

respectively.

20. Inverse Functions in general.

If y be a continuous function of x, then under certain conditions x will be a continuous function of y. This will be the case whenever the range of x admits of being divided into portions (not infinitely small) such that within each the function y steadily increases, or steadily decreases, as x increases.

Let us suppose that as x increases from a to b the value of y steadily increases from α to β. Then corresponding to any given value of y between α and β there will be one and only one value of x between a and b. Hence if we restrict

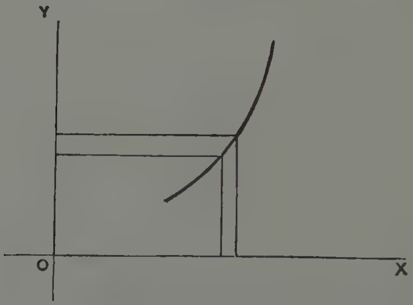

fig. 23.

ourselves to values of x within this interval, x will be a single-valued function of y. Also, if we give any positive increment ϵ to x, within the above interval, y will have a certain finite increment σ, and for all values of δy less than σ, we shall have $\delta x < \epsilon$. A similar argument holds if the increment of x be negative. Hence we can find a positive quantity σ such that, ϵ being any assigned positive quantity, however small, $|\delta x| < \epsilon$ for all values of δy such that $|\delta y| < \sigma$. But this is the condition for the continuity of x regarded as a function of y (Art. 9).

The same conclusion obviously holds if y steadily diminishes in the interval from $x = a$ to $x = b$.

If we do not limit ourselves to a range of x within which the function steadily increases, or steadily diminishes, then to any given value of y there may correspond more than one value of x; the inverse function is then said to be 'many-valued.' Again, it may (and in general will) happen that through some ranges of y there are no corresponding values of x, *i.e.* the inverse function does not exist.

Ex. Let $y = x^2$. This is a continuous function of x, and, if x be positive, continuously increases with x. Hence $x, = \surd y$, is a continuous function of y. If x be unrestricted as to sign, we have two values of x for every positive value of y; these are usually denoted by $\pm \surd y$. If y be negative, the inverse function $\surd y$ does not exist.

If
$$y = f(x) \qquad (1),$$
the inverse functional relation is sometimes expressed by
$$x = f^{-1}(y) \qquad (2).$$
We then have
$$f\{f^{-1}(y)\} = f(x) = y \qquad (3),$$
i.e. the functional symbols f and f^{-1} cancel one another. This is the reason of the notation (2).

21. The Inverse Circular Functions.

The 'goniometric' or 'inverse circular' functions
$$\sin^{-1} x, \qquad \cos^{-1} x, \qquad \tan^{-1} x, \text{ \&c.}$$
are many-valued.

The functions $\sin^{-1} x$, $\cos^{-1} x$ exist for values of x ranging from -1 to $+1$, but not for values outside these limits.

The function $\tan^{-1} x$ exists for all values of x; it is many-valued, the values forming an arithmetical progression with the common difference π.

The graph of any inverse function is derived from that of the direct function by a mere transposition of x and y. The curves for $\sin^{-1} x$ and $\tan^{-1} x$ are shewn in Art. 41.

22. The Logarithmic Function.

The 'logarithmic' function is defined as the inverse of the exponential function. Thus if

$$x = e^y,$$

we have

$$y = \log x \qquad \text{..................(1).}$$

It appears from Art. 17 that as y ranges from $-\infty$ through 0 to $+\infty$, e^y steadily increases from 0 through 1 to $+\infty$. Hence for every *positive* value of x there is one and only one value of $\log x$; moreover this value will be positive or negative, according as $x \gtrless 1$. Also for $x = 0$ we have $y = -\infty$, and for $x = +\infty$, $y = +\infty$. For negative values of x the logarithmic function does not exist.

The ordinary properties of the logarithmic function follow from the above definition in the usual manner.

The full line in Fig. 24 shews the graph of $\log x$. It is of course the same as that of e^x (Fig. 20) with x and y interchanged*.

We can now define the symbol a^x, where a is positive, for the case of x irrational. Since

$$a = e^{\log a},$$

we have, if x be rational,

$$a^x = e^{x \log a} \qquad \text{..........................(2),}$$

and the latter form may be adopted as the *definition* of a^x when x is irrational.

The logarithm of x, as above defined, is sometimes denoted by $\log_e x$ to distinguish it from the common or 'Briggian' logarithm $\log_{10} x$. The latter may be regarded as defined by the statement that

$$y = \log_{10} x, \text{ if } 10^y = x, \text{ or } e^{y \log_e 10} = x \qquad \text{...........(3).}$$

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* The function $\log x$ is tabulated in the Appendix, Table F.

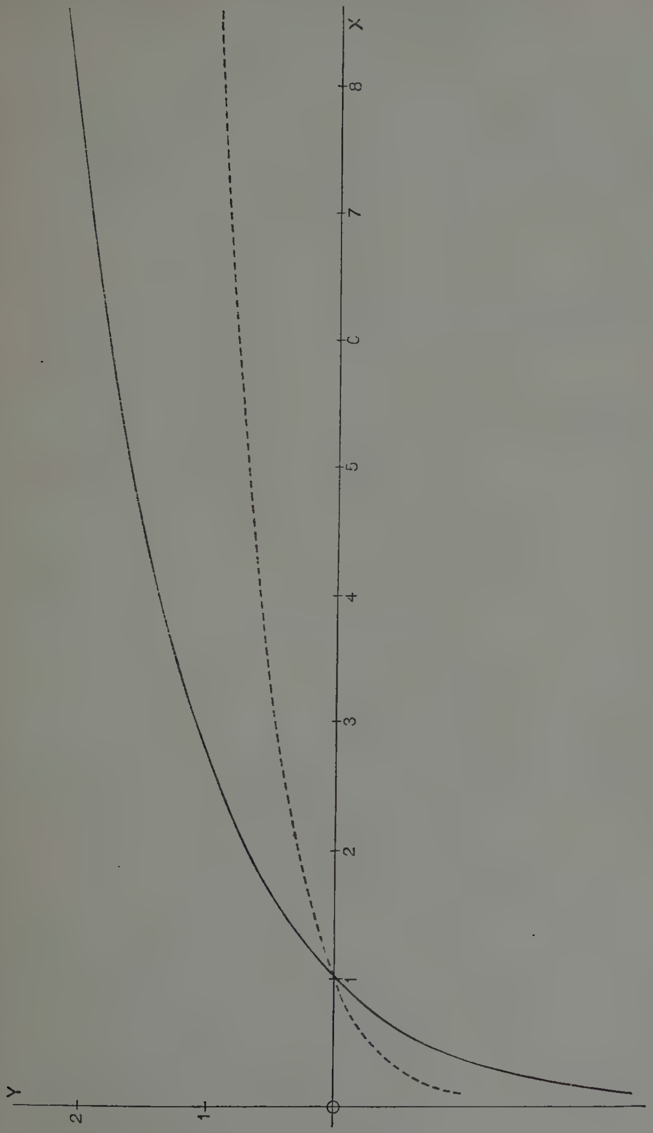

fig. 24.

Hence $$y \log_e 10 = \log_e x, \text{ or } \log_{10} x = \mu \log_e x \qquad (4),$$

where $$\mu = \frac{1}{\log_e 10} = \cdot 43429\ldots\text{*} \qquad (5).$$

Hence the graph of the function $\log_{10} x$ is obtained from that of $\log_e x$ by diminishing the ordinates in the constant ratio μ. See the dotted line in the figure.

In this book we shall always use the symbol $\log x$ in the sense of $\log_e x$.

23. The Inverse Hyperbolic Functions.

The inverse hyperbolic functions

$$\sinh^{-1} x, \quad \cosh^{-1} x, \quad \tanh^{-1} x, \text{ \&c.}$$

are defined in the manner explained in Art. 20; thus the meaning of $y = \sinh^{-1} x$ is that

$$x = \sinh y \qquad (1),$$

and so on.

These functions can all be expressed in terms of the logarithmic function. Thus if

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}) \qquad (2),$$

we have $$e^{2y} - 2xe^y - 1 = 0 \qquad (3).$$

Solving this quadratic in e^y, we find

$$e^y = x \pm \sqrt{(x^2 + 1)} \qquad (4).$$

If y is to be real, e^y must be positive, and the upper sign must be taken. Hence

$$\sinh^{-1} x = \log\{x + \sqrt{(x^2 + 1)}\} \qquad (5).$$

In a similar manner we should find that, if $x > 1$,

$$\cosh^{-1} x = \log\{x \pm \sqrt{(x^2 - 1)}\} \qquad (6).$$

Either sign is here admissible; the quantities $x \pm \sqrt{(x^2 - 1)}$ are reciprocals, and their logarithms differ simply in sign. It appears on sketching the graph of $\cosh^{-1} x$ that for every value of x which is > 1 there are two values of y, equal in magnitude, but opposite in sign.

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* The mode of calculating this quantity will be indicated in Chapter XIII.

Similarly, we should find, for $x^2 < 1$,

$$\tanh^{-1} x = \tfrac{1}{2} \log \frac{1+x}{1-x} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and, for $x^2 > 1$,

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x+1}{x-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

EXAMPLES. II.

1. Draw graphs of the following functions:

(1) $\operatorname{cosec} x$, $\cot x$, $\cot x + \tan x$.

(2) $\operatorname{cosech} x$, $\coth x$, $\coth x - \tanh x$.

(3) $\sin^2 x$, $\tan^2 x$.

(4) $\sin x + \sin 2x$, $\sin x + \cos 2x$.

(5) $\log_{10} \sin x$, $\log_{10} \tan x$, from $x = 0$ to $x = \frac{1}{2}\pi$.

(6) $\sin x^2$, $\sin \sqrt{x}$, $\sin \dfrac{1}{x}$.

2. Prove that the equation

$$\sin x - x \cos x = 0$$

has a root between π and $\frac{3}{2}\pi$.

3. Prove by calculation from the series for e^x that

$1/e = \cdot 367879$, $\cosh 1 = 1 \cdot 5430806$, $\sinh 1 = 1 \cdot 1752012$.

4. Prove that

$$\sqrt{e} = 1 \cdot 6487213, \qquad 1/\sqrt{e} = \cdot 6065307,$$
$$\cosh \tfrac{1}{2} = 1 \cdot 1276260, \qquad \sinh \tfrac{1}{2} = \cdot 5210953.$$

5. If $|a| < |b|$ the equation

$$a \cosh x + b \sinh x = 0$$

has one, and only one, real root.

6. Shew that the function

$$\tanh \frac{1}{x}$$

is discontinuous for $x = 0$.

Draw a graph of the function.

EXAMPLES. III.

Prove the following formulæ:

1. $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$.

2. $\sinh 2x = \dfrac{2\tanh x}{1-\tanh^2 x}$, $\cosh 2x = \dfrac{1+\tanh^2 x}{1-\tanh^2 x}$,

$$\tanh 2x = \frac{2\tanh x}{1+\tanh^2 x}.$$

3. $\cosh^2 x\cos^2 x + \sinh^2 x\sin^2 x = \tfrac{1}{2}(\cosh 2x + \cos 2x)$,
$\cosh^2 x\sin^2 x + \sinh^2 x\cos^2 x = \tfrac{1}{2}(\cosh 2x - \cos 2x)$.

4. $\cosh^2 x\cos^2 x - \sinh^2 x\sin^2 x = \tfrac{1}{2}(1 + \cosh 2x\cos 2x)$,
$\cosh^2 x\sin^2 x - \sinh^2 x\cos^2 x = \tfrac{1}{2}(1 - \cosh 2x\cos 2x)$.

5. $\cosh^2 u + \sinh^2 v = \sinh^2 u + \cosh^2 v = \cosh(u+v)\cosh(u-v)$,
$\cosh^2 u - \cosh^2 v = \sinh^2 u - \sinh^2 v = \sinh(u+v)\sinh(u-v)$.

6. $\sinh^{-1} x = \cosh^{-1}\sqrt{(x^2+1)}$, $\cosh^{-1} x = \sinh^{-1}\sqrt{(x^2-1)}$.

7. $\operatorname{sech}^{-1} x = \log\dfrac{1+\sqrt{(1-x^2)}}{x}$, $\operatorname{cosech}^{-1} x = \log\dfrac{1+\sqrt{(1+x^2)}}{x}$.

8. $$\tanh^{-1}\frac{x^2-1}{x^2+1} = \log x.$$

9. If $$x = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi),$$
then $\sinh x = \tan\psi$, $\cosh x = \sec\psi$, $\tanh x = \sin\psi$,
and $$\tanh\tfrac{1}{2}x = \tan\tfrac{1}{2}\psi.$$

10. $\sinh u + \sinh v = 2\sinh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v)$,
$\sinh u - \sinh v = 2\cosh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v)$,
$\cosh u + \cosh v = 2\cosh\tfrac{1}{2}(u+v)\cosh\tfrac{1}{2}(u-v)$,
$\cosh u - \cosh v = 2\sinh\tfrac{1}{2}(u+v)\sinh\tfrac{1}{2}(u-v)$.

11. $$\tanh\tfrac{1}{2}u = \frac{\sinh u}{\cosh u + 1} = \frac{\cosh u - 1}{\sinh u}$$
$$= \sqrt{\left(\frac{\cosh u - 1}{\cosh u + 1}\right)}.$$

12. $\sinh 3u = 4\sinh^3 u + 3\sinh u$,
$\cosh 3u = 4\cosh^3 u - 3\cosh u$.

24. Upper or Lower Limit of an Assemblage.

Before proceeding further with the theory of continuous functions it is convenient to extend the definitions of the terms 'upper' and 'lower' limit, and 'limiting value,' given in Arts. 2 and 5.

Consider, in the first place, any assemblage of magnitudes, infinite in number, but all less than some finite magnitude β. The assemblage may be defined in any way; all that is necessary is that there should be some criterion by which it can be determined whether a given magnitude belongs to the assemblage or not. For instance, the assemblage may consist of the values which a given function (continuous or not) assumes as the independent variable ranges over any finite or infinite interval.

In such an assemblage there may or may not be contained a 'greatest' magnitude, *i.e.* one not exceeded by any of the rest; but there will in any case be an 'upper limit' to the magnitudes of the assemblage, *i.e.* there will exist a certain magnitude μ such that no magnitude in the assemblage exceeds μ, whilst one (at least) can be found exceeding any magnitude whatever which is less than μ. And if μ be not itself one of the magnitudes of the assemblage, then an infinite number of these magnitudes can be found exceeding any magnitude which is less than μ.

The proof of these statements follows, as in Art. 2, by means of the geometrical representation.

In the same way, if we have an infinite assemblage of magnitudes, all greater than some finite quantity α, there may or may not be a 'least' magnitude in the assemblage; but there will in any case be a 'lower limit' λ such that no magnitude in the assemblage falls below λ, whilst one (at least) can be found below any magnitude whatever which is greater than λ. And if λ be not itself one of the magnitudes of the assemblage, an infinite number of these magnitudes can be found less than any magnitude which is greater than λ.

25. A Continuous Function has a Greatest and a Least Value.

An important property of a continuous function is that in any finite range of the independent variable the function has both a greatest and a least value.

More precisely, if y be a function which is continuous from $x = a$ to $x = b$, inclusively, and if μ be the upper limit of the values which y assumes in this range, there will be some value of x in the range for which $y = \mu$. Similarly for the lower limit.

The theorem is self-evident in the case of a function which steadily increases, or steadily decreases, throughout the range in question, greatest and least values obviously occurring at the extremities of the range. It is therefore true, further, when the function is such that the range can be divided into a *finite* number of intervals in each of which the function either steadily increases or steadily decreases.

The functions ordinarily met with in the applications of the subject are, as a matter of fact, found to be all of this character, but the general tests by which in any given case we ascertain this are established by reasoning which assumes the truth of the theorem of the present Art. See Art. 47. It is therefore desirable as a matter of logic to have a proof which shall assume nothing concerning the function considered except that it is continuous, according to the definition of Art. 9.

The following is an outline of such a demonstration. In the geometrical representation, let $OA = a$, $OB = b$. If at A the value of y is not equal to the upper limit μ, it will be less than μ; let us denote it by y_0. We can form, in an infinite number of ways, an ascending sequence of magnitudes

$$y_0, y_1, y_2, \ldots$$

whose upper limit is μ. For example, we may take y_1 equal to the arithmetic mean of y_0 and μ, y_2 equal to the arithmetic mean of y_1 and μ, and so on. Since, within the range AB, the value of y varies from y_0 to any quantity short of μ, there will (Art. 10) be at least one value of x for which y assumes the intermediate value y_1. Let x_1 denote this value, or (if there be more than one) the least of such

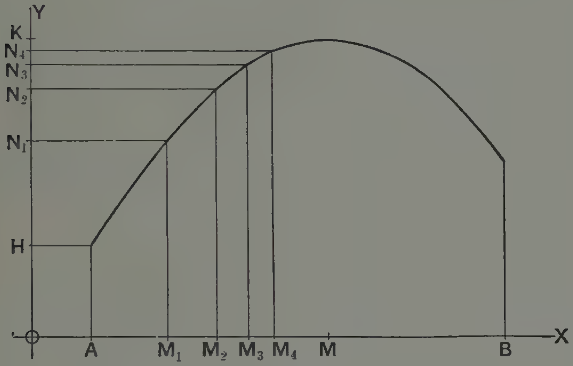

Fig. 25.

values, of x. Similarly, let x_2 be the least value of x for which $y = y_2$, and so on. It is easily seen that the quantities

$$x_1,\ x_2,\ x_3, \ldots$$

(which are represented by the points M_1, M_2, M_3,... in the figure*) must form an ascending sequence; let M represent the upper limit of this sequence. Since any range, however short, extending to the left of M contains points at which

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* The diagram is intended to be merely illustrative, and is not essential to the proof. It is of course evident that any function which can be *adequately* represented by a graph is necessarily of the special character above referred to, for which the present demonstration is superfluous.

In the figure, $OK = \mu$, $OH = y_0$, $ON_1 = y_1$, $ON_2 = y_2$,...; and, in the mode of forming the sequence

$$y_0,\ y_1,\ y_2, \ldots$$

which is suggested (as a particular case) in the text, N_1 bisects HK, N_2 bisects N_1K, N_3 bisects N_2K, and so on.

y differs from μ by less than any assignable magnitude, it follows from the continuity of the function that the value of y at the point M itself cannot be other than μ.

To see that the above theorem is not generally true of *discontinuous* functions, consider a function defined as follows. For values of x *other than* 0 let the value of the function be $(\sin x)/x$, and for $x = 0$ let the function have the value 0. This function has the upper limit 1, to which it can be made to approach as closely as we please by taking $|x|$ small enough; but it never actually attains this limit.

As another example consider the function defined as equal to 0 for all rational values of x, and equal to $\sin \pi x$ for all other values of x. (We have here an instance of a function which is discontinuous for *every* value of x.)

26. Limiting Value of a Function.

Consider the whole assemblage of values which a function y (continuous or not) assumes as the independent variable x ranges over some interval extending on one side of a fixed value x_1. Let us suppose that, as x approaches the value x_1, y approaches a certain fixed magnitude λ in such a way that by taking $|x - x_1|$ sufficiently small we can ensure that for this and for all smaller values of $|x - x_1|$ the value of $|y - \lambda|$ shall be less than σ, where σ may be any assigned magnitude however small. Under these conditions, λ is said to be the 'limiting value' of y as x approaches the value x_1 from the side in question.

The relation is often expressed thus:

$$\lim_{x=x_1} y = \lambda,$$

but in strictness the side from which x approaches the value x_1 should be specified.

If we compare with the above the definition of Art. 9 we see that in the case of a *continuous* function we have

$$\lim_{x=x_1} \phi(x) = \phi(x_1) \qquad \text{..................(1)},$$

or the 'limiting value' of the function coincides with the value of the function itself, and that if x_1 lie *within* the range of the independent variable this holds from whichever side x approaches x_1. If, on the other hand, x_1 coincides with either terminus of the range, x must be supposed to approach x_1 from within the range.

Conversely, a function is not continuous unless the condition (1) be satisfied.

Let us next take the case of a function the range of whose independent variable is unlimited on the side of x positive. If as x is continually increased, y tends to a fixed value λ in such a way that by taking x sufficiently great we can ensure that for this and for all greater values of x we shall have $|y-\lambda|$ less than σ, where σ may be any assigned positive quantity, however small, then λ is called the limiting value of y for $x=\infty$, and we write

$$\lim_{x=\infty} y = \lambda.$$

There is a similar definition of

$$\lim_{x=-\infty} y,$$

when it exists, in the case of an independent variable which is unlimited on the side of x negative.

27. General Theorems relating to Limiting Values.

1°. The limiting value of the sum of any finite number of functions is equal to the sum of the limiting values of the several functions, provided these limiting values be all finite.

2°. The limiting value of the product of any finite number of functions is equal to the product of the limiting values of the several functions, provided these limiting values be all finite.

3°. The limiting value of the quotient of two functions is equal to the quotient of the limiting values of the separate functions, provided these limiting values be finite, and that the limiting value of the divisor is not zero.

The proof is by the same method as in Art. 13, the theorems of which are in fact particular cases of the above.

Thus, let u, v be two functions of x, and let us suppose that as x approaches the value x_1, these tend to the limiting values u_1, v_1, respectively. If, then, we write

$$u = u_1 + \alpha, \quad v = v_1 + \beta,$$

α and β will be functions of x whose limiting values are zero. Now

$$(u+v)-(u_1+v_1)=\alpha+\beta,$$

$$uv-u_1v_1=\alpha v_1+\beta u_1+\alpha\beta,$$

$$\frac{u}{v}-\frac{u_1}{v_1}=\frac{\alpha v_1-\beta u_1}{v_1(v_1+\beta)}.$$

And, as in Art. 13, it appears that by making x sufficiently nearly equal to x_1 we can, under the conditions stated, make the right-hand sides less in absolute value than any assigned magnitude however small.

28. Illustrations.

We have seen in Art. 26 that the limiting value of a continuous function for any value x_1 of the independent variable x, for which the function exists, is simply the value of the function itself for $x=x_1$. It may, however, happen that for certain isolated or extreme values of the variable the function does not exist, or is undefined, whilst it is defined for values of x differing infinitely little from these. It is in such cases that the conception of a 'limiting value' becomes of special importance.

For example, consider the period of oscillation of a given pendulum, regarded as a function of the amplitude α. This function has a definite value for all values of α *between* 0 and π, but it does not exist, in any strict sense, for the extreme values 0 and π. There is, however, a definite limiting value to which the period tends as α approaches the value zero. This limiting value is known in Dynamics as the 'time of oscillation in an infinitely small arc.'

Some further illustrations are appended.

Ex. 1. Take the function

$$\frac{1-\sqrt{(1-x^2)}}{x^2}.$$

The algebraical operations here prescribed can all be performed for any value of x between ± 1, except the value 0, which gives to the fraction the form 0/0. Now the definition of a quotient

a/b is that it is a quantity which, multiplied by b, gives the result a. Since *any* finite quantity, when multiplied by 0, gives the result 0, it is evident that the quotient $0/0$ may have any value whatever. It is therefore said to be 'indeterminate.'

We may, however, multiplying numerator and denominator of the given fraction by $1+\surd(1-x^2)$, put the function in the equivalent form

$$\frac{x^2}{x^2\{1+\surd(1-x^2)\}},$$

and for all values of x between ± 1, *other than* 0, this is equal to

$$\frac{1}{1+\surd(1-x^2)}.$$

Since this function is continuous, and exists for $x=0$, its limiting value for $x=0$ is $\frac{1}{2}$.

Ex. 2. Consider the function

$$\surd(1+x)-\surd x.$$

As x is continually increased this tends to assume the indeterminate form $\infty-\infty$. But, writing the expression in the equivalent form

$$\frac{1}{\surd(1+x)+\surd x},$$

we see that its limiting value for $x=\infty$ is 0.

Ex. 3. To find

$$\lim_{x=\infty} xe^{-x}.$$

This assumes the indeterminate form $\infty\times 0$. But since

$$x/e^x = 1\Big/\left(\frac{1}{x}+1+\frac{x}{2\,!}+\frac{x^2}{3\,!}+\dots\right),$$

we see that the limiting value for $x=\infty$ is 0.

If we write z for e^{-x}, and therefore $-\log z$ for x, we infer that

$$\lim_{z=0} z\log z = 0.$$

In the same way we can prove that

$$\lim_{x=\infty} x^m e^{-x} = 0,$$

where m is any rational quantity.

29. Some Special Limiting Values.

The following examples are of special importance in the Differential Calculus.

1°. To prove that

$$\lim_{x=a} \frac{x^m - a^m}{x - a} = ma^{m-1} \ldots\ldots\ldots\ldots\ldots(1),$$

for all rational values of m.

If m be a positive integer, we have

$$\lim_{x=a} \frac{x^m - a^m}{x - a} = \lim_{x=a} (x^{m-1} + ax^{m-2} + \ldots + a^{m-2}x + a^{m-1})$$
$$= ma^{m-1},$$

since, the number (m) of terms being finite, the limiting value of the sum is equal to the sum of the limiting values of the several terms (Art. 27).

If m be a rational fraction, $=p/q$, say, we put

$$x = y^q, \qquad a = b^q,$$

and therefore

$$\frac{x^m - a^m}{x - a} = \frac{y^{mq} - b^{mq}}{y^q - b^q} = \frac{y^p - b^p}{y^q - b^q}.$$

This fraction is equal to

$$\frac{\dfrac{y^p - b^q}{y - b}}{\dfrac{y^q - b^q}{y - b}}.$$

The limiting value of the numerator is pb^{p-1}, and that of the denominator is qb^{q-1}, by the preceding case. Hence the required limit is

$$\frac{p}{q} b^{p-q}, = \frac{p}{q} a^{p/q-1}, = ma^{m-1},$$

as before.

If m be negative, $= -n$, say, we have

$$\frac{x^m - a^m}{x - a} = \frac{x^{-n} - a^{-n}}{x - a} = -\frac{1}{x^n a^n} \cdot \frac{x^n - a^n}{x - a}.$$

If n be rational, the limiting value of this is

$$-\frac{1}{a^{2n}}\cdot na^{n-1}, = -na^{-n-1}, = ma^{m-1},$$

by the preceding cases.

2°. To prove that

$$\lim_{\theta=0}\frac{\sin\theta}{\theta}=1, \qquad \lim_{\theta=0}\frac{\tan\theta}{\theta}=1 \;\ldots\ldots\ldots(2).$$

If we recall the definition of the 'length' of a circular arc, given in Art. 4, these statements are seen to be little more than truisms. If, in Fig. 4, the angle POQ be $1/n$th of four right angles, then $n \, . \, PQ$ will be the perimeter of an inscribed regular polygon of n sides, and $n(TP+TQ)$ will be the perimeter of the corresponding circumscribed polygon. Now, if

$$\theta = \angle POA = \pi/n,$$

we shall have

$$\frac{\text{chord } PQ}{\text{arc } PQ} = \frac{PN}{\text{arc } PA} = \frac{\sin\theta}{\theta},$$

and

$$\frac{TP+TQ}{\text{arc } PQ} = \frac{PT}{\text{arc } PA} = \frac{\tan\theta}{\theta}.$$

Hence the fractions

$$\frac{\sin\theta}{\theta} \quad \text{and} \quad \frac{\tan\theta}{\theta}$$

denote the ratios which the perimeters of the above-mentioned polygons respectively bear to the perimeter of the circle. Hence, as n is continually increased, each fraction tends to the limiting value unity (Art. 4).

In the above argument, it is assumed that θ is a submultiple of π. But, whatever the value of the angle POQ in the figure, we have

$$\text{chord } PQ < \text{arc } PQ, \text{ and } TP + TQ > \text{arc } PQ;$$

i.e. $(\sin\theta)/\theta$ is less than 1, and $(\tan\theta)/\theta > 1$. Hence these fractions must have respectively an upper and a lower limit; and it follows from the preceding that neither of these limits can be other than unity.

The following numerical table illustrates the way in which the above functions approach their common limiting value as θ is continually diminished.

n	θ/π	$(\sin\theta)/\theta$	$(\tan\theta)/\theta$
4	·25	·90032	1·27324
5	·20	·93549	1·15633
10	·10	·98363	1·03425
20	·05	·99589	1·00831
40	·025	·99897	1·00206
∞	0	1·00000	1·00000

The third and fourth columns give the ratios which the perimeters of the inscribed and circumscribed regular polygons of n sides respectively bear to the perimeter of the circle.

3°. To prove that

$$\lim_{h=0} \frac{e^h - 1}{h} = 1 \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We have $$\frac{e^h - 1}{h} = 1 + \frac{h}{2}\left(1 + \frac{h}{3} + \frac{h^2}{3\,.\,4} + \ldots\right).$$

The series in brackets is convergent, and its sum has the limit 1 when $h = 0$. Hence by taking h small enough, the difference between $(e^h - 1)/h$ and 1 can be made as small as we please.

If we put $$h = \log\left(1 + \frac{z}{n}\right),$$

the theorem (3) takes the form

$$\lim_{n=\infty} \frac{\dfrac{z}{n}}{\log\left(1 + \dfrac{z}{n}\right)} = 1,$$

or $$\lim_{n=\infty} n \log\left(1 + \frac{z}{n}\right) = z,$$

whence $$\lim_{n=\infty} \left(1 + \frac{z}{n}\right)^n = e^z \qquad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

30. Infinitesimals.

A variable quantity which in any process tends to the limiting value zero is said ultimately to vanish, or to be 'infinitely small.'

Two infinitely small quantities are said to be ultimately equal when the limiting value of the ratio of one to the other is unity.

Thus, in Fig. 4, p. 7, when the angle POQ is indefinitely diminished, NA and AT are ultimately equal. For, by similar triangles,

$$\frac{OP}{ON}=\frac{OT}{OP},$$

and therefore
$$\frac{OP-ON}{ON}=\frac{OT-OP}{OP},$$

or
$$\frac{NA}{AT}=\frac{ON}{OP};$$

and the limiting value of the ratio ON/OP is unity.

It is sometimes convenient to distinguish between different orders of infinitely small quantities. Thus if u, v are two quantities which tend simultaneously to the limit zero, and if the limit of the ratio v/u be finite and not zero, then v is said to be an infinitely small quantity of the same order as u. But, if the limit of the ratio v/u be zero, then v is said to be an infinitely small quantity of a higher order than u. More particularly, if the limit of v/u^m be finite and not zero, v is said to be an infinitesimal of the mth order, the standard being u.

Thus, in the figure referred to, NT is an infinitesimal of the second order, if the standard be PN. For

$$PN^2=ON\,.\,NT,$$

whence
$$\frac{NT}{PN^2}=\frac{1}{ON},\ =\frac{1}{OA}\text{ in the limit.}$$

In calculations involving quantities which are ultimately made to vanish, only infinitesimals of the lowest order present need as a rule be retained; since the neglect *ab initio* of any finite number of infinitesimals of higher order will make no difference in the accuracy of the final result. We shall have frequent exemplifications of this principle.

A quantity which in any process finally exceeds any assignable magnitude is said to be 'infinitely great.' And if one such quantity u be taken as a standard, any other v is said to be infinitely great of the mth order, when the limit of v/u^m is finite and not zero.

EXAMPLES. IV.

1. Shew geometrically that the sequence

$$1, \quad 1+\frac{p}{p+q}, \quad 1+\frac{p}{p+q}+\left(\frac{p}{p+q}\right)^2, \ \ldots$$

has the upper limit $$1+\frac{p}{q}.$$

2. Find the upper and lower limits of the magnitudes

$$\frac{2n}{n^2+1}$$

where $n=1, 2, 3, \ldots$.

3. If a and b be positive, and $a>b$, the function

$$\frac{ae^x+be^{-x}}{e^x+e^{-x}}$$

has the upper limit a and the lower limit b.

4. Find the limiting values, for $x=0$, of

$$\frac{\sin ax}{x} \quad \text{and} \quad \frac{\sinh ax}{x}.$$

5. Trace the curves

$$y=\frac{\sin x}{x}, \qquad y=\frac{\sinh x}{x}.$$

6. Prove that $$\lim_{x=0}\frac{1-\cos x}{x^2}=\tfrac{1}{2}.$$

7. Prove that

$$\lim_{x=\frac{1}{2}\pi}(\sec x-\tan x)=0.$$

8. Prove that $$\lim_{x=0}\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}=1.$$

9. Prove that

$$\lim_{x=\infty}\{\sqrt{(x^2+x+1)}-x\}=\tfrac{1}{2}.$$

10. Prove that $\lim_{x=\infty} \dfrac{\log x}{x} = 0,$

and hence shew that $\lim_{n=\infty} \sqrt[n]{n} = 1.$

11. Find the limiting values, for $x = 0$, of

$$\frac{\sin^{-1} x}{x} \text{ and } \frac{\tan^{-1} x}{x}.$$

12. A straight line AB moves so that the sum of its intercepts OA, OB on two fixed straight lines OX, OY is constant. If P be the ultimate intersection of two consecutive positions of AB, and Q the point where AB is met by the bisector of the angle XOY, then $AP = QB$.

13. Through a point A on a circle a chord AP is drawn, and on the tangent at A a point T is taken such that $AT = AP$. If TP produced meet the diameter through A in Q, the limiting value of AQ when P moves up to A is double the diameter of the circle.

14. A straight line AB moves so as to include with two fixed straight lines OX, OY a triangle AOB of constant area. Prove that the limiting position of the intersection of two consecutive positions of AB is the middle point of AB.

15. A straight line AB of constant length moves with its extremities on two fixed straight lines OX, OY which are at right angles to one another. Prove that if P be the ultimate intersection of two consecutive positions of AB, and N the foot of the perpendicular from O on AB, then $AP = NB$.

16. Tangents are drawn to a circular arc at its middle point and at its extremities; prove that the area of the triangle contained by the three tangents is ultimately one-half that of the triangle whose vertices are the three points of contact.

17. If PCP' be any fixed diameter of an ellipse, and QV any ordinate to this diameter; and if the tangent at Q meet CP produced in T, the limiting value of the ratio $TP : PV$, when PV is infinitely small, is unity.

CHAPTER II.

DERIVED FUNCTIONS.

31. Definition and Notation.

Let y be a function which is continuous over a certain range of the independent variable x; let δx be any increment of x such that $x + \delta x$ lies within the above range, and let δy be the consequent increment of y. Then, x being regarded as fixed, the ratio

$$\frac{\delta y}{\delta x} \quad \text{..............................(1)}$$

will be a function of δx. If as δx (and consequently also δy) approaches the value zero, this ratio tends to a definite and unique limiting value, the value thus arrived at is called the 'derived function,' or the 'derivative,' or the 'differential coefficient,' of y with respect to x, and is denoted by the symbol

$$\frac{dy}{dx} \quad \text{..........................(2).}$$

More concisely, the derived function (when it exists) is the limiting value of the ratio of the increment of the function to the increment of the independent variable, when both increments are indefinitely diminished.

It is to be carefully noticed that in the above definition we speak of the limiting value of a certain ratio, and not of the ratio of the limiting values of δy, δx. The latter ratio is indeterminate, being of the form $0/0$.

The symbol dy/dx is to be regarded as indecomposable, it is not a fraction, but the limiting value of a fraction. The fractional appearance is preserved merely in order to remind us of the manner in which the limiting value was approached.

When we say that the ratio $\delta y/\delta x$ tends to a unique limiting value, it is implied that (when x lies *within* the range of the independent variable) this value is the same whether δx approach the value 0 from the positive or from the negative side. It may happen that there is one limiting value when δx approaches 0 from the positive, and another when δx approaches 0 from the negative side. In this case we may say that there is a 'right-derivative,' and a 'left-derivative,' but no proper 'derivative' in the sense of the above definition.

The question whether the ratio $\delta y/\delta x$ really has a determinate limiting value depends on the nature of the original function y. Functions for which the limit is determinate and unique (save for isolated values of x) are said to be 'differentiable.' All other functions are excluded *ab initio* from the scope of the Differential Calculus.

A differentiable function is necessarily continuous, but the converse statement is now known not to be correct. Functions which are continuous without being differentiable are, however, of very rare occurrence in Mathematics, and will not be met with in this book.

There are various other notations for the derived function, in place of dy/dx. The derived function is often indicated by attaching an accent to the symbol denoting the original function. Thus if

$$y = \phi(x) \qquad\qquad (3),$$

the derived function may be denoted by y' or by $\phi'(x)$.

Since
$$\frac{\delta y}{\delta x} = \frac{\phi(x+\delta x) - \phi(x)}{\delta x},$$

we have, writing h for δx,

$$\phi'(x) = \lim_{h=0} \frac{\phi(x+h) - \phi(x)}{h} \qquad\qquad (4).$$

The operation of finding the differential coefficient of a given function is called 'differentiating.' If x be the independent variable, we may look upon d/dx as a symbol denoting this operation. It is often convenient to replace this by the single letter D; thus we may write, indifferently,

$$\frac{dy}{dx}, \quad \frac{d}{dx}y, \quad Dy,$$

for the differential coefficient of y with respect to x.

32. Geometrical meaning of the Derived Function.

In the annexed figure, let

$$OM = x, \quad ON = x + \delta x, \quad PM = y, \quad QN = y + \delta y,$$

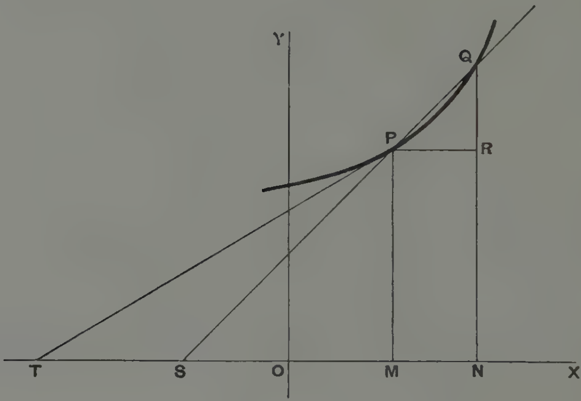

Fig. 26.

and draw PR parallel to OX. Let QP produced cut the axis of x in S. Then

$$\frac{\delta y}{\delta x} = \frac{QR}{PR} = \frac{PM}{SM} = \tan PSX \;\ldots\ldots\ldots\ldots(1).$$

As δx is indefinitely diminished, Q approaches P, and it follows that if the derived function exist the line PQ tends to a definite limiting position PT, such that

$$\tan PTX = \frac{dy}{dx} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

It appears then that the assemblage of points (Art. 11) which represents any differentiable function has at each of its points a definite direction, or a definite 'tangent-line.' And the derived function is the tangent of the angle which this line makes with the axis of x.

The question as to whether a continuous function can be represented by a curve depends, as already stated (Art. 11), on the meaning which we attribute to the term 'curve.' In its ordinary acceptation, the word implies not merely the idea of a connected assemblage of points, but also the existence of a definite tangent-line at every point, and (further) that the direction of this tangent-line varies continuously as we pass along the assemblage. That is, it is implied that the ordinate y is a differentiable function of the abscissa x, and that the derived function dy/dx is itself a continuous function of x. These conditions will be found to be satisfied, save occasionally at isolated points, by all the functions met with in the ordinary applications of the Calculus. And whenever we speak of a 'curve,' we shall, for the present, not attach to the term any connotation beyond what is contained in the above statements.

It is convenient to have a name for the property of a curve which is measured by the derived function. We shall use the term 'gradient' in this sense, viz. if from any point P on the curve, we draw the tangent-line, *to the right*, the gradient is the tangent of the angle which this line makes with the positive direction of the axis of x.

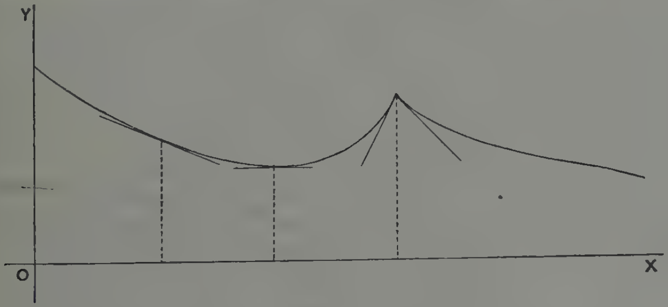

Fig. 27.

If this angle be negative, the gradient is negative. If the tangent-line be parallel to the axis of x, the gradient is zero. If it be perpendicular to the axis of x, the gradient is infinite. When for a particular value of x we have a right-derivative and a left-derivative, different from one another, then on the corresponding curve there are two branches making an angle with one another. The value of dy/dx is then discontinuous. The figure illustrates some of these cases.

33. Physical Illustrations.

The importance of the derived function in the various applications of the subject rests on the fact that it gives us a measure of the *rate of increase* of the original function, per unit increase of the independent variable.

To illustrate this, we may consider, first, the rectilinear motion of a point. The distance s of the point from some fixed origin in the line of motion will be a function of the time t reckoned from some fixed epoch. The relation between these variables is often exhibited graphically by a 'curve of positions,' in which the abscissæ are proportional to t and the ordinates to s; see Fig. 7, p. 14. If in the interval δt the space δs is described, the ratio $\delta s/\delta t$ is called the 'mean velocity' during the interval δt; *i.e.* a point moving with a constant velocity equal to this would accomplish the same space δs in the same time δt. In the limit, when δt (and consequently also δs) is indefinitely diminished, the limiting value to which this mean velocity tends is adopted, by definition, as the measure of the 'velocity at the instant t.' In the notation of the calculus, therefore, this velocity v is given by the formula

$$v = \frac{ds}{dt} \quad \text{.............................(1).}$$

In the graphical representation aforesaid, v is the gradient of the curve of positions.

Again, the velocity v is itself a function of t. The curve representing this relation is called the 'curve of velocities.' If δv be the increase of velocity in the interval δt, then $\delta v/\delta t$ is called the 'mean rate of increase of velocity,' or the 'mean acceleration' in this interval. The limiting value to which the mean acceleration tends when δt is indefinitely diminished is called the 'acceleration at the instant t.' If this acceleration be denoted by a, we have

$$a = \frac{dv}{dt} \quad \text{.............................(2).}$$

In the graphical representation, α is the gradient of the curve of velocities.

In the case of a rigid body revolving about a fixed axis, if θ be the angle through which the body has revolved from some standard position, the 'mean angular velocity' in any interval δt is denoted by $\delta\theta/\delta t$, and the 'angular velocity at the instant t,' by

$$\frac{d\theta}{dt} \quad\text{..............................}(3).$$

Again, if ω denote this angular velocity, the 'mean angular acceleration in the interval δt' is denoted by $\delta\omega/\delta t$, and the 'angular acceleration at the instant t' by

$$\frac{d\omega}{dt} \quad\text{..............................}(4).$$

Again, the length of a bar of given material is a function of the temperature (θ). If x be the length at temperature θ of a bar whose length at some standard temperature (say 0°) is unity, then $\delta x/\delta\theta$ represents the mean coefficient of (linear) expansion from temperature θ to temperature $\theta + \delta\theta$, and $dx/d\theta$ represents the coefficient of expansion at temperature θ.

As another example, suppose we have a fluid which is free to assume a series of states such that the pressure (p) is a definite function of the volume (v) of unit mass. If the volume change from v to $v + \delta v$, the fraction $-\delta v/v$ measures the ratio of the diminution of volume to the original volume, and gives therefore the 'compression.' The ratio of the increment of pressure δp required to produce this compression, to the compression, is $-v\delta p/\delta v$. The limiting value of this when δv is infinitely small, viz. $-vdp/dv$, is defined to be the 'elasticity of volume' of the fluid under the given conditions.

34. **Differentiations** *ab initio.*

Before investigating general rules for calculating the derivatives of given analytical functions, we may discuss a few examples independently from first principles.

Ex. 1. If $y = x$, we have $\delta y = \delta x$, and therefore

$$\frac{\delta y}{\delta x} = 1, \text{ whence } \frac{dy}{dx} = 1.$$

Ex. 2. Let $$y = x^2 \text{ ....(1).}$$

We have, writing h for δx,

$$\frac{\delta y}{\delta x} = \frac{(x+h)^2 - x^2}{h} = 2x + h.$$

Proceeding to the limit, when $h = 0$, we find

$$\frac{dy}{dx} = 2x \text{ ....(2).}$$

Ex. 3. Let $$y = \frac{1}{x} \text{ ....(3).}$$

We have $$\delta y = \frac{1}{x+h} - \frac{1}{x} = -\frac{h}{x(x+h)},$$

and therefore $$\frac{\delta y}{\delta x} = -\frac{1}{x(x+h)}.$$

Hence $$\frac{dy}{dx} = -\lim_{h=0} \frac{1}{x(x+h)} = -\frac{1}{x^2} \text{ ....(4).}$$

The negative sign indicates that y diminishes as x increases.

Ex. 4. If $$y = \sqrt{x} \text{ ....(5),}$$

we have $$\delta y = \sqrt{(x+h)} - \sqrt{x} = \frac{h}{\sqrt{(x+h)} + \sqrt{x}},$$

$$\frac{\delta y}{\delta x} = \frac{1}{\sqrt{(x+h)} + \sqrt{x}}.$$

Proceeding to the limit ($h = 0$), we find

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \text{ ....(6).}$$

EXAMPLES. V.

1. Find, from first principles, the derived functions of

$$x^3, \quad x^4, \quad \frac{1}{x^2}, \quad \frac{1}{\sqrt{x}}.$$

2. Also of $$\frac{1}{x+a}, \quad \frac{x+a}{x-a}, \quad \frac{1}{x^2-a^2}.$$

3. Also of $$\sqrt{(x^2+a^2)}, \quad \frac{1}{\sqrt{(x^2+a^2)}}.$$

4. Also of $\cot x$, $\sec x$, $\operatorname{cosec} x$.

5. Also of $\sin^2 x$, $\cos^2 x$, $\sin 2x$, $\cos 2x$.

6. If, in the rectilinear motion of a point,

$$s = ut + \tfrac{1}{2}at^2,$$

where u, a are constants, prove that the velocity at time t is $u + at$, and that the acceleration is constant.

7. If the pressure and the volume of a gas kept at constant temperature be connected by the relation

$$pv = \text{const.},$$

the cubical elasticity is equal to p.

8. If the radius of a circle be increasing at the rate of one foot per second, find the rate of increase of the area, in square feet per second, at the instant when the radius is 10 feet.

9. If the area of a circle increase at a uniform rate, the rate of increase of the perimeter varies inversely as the radius.

10. If the volume of a gramme of water varies as

$$1 + \frac{(\theta - 4)^2}{144000},$$

where θ is the temperature centigrade, find the coefficients of cubical expansion for $\theta = 0°$ and $\theta = 20°$.

35. Differentiation of Standard Functions.

1°. If $$y = x^m \qquad\text{.........................}(1),$$

we have $$\frac{\delta y}{\delta x} = \frac{(x + \delta x)^m - x^m}{(x + \delta x) - x}.$$

It has been shewn in Art. 29, 1°, that, for all rational values of m, the limiting value of this fraction when $\delta x = 0$ is mx^{m-1}. Hence

$$\frac{dy}{dx} = mx^{m-1} \qquad\text{.....................}(2).$$

Ex. If $m = 2$, $dy/dx = 2x$; if $m = \frac{1}{2}$, $dy/dx = \frac{1}{2}x^{-\frac{1}{2}}$. Cf. Art. 34.

2°. If $$y = \sin x \qquad\text{.......................}(3)$$

we have, writing h for δx,

$$\frac{\delta y}{\delta x} = \frac{\sin(x + h) - \sin x}{h} = \frac{\sin \frac{1}{2}h}{\frac{1}{2}h} \cdot \cos(x + \tfrac{1}{2}h).$$

If the angles be expressed in 'circular measure,' we have

$$\lim_{h=0}\left(\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right)=1,$$

by Art. 29, 2°; and the limiting value of the second factor is $\cos x$. Hence

$$\frac{dy}{dx}=\cos x \quad\text{.......................(4).}$$

The student should refer to the graph of $\sin x$ on p. 34, and notice how the gradient of the curve varies in accordance with this formula.

3°. If
$$y=\cos x \quad\text{.......................(5),}$$
we have
$$\frac{\delta y}{\delta x}=\frac{\cos(x+h)-\cos x}{h}$$
$$=-\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}.\sin(x+\tfrac{1}{2}h);$$
the limiting value of which is, on the same understanding as before,

$$\frac{dy}{dx}=-\sin x \quad\text{....................(6).}$$

4°. If
$$y=\tan x \quad\text{.......................(7),}$$
we have
$$\frac{\delta y}{\delta x}=\frac{\tan(x+h)-\tan x}{h}=\frac{\sin(x+h)\cos x-\cos(x+h)\sin x}{h\cos x\cos(x+h)}$$
$$=\frac{\sin h}{h}.\frac{1}{\cos x\cos(x+h)}.$$
Hence, in the limit,

$$\frac{dy}{dx}=\frac{1}{\cos^2 x}=\sec^2 x \quad\text{..................(8).}$$

This shews that the gradient of the curve $y=\tan x$, between the points of discontinuity, is always positive; see Fig. 19, p. 34.

5°. If
$$y=e^x \quad\text{.......................(9),}$$
we have
$$\frac{\delta y}{\delta x}=\frac{e^{x+h}-e^x}{h}=\frac{e^h-1}{h}e^x.$$

It was shewn in Art. 29, 3° that

$$\lim_{h=0} \frac{e^h - 1}{h} = 1.$$

Hence $$\frac{dy}{dx} = e^x \quad \text{......................(10).}$$

More generally, if $$y = e^{kx} \quad \text{......................(11),}$$

we have

$$\frac{dy}{dx} = \lim_{h=0} \frac{e^{k(x+h)} - e^{kx}}{h} = \lim_{kh=0} \frac{e^{kh} - 1}{kh} \, . \, ke^{kx}$$

$$= ke^{kx} \quad \text{......................................(12).}$$

In particular, if $$y = e^{-x} \quad \text{......................(13),}$$

we have $$\frac{dy}{dx} = -e^{-x} \quad \text{......................(14).}$$

Again, if a be any positive quantity, we have by the definition of Art. 22 (2),

$$a^x = e^{x \log a}.$$

Hence if $$y = a^x \quad \text{.........................(15),}$$

we have, by (12),

$$\frac{dy}{dx} = \log a \, . \, e^{x \log a} = \log a \, . \, a^x \quad \text{...........(16).}$$

36. Rules for differentiating combinations of simple types. Differentiation of a Sum.

1°. Let $$y = u + C \quad \text{......................(1),}$$

where u is a known function of x, and C is a constant. We have

$$y + \delta y = u + \delta u + C,$$

and therefore $$\delta y = \delta u,$$

$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x},$$

or, in the limit, $$\frac{dy}{dx} = \frac{du}{dx} \quad \text{........................(2).}$$

This fact, that an *additive* constant disappears on differentiation, obvious as it is, is very important. The geometrical meaning is that shifting a curve bodily parallel to the axis of y does not alter the gradient.

2°. Let
$$y = u + v \quad\text{......................}(3),$$
where u, v are given functions of x. As in Art. 13, we find
$$\delta y = \delta u + \delta v,$$
and therefore
$$\frac{\delta y}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}.$$
Hence, since the limiting value of a sum is the sum of the limiting values of the several terms,
$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad\text{......................}(4).$$

Again, if
$$y = u + v + w \quad\text{......................}(5),$$
we have
$$\frac{dy}{dx} = \frac{d}{dx}(u + v) + \frac{dw}{dx}$$
$$= \frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} \quad\text{..................}(6),$$
by a double application of the preceding result. In this way we can prove, step by step, that the derived function of the sum of any *finite* number of given functions is the sum of the derivatives of the separate functions.

Ex. The derived function of
$$A_0x^m + A_1x^{m-1} + \dots + A_{m-1}x + A_m$$
is
$$mA_0x^{m-1} + (m-1)A_1x^{m-2} + \dots + A_{m-1}.$$

37. Differentiation of a Product.

1°. If
$$y = Cu \quad\text{........................}(1),$$
where C is a constant, and u a function of x, we have
$$y + \delta y = C(u + \delta u),$$
and therefore
$$\delta y = C\delta u.$$
Hence
$$\frac{\delta y}{\delta x} = C\frac{\delta u}{\delta x},$$
and, proceeding to the limit,
$$\frac{dy}{dx} = C\frac{du}{dx} \quad\text{.......... ....}(2).$$

Hence a constant factor remains attached after the differentiation.

The geometrical meaning of this result is that if all the ordinates of a curve be altered in a given ratio, the gradient is altered in the same ratio. Cf. Fig. 24, p. 47.

2°. Let $$y = uv \qquad\qquad (3),$$ where u, v are both functions of x. As in Art. 13, we find

$$\begin{aligned}\delta y &= (u+\delta u)(v+\delta v) - uv\\ &= v\delta u + u\delta v + \delta u\delta v,\end{aligned}$$

and therefore $$\frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + (u+\delta u)\frac{\delta v}{\delta x}.$$

Hence, proceeding to the limit, and making use of the principle that the limit of a product is the product of the limits, we have

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \qquad\qquad (4).$$

If we divide both sides of this equation by $y, = uv$, we obtain the form

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx}.$$

This result is easily extended; thus if $y = uvw$, we have, writing $y = zw$, where $z = uv$,

$$\begin{aligned}\frac{1}{y}\frac{dy}{dx} &= \frac{1}{z}\frac{dz}{dx} + \frac{1}{w}\frac{dw}{dx}\\ &= \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx} \qquad\qquad (5),\end{aligned}$$

by a double application of the preceding result. And so on for any finite number of factors.

If we multiply both sides by $y, = uvw\ldots$, the generalized form of the last result becomes

$$\frac{dy}{dx} = vw\ldots\frac{du}{dx} + uw\ldots\frac{dv}{dx} + uv\ldots\frac{dw}{dx} + \ldots \qquad\qquad (6),$$

or, in words:

The derived function of a product is found by differentiating with respect to x, so far as it is involved in each factor separately, the other factors being treated as constants, and adding the results.

Ex. 1. If $y = u.u.u \ldots\ldots$ to m factors $= u^m$ ....(7),

we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{u}\frac{du}{dx} + \ldots \text{ to } m \text{ terms} = \frac{m}{u}\frac{du}{dx},$$

whence $$\frac{dy}{dx} = mu^{m-1}\frac{du}{dx} \qquad \text{.......................(8).}$$

A general proof of this result, free from the restriction that m is a positive integer, is given in Art. 39.

Ex. 2. If $$y = \sin x \cos x \qquad \text{........................(9),}$$

we have $$\frac{dy}{dx} = \cos x \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(\cos x)$$
$$= \cos x \,.\, \cos x - \sin x \,.\, \sin x$$
$$= \cos^2 x - \sin^2 x = \cos 2x \qquad \text{....................(10).}$$

Ex. 3. If $$y = x^2 \sin x \qquad \text{........................(11),}$$

we have $$\frac{dy}{dx} = x^2 \frac{d}{dx}(\sin x) + \sin x \frac{d}{dx}(x^2)$$
$$= x^2 \cos x + 2x \sin x \qquad \text{.....................(12).}$$

38. Differentiation of a Quotient.

Let $$y = \frac{u}{v} \qquad \text{..........................(1),}$$

where u, v are given functions of x. As in Art. 13, we find

$$\delta y = \frac{u + \delta u}{v + \delta v} - \frac{u}{v} = \frac{v\delta u - u\delta v}{v(v + \delta v)},$$

whence $$\frac{\delta y}{\delta x} = \frac{v\dfrac{\delta u}{\delta x} - u\dfrac{\delta v}{\delta x}}{v(v + \delta v)}.$$

Hence, in the limit, $$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{....................(2).}$$

In words: To find the derived function of a quotient, from the product of the denominator into the derived function of the numerator subtract the product of the numerator into the derived function of the denominator, and divide the result by the square of the denominator.

Ex. 1. If
$$y=\frac{1+x+x^2}{1-x+x^2}\qquad\text{.....................(3)},$$

we have
$$\frac{du}{dx}=1+2x,\quad \frac{dv}{dx}=-1+2x,$$

whence
$$\frac{dy}{dx}=\frac{(1+2x)(1-x+x^2)+(1-2x)(1+x+x^2)}{(1-x+x^2)^2}$$
$$=\frac{2(1-x^2)}{(1-x+x^2)^2}\qquad\text{.....................(4)}.$$

The particular case
$$y=\frac{1}{v}\qquad\text{.....................(5)}$$

is worthy of separate notice. We then have
$$\delta y=\frac{1}{v+\delta v}-\frac{1}{v}=-\frac{\delta v}{v(v+\delta v)},$$
$$\frac{\delta y}{\delta x}=-\frac{\dfrac{\delta v}{\delta x}}{v(v+\delta v)},$$
$$\frac{dy}{dx}=-\frac{1}{v^2}\frac{dv}{dx}\qquad\text{.....................(6)}.$$

This might of course have been deduced by putting $u=1$, $du/dx=0$ in the general formula (2).

Ex. 2. If
$$y=\frac{1}{u^m}\qquad\text{.....................(7)},$$

where m is a positive integer, we have
$$\frac{dy}{dx}=-\frac{1}{u^{2m}}\cdot mu^{m-1}\frac{du}{dx}=-mu^{-m-1}\frac{du}{dx}\qquad\text{...........(8)};$$

see Art. 39.

The formula of this Art. might also be deduced from the result of Art. 37, 2°. If $y=u/v$, we have $u=vy$, and therefore
$$\frac{1}{u}\frac{du}{dx}=\frac{1}{v}\frac{dv}{dx}+\frac{1}{y}\frac{dy}{dx}\qquad\text{.....................(9)},$$

whence
$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}-\frac{1}{v}\frac{dv}{dx} \quad\text{..............}(10);$$
this is equivalent to (2) above.

The following examples are important:

1°. If
$$y=\tan x=\frac{\sin x}{\cos x} \quad\text{.................}(11),$$
we find
$$\frac{dy}{dx}=\frac{\cos x\,\dfrac{d}{dx}(\sin x)-\sin x\,\dfrac{d}{dx}(\cos x)}{\cos^2 x}$$
$$=\frac{\cos^2 x+\sin^2 x}{\cos^2 x}=\sec^2 x\text{.........}(12).$$
This agrees with Art. 35, 4°.

Similarly, if
$$y=\cot x \quad\text{.........................}(13),$$
we find
$$\frac{dy}{dx}=-\operatorname{cosec}^2 x \quad\text{...................}(14).$$

2°. If
$$y=\sec x=\frac{1}{\cos x} \quad\text{.................}(15),$$
we have
$$\frac{dy}{dx}=-\frac{1}{\cos^2 x}\frac{d}{dx}(\cos x)=\frac{\sin x}{\cos^2 x} \quad\text{.........}(16).$$

Similarly, if
$$y=\operatorname{cosec} x\text{.......................}(17),$$
we find
$$\frac{dy}{dx}=-\frac{\cos x}{\sin^2 x}\text{.....................}(18).$$

If, as explained in Art. 31, we employ the symbol D to denote the operation of differentiating with respect to x, the results of Arts. 36—38 may be summed up as follows:
$$D(u+v)=Du+Dv \quad\text{................}(19),$$
$$D(uv)=vDu+uDv \quad\text{.............}(20),$$
$$D\left(\frac{u}{v}\right)=\frac{vDu-uDv}{v^2} \quad\text{...........}(21).$$

EXAMPLES. VI.

Verify the following differentiations:

1. $y=x(1-x)$, $\quad Dy=1-2x$.
2. $y=x(1-x)^2$, $\quad Dy=(1-x)(1-3x)$.
3. $y=x^m(1-x)^n$, $\quad Dy=x^{m-1}(1-x)^{n-1}\{m-(m+n)x\}$.
4. $y=(x-1)(x-2)(x-3)$, $\quad Dy=3x^2-12x+11$.
5. $y=x(1-x)^2(1+x)^3$, $\quad Dy=(1-x)(1+x)^2(1+3x)(1-2x)$.
6. $y=(1+x^2)(1-2x^2)$, $\quad Dy=-2x(1+4x^2)$.
7. $y=\left(x+\dfrac{1}{x}\right)^2$, $\quad Dy=2\left(x+\dfrac{1}{x}\right)\left(1-\dfrac{1}{x^2}\right)$.
8. $y=\dfrac{x}{1-x}$, $\quad Dy=\dfrac{1}{(1-x)^2}$.
9. $y=\dfrac{x}{1+x^2}$, $\quad Dy=\dfrac{1-x^2}{(1+x^2)^2}$.
10. $y=\dfrac{1+x^2}{1-x^2}$, $\quad Dy=\dfrac{4x}{(1-x^2)^2}$.
11. $y=\dfrac{x^m}{(1-x)^n}$, $\quad Dy=\dfrac{x^{m-1}}{(1-x)^{n+1}}\{m-(m-n)x\}$.
12. $y=x\sin x$, $\quad Dy=\sin x+x\cos x$.
13. $y=x^2\cos x$, $\quad Dy=2x\cos x-x^2\sin x$.
14. $y=\sin^2 x\cos x$, $\quad Dy=2\sin x-3\sin^3 x$.
15. $y=\dfrac{\sin x}{x}$, $\quad Dy=\dfrac{x\cos x-\sin x}{x^2}$.
16. $y=\dfrac{x}{\sin x}$, $\quad Dy=\dfrac{\sin x-x\cos x}{\sin^2 x}$.
17. $y=\dfrac{\tan x}{x}$, $\quad Dy=\dfrac{x-\sin x\cos x}{x^2\cos^2 x}$.
18. $y=\tan^2 x$, $\quad Dy=2\tan x\sec^2 x$.
19. $y=\sec^2 x$, $\quad Dy=2\tan x\sec^2 x$.
20. $y=\dfrac{\sin x}{1+\tan x}$, $\quad Dy=\dfrac{\cos^3 x-\sin^3 x}{(\sin x+\cos x)^2}$.

21. $y = \dfrac{1 + \sin x}{1 - \sin x}$, $\qquad Dy = \dfrac{2 \cos x}{(1 - \sin x)^2}$.

22. $y = \dfrac{1 - \cos x}{1 + \cos x}$, $\qquad Dy = \dfrac{2 \sin x}{(1 + \cos x)^2}$.

23. $y = xe^x$, $\qquad Dy = (x + 1)\, e^x$.

24. $y = xe^{-x}$, $\qquad Dy = (1 - x)\, e^{-x}$.

25. $y = x^m e^x$, $\qquad Dy = (x + m)\, x^{m-1} e^x$.

26. $y = e^x \sin x$, $\qquad Dy = e^x (\sin x + \cos x)$.

27. $y = e^x \cos x$, $\qquad Dy = e^x (\cos x - \sin x)$.

28. $y = \dfrac{e^x - 1}{e^x + 1}$, $\qquad Dy = \dfrac{2e^x}{(e^x + 1)^2}$.

39. Differentiation of a Function of a Function.

If
$$y = F(u) \quad \ldots\ldots\ldots\ldots(1),$$
where
$$u = f(x) \quad \ldots\ldots\ldots\ldots(2),$$
the symbols F, f denoting given functions, then
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = F'(u) \,.\, f'(x) \quad \ldots\ldots\ldots(3).$$

For, if δx, δy, δu be simultaneous increments, we have
$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta u} \cdot \frac{\delta u}{\delta x},$$
identically; and therefore, since the limit of a product is the product of the limits,
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

A useful application of the formula (3) occurs in the theory of rectilinear motion. Thus if, as in Art. 33, we denote by v and a the velocity and the acceleration, respectively, of a moving point, we have
$$v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt} \quad \ldots\ldots\ldots\ldots(4).$$
Hence if v be regarded as a function of the space described (s), we have
$$a = \frac{dv}{ds} \frac{ds}{dt} = v \frac{dv}{ds} \quad \ldots\ldots\ldots\ldots(5).$$

Similarly, in the case of a rigid body rotating about an axis, the angular acceleration, when the angular velocity is regarded as a function of θ, will be given by

$$\frac{d\omega}{d\theta}\frac{d\theta}{dt}, \text{ or } \omega\frac{d\omega}{d\theta} \qquad\ldots\ldots(6).$$

The following deductions from (3) are important:

1°. If
$$y = F(x+a) \qquad\ldots\ldots(7),$$
then, putting $u = x + a$, $du/dx = 1$, the formula (2) gives
$$\frac{dy}{dx} = F'(x+a) \qquad\ldots\ldots(8).$$

The geometrical meaning of this is that shifting a curve bodily parallel to the axis of x does not alter the gradient.

2°. If
$$y = F(kx) \qquad\ldots\ldots(9),$$
we have, putting $u = kx$, $du/dx = k$,
$$\frac{dy}{dx} = kF'(kx) \qquad\ldots\ldots(10).$$

3°. If
$$y = u^m \qquad\ldots\ldots(11),$$
where m is any rational quantity, we have
$$F(u) = u^m, \quad F'(u) = mu^{m-1},$$
and therefore
$$\frac{dy}{dx} = mu^{m-1}\frac{du}{dx} \qquad\ldots\ldots(12).$$

In particular, in the cases $m = \frac{1}{2}$, $m = -\frac{1}{2}$, we find
$$\left.\begin{aligned} \frac{d}{dx}\sqrt{u} &= \frac{1}{2\sqrt{u}}\frac{du}{dx}, \\ \frac{d}{dx}\frac{1}{\sqrt{u}} &= -\frac{1}{2u^{\frac{3}{2}}}\frac{du}{dx} \end{aligned}\right\} \qquad\ldots\ldots(13),$$
respectively.

We add a few examples on the above rules.

Ex. 1. If
$$y = \sin^m x \qquad\ldots\ldots(14),$$
that is,
$$y = u^m, \text{ where } u = \sin x,$$
we have
$$Dy = mu^{m-1}\frac{du}{dx} = m\sin^{m-1}x\cos x \qquad\ldots\ldots(15).$$

Ex. 2. If $$y=\sqrt{(a^2-x^2)} \quad\text{.........................(16)},$$
we have $$Dy=D\,(a^2-x^2)^{\frac{1}{2}}=\tfrac{1}{2}\,(a^2-x^2)^{-\frac{1}{2}}\,.\,D\,(a^2-x^2)$$
$$=-\frac{x}{\sqrt{(a^2-x^2)}} \quad\text{.................................(17)}.$$

Ex. 3. Let $$y=\frac{x}{\sqrt{(a^2-x^2)}} \quad\text{......................(18)}.$$

If we put $$u=x,\quad v=\sqrt{(a^2-x^2)},$$
we have $$Du=1,\ \text{and}\ Dv=\frac{-x}{\sqrt{(a^2-x^2)}},$$
as in the preceding Ex. The rule for differentiating a fraction then gives
$$Dy=\frac{vDu-uDv}{v^2}=\frac{\sqrt{(a^2-x^2)}+\dfrac{x^2}{\sqrt{(a^2-x^2)}}}{a^2-x^2}$$
$$=\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}} \quad\text{.................................(19)}.$$

EXAMPLES. VII.

1. $y=(x+a)^m\,(x+b)^n$, $\quad Dy=(x+a)^{m-1}\,(x+b)^{n-1}\,\{(m+n)\,x+mb+na\}$.

2. $y=\dfrac{x^n}{(1+x)^n}$, $\quad Dy=\dfrac{nx^{n-1}}{(1+x)^{n+1}}$.

3. $y=x\sqrt{(1+x)}$, $\quad Dy=\dfrac{2+3x}{2\sqrt{(1+x)}}$.

4. $y=\sqrt{\{(x+1)\,(x+2)\}}$, $\quad Dy=\dfrac{2x+3}{2\sqrt{\{(x+1)\,(x+2)\}}}$.

✓ 5. $y=(1+x)\sqrt{(1-x)}$, $\quad Dy=\dfrac{1-3x}{2\sqrt{(1-x)}}$.

6. $y=(1-x)\sqrt{(1+x^2)}$, $\quad Dy=-\dfrac{(1-x+2x^2)}{\sqrt{(1+x^2)}}$.

7. $y=\dfrac{1}{\sqrt{(1-x^2)}}$, $\quad Dy=\dfrac{x}{(1-x^2)^{\frac{3}{2}}}$.

✓ 8. $y=\dfrac{x}{\sqrt{(1+x^2)}}$, $\quad Dy=\dfrac{1}{(1+x^2)^{\frac{3}{2}}}$.

9. $y=\dfrac{\sqrt{(1+x^2)}}{x}, \qquad Dy=-\dfrac{1}{x^2\sqrt{(1+x^2)}}.$

10. $y=\sqrt{\left(\dfrac{1+x}{1-x}\right)}, \qquad Dy=\dfrac{1}{(1-x)\sqrt{(1-x^2)}}.$

11. $y=\dfrac{1}{\sqrt{(1+x^2)}+x}, \qquad Dy=\dfrac{x}{\sqrt{(x^2+1)}}-1.$

12. $y=\dfrac{x}{\sqrt{(1+x^2)}+x}, \qquad Dy=\dfrac{1+2x^2}{\sqrt{(1+x^2)}}-2x.$

13. $y=\sqrt{\left(\dfrac{1+x+x^2}{1-x+x^2}\right)}, \qquad Dy=\dfrac{1-x^2}{(1+x+x^2)^{\frac{1}{2}}(1-x+x^2)^{\frac{3}{2}}}.$

14. $y=\sin 2(x-a), \qquad Dy=2\cos 2(x-a).$

15. $y=\sin^2 2x, \qquad Dy=2\sin 4x.$

16. $y=\sqrt{(1+\sin x)}, \qquad Dy=\tfrac{1}{2}\sqrt{(1-\sin x)}.$

17. $y=\tan^2 x, \qquad Dy=\dfrac{2\sin x}{\cos^3 x}.$

18. $y=\sec^2 x, \qquad Dy=\dfrac{2\sin x}{\cos^3 x}.$

19. $y=\tan^3 x, \qquad Dy=\dfrac{3\sin^2 x}{\cos^4 x}.$

20. $y=\sin^3 x+\cos^3 x, \qquad Dy=3\sin x\cos x(\sin x-\cos x).$

21. $y=\sin x-\tfrac{1}{3}\sin^3 x, \qquad Dy=\cos^3 x.$

22. $y=\tan x+\tfrac{1}{3}\tan^3 x, \qquad Dy=\sec^4 x.$

23. $y=\dfrac{\sin^2 x}{x^2}, \qquad Dy=\dfrac{2\sin x}{x^3}(x\cos x-\sin x).$

24. $y=e^{x^2}, \qquad Dy=2xe^{x^2}.$

25. $y=\dfrac{e^{x^2}}{x}, \qquad Dy=e^{x^2}\left(2-\dfrac{1}{x^2}\right).$

26. $y=e^{\sin x}, \qquad Dy=\cos x\, e^{\sin x}.$

27. $y=e^{\sin^2 x}, \qquad Dy=\sin 2x\, e^{\sin^2 x}.$

28. $y=e^{ax}\sin\beta x, \qquad Dy=e^{ax}(a\sin\beta x+\beta\cos\beta x).$

29. $y=e^{ax}\cos\beta x, \qquad Dy=e^{ax}(a\cos\beta x-\beta\sin\beta x).$

30. $y = \sin mx \sin nx$, $\quad Dy = n \sin mx \cos nx + m \cos mx \sin nx.$

31. $y = \sqrt{(\alpha \sin^2 x + \beta \cos^2 x)}$, $\quad Dy = \frac{1}{2}(\alpha - \beta) \frac{\sin 2x}{\sqrt{(\alpha \sin^2 x + \beta \cos^2 x)}}.$

40. Differentiation of the Hyperbolic Functions.

1°. If
$$y = \sinh x \qquad \text{.......................(1)},$$
we have
$$\frac{dy}{dx} = D\left(\frac{e^x - e^{-x}}{2}\right) = \tfrac{1}{2}(De^x - De^{-x})$$
$$= \tfrac{1}{2}(e^x + e^{-x}) = \cosh x \qquad \text{.................(2)}.$$

Similarly, if
$$y = \cosh x \qquad \text{.......................(3)},$$
we find
$$\frac{dy}{dx} = \sinh x \qquad \text{.......................(4)}.$$

2°. If
$$y = \tanh x \qquad \text{.......................(5)},$$
we have
$$\frac{dy}{dx} = D\frac{\sinh x}{\cosh x} = \frac{\cosh x\, D \sinh x - \sinh x\, D \cosh x}{\cosh^2 x}$$
$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \operatorname{sech}^2 x \qquad \text{....................(6)},$$
by Art. 19 (4).

Similarly, if
$$y = \coth x \qquad \text{......................(7)},$$
we find
$$\frac{dy}{dx} = -\operatorname{cosech}^2 x \qquad \text{.................(8)}.$$

3°. If
$$y = \operatorname{sech} x \qquad \text{......................(9)},$$
we have, by Art. 38 (6),
$$\frac{dy}{dx} = D\left(\frac{1}{\cosh x}\right) = \frac{-D \cosh x}{\cosh^2 x}$$
$$= -\frac{\sinh x}{\cosh^2 x} \qquad \text{.............................(10)}.$$

Similarly, if
$$y = \operatorname{cosech} x \qquad \text{....................(11)},$$
we find
$$\frac{dy}{dx} = -\frac{\cosh x}{\sinh^2 x} \qquad \text{....................(12)}.$$

EXAMPLES. VIII.

Verify the following differentiations:

1. $y = \sinh^2 x,\qquad Dy = \sinh 2x.$
2. $y = \cosh^2 x,\qquad Dy = \sinh 2x.$
3. $y = \tanh^2 x,\qquad Dy = \dfrac{2\sinh x}{\cosh^3 x}.$
4. $y = \sinh x + \frac{1}{3}\sinh^3 x,\qquad Dy = \cosh^3 x.$
5. $y = \tanh x - \frac{1}{3}\tanh^3 x,\qquad Dy = \operatorname{sech}^4 x.$
6. $y = \cosh x \cos x + \sinh x \sin x,\qquad Dy = 2\sinh x \cos x.$
7. $y = \cosh x \sin x + \sinh x \cos x,\qquad Dy = 2\cosh x \cos x.$
8. $y = \dfrac{\cosh x - \cos x}{\sinh x + \sin x},\qquad Dy = \dfrac{2\sin x \sinh x}{(\sinh x + \sin x)^2}.$

41. Differentiation of Inverse Functions.

If y be a continuous function of x, then under a certain condition (see Art. 20), which is fulfilled in the case of most ordinary mathematical functions, x will be a continuous function of y.

If δx, δy be corresponding increments of x and y, we have

$$\frac{\delta y}{\delta x}\cdot\frac{\delta x}{\delta y} = 1,$$

identically. Hence, since the limit of the product is equal to the product of the limits,

$$\frac{dy}{dx}\cdot\frac{dx}{dy} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Hence, it being presupposed that y is a differentiable function of x, it follows that x is in general a differentiable function of y, and that the two derived functions are reciprocals.

The geometrical meaning of this is that the tangent to a curve makes complementary angles with the axes of x and y.

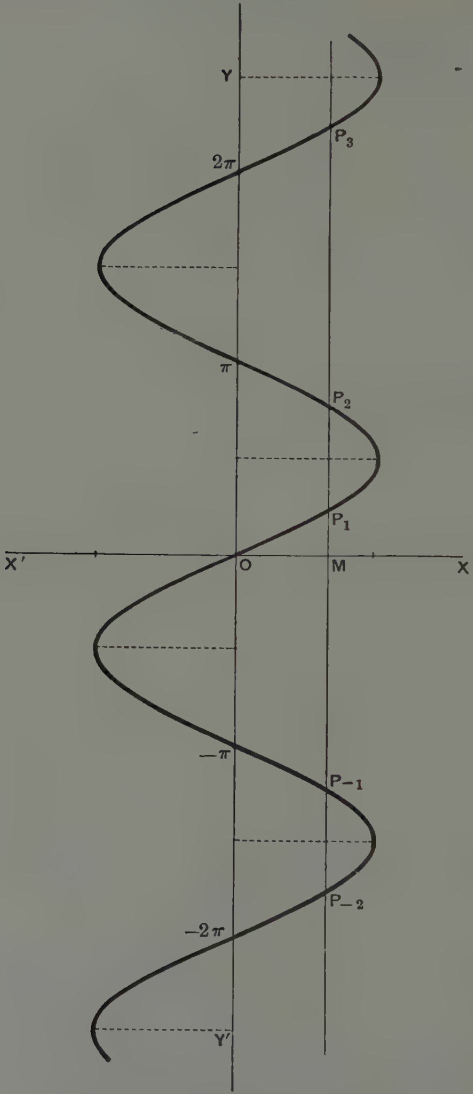

Fig. 28.

The following cases are important:

1°. If
$$y = \sin^{-1} x \quad \text{.......................(2)},$$

we have
$$x = \sin y, \quad \frac{dx}{dy} = \cos y.$$

Hence
$$\frac{dy}{dx} = \frac{1}{\cos y} = \pm \frac{1}{\sqrt{(1 - x^2)}} \quad \text{..............(3)}.$$

2°. If
$$y = \cos^{-1} x \quad \text{.......................(4)},$$

we have
$$x = \cos y, \quad \frac{dx}{dy} = -\sin y,$$

and therefore
$$\frac{dy}{dx} = -\frac{1}{\sin y} = \mp \frac{1}{\sqrt{(1 - x^2)}} \quad \text{..............(5)}.$$

The ambiguity of sign in these results is to be accounted for as follows. We have seen that if $y = \sin^{-1} x$, then y is a many-valued function of x; viz. for any assigned value of x (between the limits ± 1) there is a series of values of y, and for some of these dy/dx is positive, for others negative. See Fig. 28. Similarly for $\cos^{-1} x$.

If, when x is positive, we agree to understand by $\sin^{-1} x$ the angle between 0 and $\frac{1}{2}\pi$ whose sine is x, we must write

$$\frac{d}{dx} \sin^{-1} x = + \frac{1}{\sqrt{(1 - x^2)}} \quad \text{..................(6)}.$$

Similarly if, x being positive, $\cos^{-1} x$ be restricted to lie between 0 and $\frac{1}{2}\pi$, we have

$$\frac{d}{dx} \cos^{-1} x = - \frac{1}{\sqrt{(1 - x^2)}} \quad \text{..................(7)}.$$

3°. If
$$y = \tan^{-1} x \quad \text{.......................(8)},$$

we have
$$x = \tan y, \quad \frac{dx}{dy} = \sec^2 y,$$

and therefore
$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + x^2} \quad \text{....................(9)}.$$

There is here no ambiguity of sign. For each value of x there is an infinite series of values of y, but the value of dy/dx is the same for all, the tangent lines at the corresponding points of the curve $y = \tan^{-1} x$ being parallel. See Fig. 29.

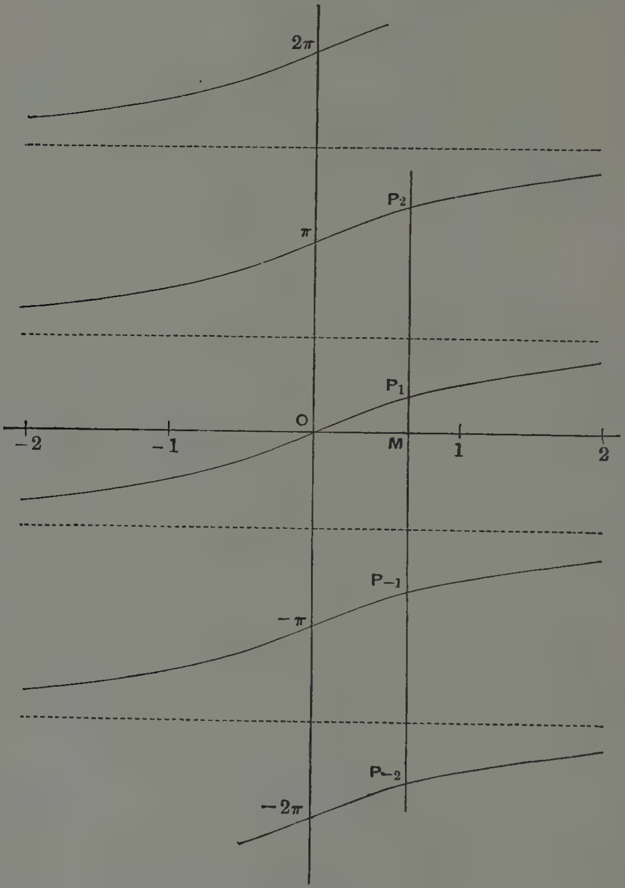

Fig. 29.

Ex. 1. Let $$y = \sin^{-1} \frac{x}{\sqrt{(1+x^2)}} \quad \text{.....................(10),}$$

or $$y = \sin^{-1} u, \text{ where } u = \frac{x}{\sqrt{(1+x^2)}}.$$

We have $$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{\surd(1-u^2)}\frac{du}{dx}.$$

We easily find
$$\surd(1-u^2)=\frac{1}{\surd(1+x^2)},\quad \frac{du}{dx}=\frac{1}{(1+x^2)^{\frac{3}{2}}},$$

whence $$\frac{dy}{dx}=\frac{1}{1+x^2}\text{ ....(11).}$$

It is easily proved (putting $x=\tan\theta$) that
$$\sin^{-1}\frac{x}{\surd(1+x^2)}=\tan^{-1}x,$$
so that the above result is in accordance with (9) above.

Ex. 2. Let $$y=\tan^{-1}\frac{1+x+x^2}{1-x+x^2}\text{ ....(12).}$$

If we write $$u=\frac{1+x+x^2}{1-x+x^2},$$

we have $$\frac{dy}{dx}=\frac{1}{1+u^2}\frac{du}{dx}.$$

We find
$$1+u^2=\frac{2(1+3x^2+x^4)}{(1-x+x^2)^2},\quad \frac{du}{dx}=\frac{2(1-x^2)}{(1-x+x^2)^2},$$

whence $$\frac{dy}{dx}=\frac{1-x^2}{1+3x^2+x^4}\text{ ....(13).}$$

42. Differentiation of a Logarithm.

1°. If $$y=\log x\text{ ....(1),}$$

we have $$x=e^y,\quad \frac{dx}{dy}=e^y=x,$$

and therefore $$\frac{dy}{dx}=\frac{1}{x}\text{ ....(2).}$$

This diminishes as x increases, so that the representative curve becomes less and less inclined to the axis of x. See Fig. 24, p. 47.

2°. If $$y=\log_a x\text{ .... (3),}$$

we have $$x=a^y,\quad \frac{dx}{dy}=a^y.\log_e a=x.\log_e a,$$

whence $$\frac{dy}{dx}=\frac{1}{\log_e a}\cdot\frac{1}{x}\quad\text{......................(4).}$$

For instance, if $$y=\log_{10} x\quad\text{......................(5),}$$

we have $$\frac{dy}{dx}=\frac{\mu}{x}\quad\text{......................(6),}$$

where $\mu=\cdot43429\ldots$ as in Art. 22.

3°. If $$y=\log u\quad\text{......................(7),}$$

where u is a given function of x, we have, by Art. 39,

$$\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}=\frac{1}{u}\frac{du}{dx}\quad\text{......................(8).}$$

Ex. 1. If $$y=\log\sin x\quad\text{......................(9),}$$

we have $$\frac{dy}{dx}=\frac{1}{\sin x}\cdot D\sin x=\cot x\quad\text{......................(10).}$$

Similarly, if $$y=\log\sec x=-\log\cos x\quad\text{......................(11),}$$

we find $$\frac{dy}{dx}=\tan x\quad\text{......................(12).}$$

Ex. 2. If $$y=\log\tan\tfrac{1}{2}x\quad\text{......................(13),}$$

we have

$$\frac{dy}{dx}=\frac{1}{\tan\frac{1}{2}x}\cdot D\tan\tfrac{1}{2}x=\frac{1}{\tan\frac{1}{2}x}\cdot\sec^2\tfrac{1}{2}x\cdot\tfrac{1}{2}$$
$$=\frac{1}{\sin x}\quad\text{......................(14).}$$

Similarly, if $$y=\log\tan(\tfrac{1}{4}\pi+\tfrac{1}{2}x)\quad\text{......................(15),}$$

we should find $$\frac{dy}{dx}=\frac{1}{\cos x}\quad\text{......................(16).}$$

Ex. 3. Let $$y=\tfrac{1}{2}\log\frac{1+x}{1-x}\quad\text{......................(17),}$$
$$=\tfrac{1}{2}\log(1+x)-\tfrac{1}{2}\log(1-x).$$

Hence $$\frac{dy}{dx}=\tfrac{1}{2}\,\frac{1}{1+x}+\tfrac{1}{2}\,\frac{1}{1-x}=\frac{1}{1-x^2}\quad\text{......................(18).}$$

Ex. 4. Let $$y=\log\{x+\surd(x^2\pm 1)\}\quad\text{......................(19)}$$

We have
$$\frac{dy}{dx}=\frac{1}{x+\sqrt{(x^2\pm 1)}}\cdot D\{x+\sqrt{(x^2\pm 1)}\}$$
$$=\frac{1}{x+\sqrt{(x^2\pm 1)}}\left\{1+\frac{x}{\sqrt{(x^2\pm 1)}}\right\}$$
$$=\frac{1}{\sqrt{(x^2\pm 1)}}\;\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots(20).$$

43. Logarithmic Differentiation.

In the case of a function consisting of a number of factors it is sometimes convenient to take the logarithm before differentiating. Thus if
$$y=\frac{u_1u_2u_3\dots}{v_1v_2v_3\dots}\;\dots\dots\dots\dots\dots\dots(1),$$
we have
$$\log y=\log u_1+\log u_2+\log u_3+\dots$$
$$-\log v_1-\log v_2-\log v_3-\dots(2),$$
and therefore, by Art. 42, 3°,
$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u_1}\frac{du_1}{dx}+\frac{1}{u_2}\frac{du_2}{dx}+\frac{1}{u_3}\frac{du_3}{dx}+\dots$$
$$-\frac{1}{v_1}\frac{dv_1}{dx}-\frac{1}{v_2}\frac{dv_2}{dx}-\frac{1}{v_3}\frac{dv_3}{dx}-\dots(3).$$

This is a generalization of the results of Arts. 37, 38.

Ex. If
$$y=\sqrt{\left\{\frac{(a+x)(b+x)}{(a-x)(b-x)}\right\}},$$
we have
$$\log y=\tfrac{1}{2}\log(a+x)+\tfrac{1}{2}\log(b+x)-\tfrac{1}{2}\log(a-x)-\tfrac{1}{2}\log(b-x).$$

Hence
$$\frac{1}{y}\frac{dy}{dx}=\tfrac{1}{2}\left\{\frac{1}{a+x}+\frac{1}{b+x}+\frac{1}{a-x}+\frac{1}{b-x}\right\}$$
$$=\frac{a}{a^2-x^2}+\frac{b}{b^2-x^2}=\frac{(a+b)(ab-x^2)}{(a^2-x^2)(b^2-x^2)},$$
$$\frac{dy}{dx}=\frac{(a+b)(ab-x^2)}{(a-x)^{\frac{3}{2}}(b-x)^{\frac{3}{2}}(a+x)^{\frac{1}{2}}(b+x)^{\frac{1}{2}}}$$

EXAMPLES. IX.

Verify the following differentiations:

1. $y = x \log x,$ $\qquad Dy = 1 + \log x.$

2. $y = x^m \log x,$ $\qquad Dy = x^{m-1}(1 + m \log x).$

3. $y = \log \sin x,$ $\qquad Dy = \cot x.$

4. $y = \log \cos x,$ $\qquad Dy = -\tan x.$

5. $y = \log \tan x,$ $\qquad Dy = 2 \operatorname{cosec} 2x.$

6. $y = \log \sinh x,$ $\qquad Dy = \coth x.$

7. $y = \log \cosh x,$ $\qquad Dy = \tanh x.$

8. $y = \log \tanh x,$ $\qquad Dy = 2 \operatorname{cosech} 2x.$

9. $y = \log \dfrac{x}{1+x},$ $\qquad Dy = \dfrac{1}{x(1+x)}.$

10. $y = \log \dfrac{1-x}{1+x},$ $\qquad Dy = -\dfrac{2}{1-x^2}.$

11. $y = \log \dfrac{x}{\sqrt{(x^2+1)}},$ $\qquad Dy = \dfrac{1}{x(x^2+1)}.$

12. $y = \log \dfrac{1+\sqrt{x}}{1-\sqrt{x}},$ $\qquad Dy = \dfrac{1}{(1-x)\sqrt{x}}.$

13. $y = \log \{\sqrt{(x+1)} + \sqrt{(x-1)}\},$ $\qquad Dy = \dfrac{1}{2\sqrt{(x^2-1)}}.$

14. $y = \log \dfrac{x}{\sqrt{(x^2+1)} - x},$ $\qquad Dy = \dfrac{1}{x} + \dfrac{1}{\sqrt{(x^2+1)}}.$

15. $y = \log (x-1) - \dfrac{2x-1}{(x-1)^2},$ $\qquad Dy = \dfrac{x^2+1}{(x-1)^3}.$

16. $y = \log \dfrac{1+x+x^2}{1-x+x^2},$ $\qquad Dy = \dfrac{2(1-x^2)}{1+x^2+x^4}.$

17. $y = \sin^{-1}(1-x),$ $\qquad Dy = -\dfrac{1}{\sqrt{(2x-x^2)}}.$

18. $y = x \sin^{-1} x,$ $\qquad Dy = \sin^{-1} x + \dfrac{x}{\sqrt{(1-x^2)}}.$

19. $y=\cot^{-1}x,\qquad Dy=-\dfrac{1}{1+x^2}.$

20. $y=\sec^{-1}x,\qquad Dy=\dfrac{1}{x\sqrt{(x^2-1)}}.$

21. $y=\operatorname{cosec}^{-1}x,\qquad Dy=-\dfrac{1}{x\sqrt{(x^2-1)}}.$

22. $y=\sin^{-1}x+\sin^{-1}\sqrt{(1-x^2)},\qquad Dy=0.$

23. $y=\tan^{-1}x+\tan^{-1}\dfrac{1}{x},\qquad Dy=0.$

24. $y=\sin^{-1}\{2x\sqrt{(1-x^2)}\},\qquad Dy=\dfrac{2}{\sqrt{(1-x^2)}}.$

25. $y=\tan^{-1}\dfrac{2x}{1-x^2},\qquad Dy=\dfrac{2}{1+x^2}.$

26. $y=\tan^{-1}\{\sqrt{(x^2+1)}-x\},\qquad Dy=-\dfrac{1}{2(x^2+1)}.$

27. $y=\sin^{-1}(\cos x),\qquad Dy=-1.$

28. $y=\cos^{-1}\dfrac{1-x^2}{1+x^2},\qquad Dy=\dfrac{2}{1+x^2}.$

29. $y=\sqrt{\left(\dfrac{1+x^2}{1-x^2}\right)},\qquad Dy=\dfrac{2x}{(1+x^2)^{\frac{1}{2}}(1-x^2)^{\frac{3}{2}}}.$

30. $y=\dfrac{\sqrt{(1+x^2)}-x}{\sqrt{(1+x^2)}+x},\qquad Dy=\dfrac{-2}{\sqrt{(1+x^2)}\{\sqrt{(1+x^2)}+x\}^2}.$

31. $y=\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}},\qquad Dy=-\dfrac{1}{x^2\sqrt{(1-x^2)}}-\dfrac{1}{x^2}.$

32. $y=\dfrac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}},\qquad Dy=-\dfrac{2}{x^3\sqrt{(1-x^4)}}-\dfrac{2}{x^3}.$

44. Differentiation of the Inverse Hyperbolic Functions.

1°. If
$$y=\sinh^{-1}x\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
we have
$$x=\sinh y,\qquad \frac{dx}{dy}=\cosh y=\sqrt{(x^2+1)},$$

and therefore $$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)}} \quad \text{.................(2).}$$

There is no ambiguity of sign, for cosh y is essentially positive.

2°. If $$y = \cosh^{-1} x \quad \text{....................(3),}$$

we have $$x = \cosh y, \quad \frac{dx}{dy} = \sinh y = \pm\sqrt{(x^2-1)},$$

whence $$\frac{dy}{dx} = \pm\frac{1}{\sqrt{(x^2-1)}} \quad \text{.................(4).}$$

For any given value of x, greater than unity, there are two possible values of y, one positive, the other negative, and for these dy/dx has opposite signs. [Cf. Fig. 21, Art. 19, interchanging x and y.]

3°. If $$y = \tanh^{-1} x \text{.......................(5),}$$

we have $$x = \tanh y, \quad \frac{dx}{dy} = \operatorname{sech}^2 y = 1 - x^2,$$

and therefore $$\frac{dy}{dx} = \frac{1}{1-x^2} \text{.......................(6).}$$

This agrees with Art. 42, Ex. 3. It is to be noticed that y is real only when $x < 1$. See Fig. 22.

Similarly, if $$y = \coth^{-1} x \quad \text{....................(7),}$$

we find $$\frac{dy}{dx} = -\frac{1}{x^2-1} \quad \text{....................(8),}$$

x being necessarily > 1, if y is real.

EXAMPLES. X.

Verify the following differentiations:

1. $y = \operatorname{sech}^{-1} x$, $\quad Dy = -\dfrac{1}{x\sqrt{(1-x^2)}}$.
2. $y = \operatorname{cosech}^{-1} x$, $\quad Dy = -\dfrac{1}{x\sqrt{(1+x^2)}}$.
3. $y = \sin^{-1}(\tanh x)$, $\quad Dy = \operatorname{sech} x$.
4. $y = \tan^{-1}(\sinh x)$, $\quad Dy = \operatorname{sech} x$.
5. $y = \tan^{-1}(\tanh \frac{1}{2}x)$, $\quad Dy = \frac{1}{2}\operatorname{sech} x$.
6. $y = \tanh^{-1}(\tan \frac{1}{2}x)$, $\quad Dy = \frac{1}{2}\sec x$.

EXAMPLES. XI.

1. What is the geometrical meaning of the theorem

$$\frac{d}{dx}\phi(kx) = k\phi'(kx)?$$

2. If

$$y = \tan^{-1}\frac{u}{v},$$

prove that

$$Dy = \frac{vDu - uDv}{u^2 + v^2}.$$

3. Assuming that $\dfrac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \ldots + x^n$,

deduce, by differentiation, the sum of the series

$$1 + 2x + 3x^2 + \ldots + nx^{n-1},$$

and test the result by putting $x = 1$.

Hence shew that, if $|x| < 1$,

$$1 + 2x + 3x^2 + 4x^3 + \ldots \text{ to } \infty = (1 - x)^{-2}.$$

4. If, in the rectilinear motion of a point, v^2 be a linear function of s, the acceleration is constant.

5. If v^2 be a quadratic function of s, the acceleration varies as the distance from a fixed point in the line of motion.

6. If the time be a quadratic function of the space described, the acceleration varies as the cube of the velocity.

7. If

$$v^2 = A + \frac{B}{s},$$

the acceleration varies inversely as the square of the distance from a fixed point in the line of motion.

8. If s^2 be a quadratic function of t, the acceleration varies as $1/s^3$.

9. If the pressure and the volume of a gas be connected by the relation

$$pv^\gamma = \text{const.},$$

the cubical elasticity is γp.

45. Functions of two or more independent variables. Partial Derivatives.

Although in this treatise we are concerned mainly with functions of a single independent variable, it will occasionally be useful to have at our command ideas and notations borrowed from the more general theory.

One quantity u is said to be a function of two or more independent variables $x, y, \ldots$, when its value is determined by those of the latter, which may be assigned arbitrarily and independently within (in each case) a certain range. Thus if P be any point of a given surface, and a perpendicular PN be drawn to any fixed horizontal plane, the altitude PN is a function of the coordinates (x, y) of the point N.

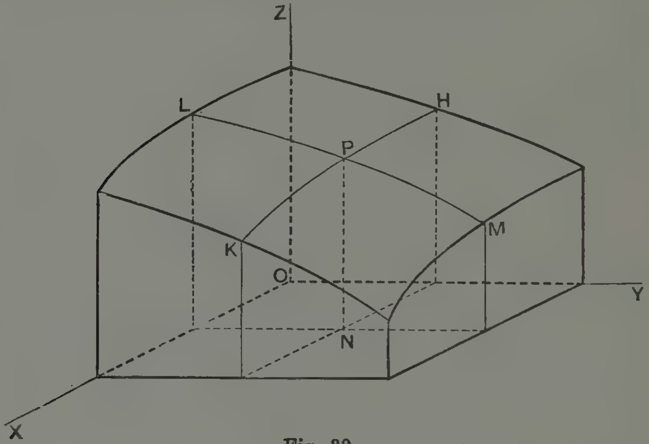

Fig. 30.

So again, in Physics, the pressure of a gas is a function of two independent variables, viz. the volume (per unit mass) and the temperature.

The functional relation is expressed by an equation of the form

$$u = \phi(x, y, \ldots) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

In particular, in the aforesaid case of a surface, if we denote the altitude PN by z, we have

$$z = \phi(x, y) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Let us now suppose that all the independent variables save one (x) are kept constant. Then the function u may or may not be a differentiable function of x; if it is differentiable, its derived function with respect to x is called the 'partial differential coefficient' or 'partial derivative' of u with respect to x, and is conveniently denoted by $\partial u/\partial x$. Thus

$$\frac{\partial u}{\partial x} = \lim_{\delta x = 0} \frac{\phi(x + \delta x, y, \dots) - \phi(x, y, \dots)}{\delta x} \quad \dots(3).$$

In like manner

$$\frac{\partial u}{\partial y} = \lim_{\delta y = 0} \frac{\phi(x, y + \delta y, \dots) - \phi(x, y, \dots)}{\delta y} \quad \dots(4).$$

In the case of the surface (2) it is plain that the partial derivatives

$$\frac{\partial z}{\partial x}, \quad \frac{\partial z}{\partial y}$$

are the gradients of the sections (HK, LM, in the figure) of the surface by planes parallel to the planes ZOX, ZOY respectively.

Ex. 1. If

$$z = x^m y^n \quad \dots\dots(5),$$

we have

$$\frac{\partial z}{\partial x} = m x^{m-1} y^n, \quad \frac{\partial z}{\partial y} = n x^m y^{n-1} \quad \dots\dots(6).$$

Ex. 2. Assuming that in a gas the pressure (p), volume (v), and temperature (θ) are connected by the relation

$$p = \frac{R\theta}{v} \quad \dots\dots(7),$$

we have

$$\frac{\partial p}{\partial v} = -\frac{R\theta}{v^2}, \quad \frac{\partial p}{\partial \theta} = \frac{R}{v} \quad \dots\dots(8).$$

46. Implicit Functions.

An equation of the type

$$\phi(x, y) = 0 \quad \dots\dots(1)$$

in general determines y as a function of x; for if we assign any arbitrary value to x, the resulting equation in y has in general one or more definite roots. These roots may be real or imaginary, but we shall only contemplate cases where, for values of x within a certain range, one value (at least) of y is real. The term 'implicit' is applied to functions determined in this manner, by way of contrast with cases where y is given 'explicitly' in the form

$$y = f(x) \quad \text{........................(2).}$$

If we regard

$$z = \phi(x, y) \quad \text{........................(3)}$$

as the equation of a surface, then (1) is the equation of the section of this surface by the plane $z = 0$. If the plane xy be regarded as horizontal, the sections $z = C$, where C may have different constant values, are the 'contour-lines.'

If we require to differentiate an implicit function, we may seek, first, to solve the equation (1) with respect to y, so as to bring it into the form (2). It is useful, however, to have a rule to meet cases when this process would be inconvenient or impracticable. It will be sufficient, for the present, to consider the case where $\phi(x, y)$ is a rational integral function of x and y, *i.e.* it is the sum of a series of terms of the type $A_{m,n}x^m y^n$, where m, n may have the values 0, 1, 2, 3,.... Since, by hypothesis, $\phi(x, y)$ is constantly null, its derived function with respect to x will be zero. Now by Arts. 35, 37, 39, we have

$$\frac{d}{dx}(x^m y^n) = m x^{m-1} y^n + n x^m y^{n-1} \frac{dy}{dx}.$$

Hence, if $\phi(x, y) = \Sigma A_{m,n} x^m y^n$ ....(4),

we have $\Sigma A_{m,n} m x^{m-1} y^n + \Sigma A_{m,n} n x^m y^{n-1} \dfrac{dy}{dx} = 0$........(5).

In the notation of Art. 45, this may be written

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \quad \text{....................(6),}$$

or

$$\frac{dy}{dx} = -\frac{\dfrac{\partial \phi}{\partial x}}{\dfrac{\partial \phi}{\partial y}} \quad \text{........................(7).}$$

It will be shewn in the next Chapter that the results (6) and (7) are not limited to the above special form of $\phi(x, y)$; but the present case is sufficient for most geometrical applications.

EXAMPLES. XII.

1. Sketch the contour-lines of the surface

$$az = x^2 + y^2,$$

and describe the general form of the surface.

2. Also of the surface $az = xy$.

3. If

$$z = f(x+y),$$

prove that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y},$$

and give the geometrical interpretation of this result.

4. If

$$r = \sqrt{(x^2+y^2)},$$

prove that

$$r\frac{\partial r}{\partial x} = x, \qquad r\frac{\partial r}{\partial y} = y.$$

5. If

$$z = a\tan^{-1}\frac{y}{x},$$

prove that

$$\frac{\partial z}{\partial x} = -\frac{ay}{x^2+y^2}, \qquad \frac{\partial z}{\partial y} = \frac{ax}{x^2+y^2}.$$

6. If

$$z = f(x^2+y^2),$$

prove that

$$\frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = x : y.$$

7. If

$$z = f\left(\frac{y}{x}\right),$$

prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

8. If

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

prove that

$$\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}.$$

CHAPTER III.

APPLICATIONS OF THE DERIVED FUNCTION.

47. Inferences from the sign of the Derived Function.

If $y = \phi(x)$, and if δx, δy be simultaneous increments of x and y, the limiting value of the ratio $\delta y/\delta x$ when δx is indefinitely diminished is, by definition, $\phi'(x)$. Hence, before the limit, we may write

$$\frac{\delta y}{\delta x} = \phi'(x) + \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where σ is an ultimately vanishing quantity.

A numerical example of the manner in which the ratio $\delta y/\delta x$ approximates to its limiting value may be of interest. We take the case of $y = \log_{10} x$, for the neighbourhood of $x = 1$. The limiting value is here

$$\frac{dy}{dx} = \frac{\mu}{x} = \cdot 43429 \ldots .$$

The numbers in the second column are taken from the printed tables.

δx	δy	$\delta y/\delta x$
·1000	·041393	·41393
·0500	·021189	·42379
·0100	·0043214	·43214
·0050	·0021661	·43321
·0010	·00043408	·43408
·0005	·00021709	·43419
·0001	·000043427	·43427

Let us first suppose that

$$\phi'(x) > 0.$$

Since the limiting value of σ is zero, we can by taking δx small enough ensure that

$$\phi'(x) + \sigma > 0.$$

That is, by (1), δy will have the same sign as δx for all admissible values of δx which are less in absolute value than a certain magnitude ϵ.

In the same way, if

$$\phi'(x) < 0,$$

δy will have the opposite sign to δx for all admissible values of δx which are less in absolute value than a certain quantity ϵ.

If the independent variable be represented geometrically as in Fig. 1, Art. 1, and if $x = OM$, where M is a point within the range considered, we may say that if $\phi'(x)$ be positive there is a certain interval to the right of M for every point of which the value of the function $\phi(x)$ is greater than its value at M, and a certain interval to the left of M at every point of which the value of the function is less than its value at M. If $\phi'(x)$ be negative, the words 'greater' and 'less' must be interchanged in this statement. When M is at the beginning or end of the range of x, the intervals referred to lie of course to the right or left of M, respectively.

It follows that if $\phi'(x)$ be positive over any finite range, the value of $\phi(x)$ will steadily increase with x throughout the range; *i.e.* if x_1 and x_2 be any two values of x belonging to the range, such that $x_2 > x_1$, then

$$\phi(x_2) > \phi(x_1).$$

For $\phi(x)$, being by hypothesis differentiable, and therefore continuous, must have (Art. 25) a greatest and a least value in the interval from x_1 to x_2 (inclusive). And the preceding argument shews that the *greatest* value cannot occur at the beginning of the interval, or in the interior; it must therefore occur at the end. Similarly the *least* value of $\phi(x)$ must occur at the beginning of the interval.

In the same way it appears that if $\phi'(x)$ be negative over any finite range, then $\phi(x)$ will steadily decrease as x increases, throughout this range; *i.e.* if x_1 and x_2 be any two values of x belonging to the range, such that $x_2 > x_1$, then

$$\phi(x_2) < \phi(x_1).$$

The geometrical meaning of these results is obvious. When the gradient of a curve is positive the ordinates increase with x; when the gradient is negative the ordinates decrease as x increases. The graphs of various functions given in Chapter I. will serve as illustrations.

The converse statements that if $\phi(x)$ steadily increases with x throughout any range, $\phi'(x)$ cannot be negative for any value of x belonging to this range, and that, if $\phi(x)$ steadily decreases as x increases, $\phi'(x)$ cannot be positive, follow immediately from the definition of $\phi'(x)$.

Again, even if $\phi'(x)$ vanish at a finite number of isolated points, provided it be elsewhere uniformly positive, $\phi(x)$ will steadily increase. Suppose, for example, that $\phi'(x_1) = 0$, and that with this exception $\phi'(x)$ is positive in the interval from $x = x_1$ to $x = x_2$, where $x_2 > x_1$. The least value of $\phi(x)$ cannot then occur within this interval, or at the upper extremity $(x = x_2)$. It must therefore occur at the lower extremity $(x = x_1)$. Hence

$$\phi(x_2) > \phi(x_1).$$

The same conclusion is arrived at if $\phi'(x)$ is positive from $x = x_1$ to $x = x_2$, except for $x = x_2$, where it vanishes.

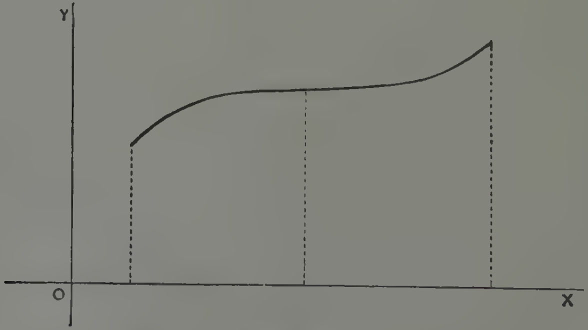

Fig. 31.

In the same way, if $\phi'(x)$ vanish at a finite number of isolated points, but is otherwise negative, $\phi(x)$ will steadily decrease.

Ex. If
$$y = \tan x - x,$$
we have
$$\frac{dy}{dx} = \sec^2 x - 1 = \tan^2 x.$$

Hence dy/dx is positive, except for $x = 0, \pi, 2\pi, \ldots$. Hence y steadily increases with x throughout any range which does not include one of the points of discontinuity $(x = \frac{1}{2}\pi, \frac{3}{2}\pi, \ldots)$.

It easily follows that the equation
$$\tan x - x = 0$$
has no root *between* 0 and $\frac{1}{2}\pi$; one, and only one, root between $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$; and so on.

These results may be verified by a graphical construction. If we draw the lines
$$y = \tan x, \quad y = x,$$
their intersections will determine the values of x which make
$$\tan x = x.$$

48. The Derivative vanishes in the interval between two equal values of the Function.

If $\phi(x)$ vanish for $x = a$ and $x = b$, and if $\phi'(x)$ be finite for all values of x between a and b, then $\phi'(x)$ will vanish for some value of x *between* a and b.

For, either $\phi(x)$ is constantly zero throughout the interval from a to b, or it will have (Art. 25) a greatest or a least value for some value (x_1) of x within this interval. In the former case we shall have $\phi'(x) = 0$ throughout the interval; in the latter case $\phi'(x_1)$ cannot be either positive or negative (Art. 47) and must therefore vanish, since it is by hypothesis finite.

The geometrical statement of this theorem is that if a curve meets the axis of x at two points, and if the gradient is everywhere finite, there must be at least one intervening point at which the tangent is parallel to the axis of x. See, for example, the graph of $\sin x$ on p. 34; also Fig. 13, p. 27.

Ex. 1. If $$\phi(x) = (x-a)(x-b),$$
we have $$\phi'(x) = 2x - (a+b).$$
Hence $\phi'(x)$ vanishes for $x = \frac{1}{2}(a+b)$, which lies between a and b.

Ex. 2. If $$\phi(x) = \frac{\sin x}{x},$$
we have $$\phi'(x) = \frac{x \cos x - \sin x}{x^2}.$$
Here $\phi(x) = 0$ for $x = \pi$ and $x = 2\pi$; hence $\phi'(x)$ must vanish for some intermediate value of x. This is in agreement with Art. 47, where it was shewn that the equation $x = \tan x$ has a root between π and $\frac{3}{2}\pi$.

It is to be carefully noticed that, in the above demonstration, the conditions that $\phi(x)$ and $\phi'(x)$ should each have a definite (and therefore finite) value throughout the interval from $x = a$ to $x = b$ are essential. The annexed figures exhibit various cases when the conclusion does not hold, owing to the violation of one or other of these conditions.

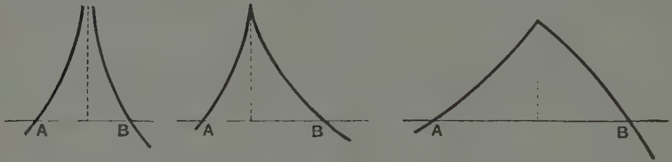

Fig. 32.

A slightly more general form of the theorem of this Art. is that if $\phi(x)$ has the same value (β) for $x = a$ and $x = b$, then under the same conditions as to the continuity and finiteness of $\phi(x)$ and $\phi'(x)$, the derived function $\phi'(x)$ will vanish for some intermediate value of x. This follows by the same argument, applied now to the function $\phi(x) - \beta$.

49. Application to the Theory of Equations.

If $\phi(x)$ be a rational integral function of x, then $\phi(x)$ and its derivative $\phi'(x)$ are both of them continuous (and finite) for all finite values of x. Hence at least one real root of the equation
$$\phi'(x) = 0 \text{..........................} (1)$$

will lie between any two real roots of

$$\phi(x) = 0 \qquad\qquad (2).$$

This result, which is known as 'Rolle's Theorem,' is important in the Theory of Equations. It is an immediate consequence that at most one real root of (2) lies between any two *consecutive* roots of (1). That is, the roots of (1) separate those of (2).

Ex. 1. If $\phi(x) = 4x^3 - 21x^2 + 18x + 20$,

we have $\phi'(x) = 12x^2 - 42x + 18 = 6(2x - 1)(x - 3)$.

Hence the real roots of $\phi(x) = 0$, if any, will lie in the intervals between $-\infty$ and $\frac{1}{2}$, $\frac{1}{2}$ and 3, 3 and $+\infty$, respectively. Now, for

$$x = -\infty, \quad \tfrac{1}{2}, \quad 3, \quad +\infty,$$

the signs of $\phi(x)$ are $\quad -, \quad +, \quad -, \quad +,$

respectively, so that $\phi(x)$ must in fact vanish once (by Art. 10) in each of the above intervals. Hence there are three real roots. The figure shews the graph of $\phi(x)$.

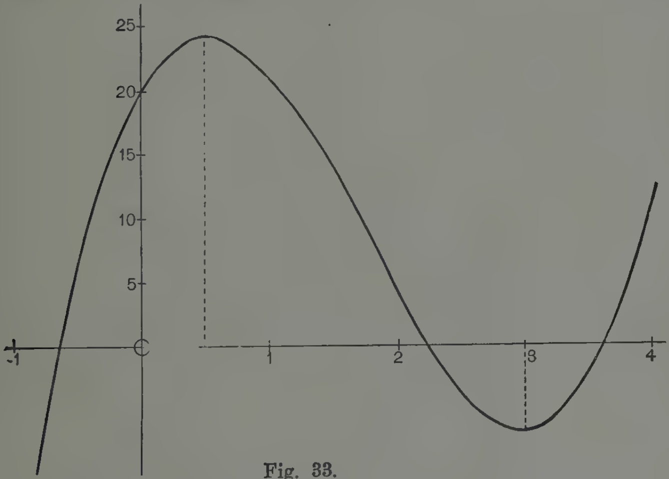

Fig. 33.

If by continuous modification of the form of $\phi(x)$, for example by the addition or subtraction of a constant, two roots are made to coalesce, the root of $\phi'(x) = 0$ which lies between must coalesce with them. Hence a double root of $\phi(x) = 0$ is also a root of $\phi'(x) = 0$.

More generally, an r-fold root of $\phi(x)=0$ being regarded as due to the coalescence of r distinct roots, the equation $\phi'(x)=0$ will have $r-1$ intervening roots which coalesce.

This suggests a method of ascertaining the multiple roots, if any, of a proposed algebraic equation. If α be an r-fold root of $\phi(x)$, we have

$$\phi(x)=(x-\alpha)^r\chi(x) \qquad\qquad (3),$$

where $\chi(x)$ is a rational integral function. Hence

$$\phi'(x)=(x-\alpha)^{r-1}\{r\chi(x)+(x-\alpha)\chi'(x)\} \qquad (4);$$

i.e. $(x-\alpha)^{r-1}$ will be a common factor of $\phi(x)$ and $\phi'(x)$. And it is easily seen that $(x-\alpha)^{r-1}$ will not be a common factor unless $\phi(x)$ is divisible by $(x-\alpha)^r$. Hence the multiple roots of $\phi(x)$, if any, are to be detected by finding the common factors of $\phi(x)$ and $\phi'(x)$ by the usual algebraical process.

Ex. 2. If $$\phi(x)=x^4-9x^2+4x+12,$$
we have $$\phi'(x)=4x^3-18x+4.$$
The usual method leads to the conclusion that $x-2$ is a common factor of $\phi(x)$ and $\phi'(x)$; whence we infer that $(x-2)^2$ is a factor of $\phi(x)$. The remaining factors are then easily ascertained; thus we find

$$\phi(x)=(x-2)^2(x+1)(x+3).$$

Ex. 3. To find the condition that the cubic

$$x^3+qx+r=0 \qquad\qquad (5)$$

should have a double root.

The double root, if it exists, must satisfy

$$3x^2+q=0 \quad \text{or} \quad x=\pm\sqrt{(-\tfrac{1}{3}q)} \qquad (6).$$

Substituting in (5), we find

$$r=\pm\tfrac{2}{3}\sqrt{(-\tfrac{1}{3}q^3)}, \quad \text{or} \quad r^2=-\tfrac{4}{27}q^3 \qquad (7),$$

which is the required condition.

50. Maxima and Minima.

A 'maximum' value of a continuous function is one which is greater, and a 'minimum' value is one which is less than the values in the immediate neighbourhood, on either side.

More precisely, the function $\phi(x)$ is a maximum for

$x = x_1$, if two positive quantities, ϵ and ϵ', can be found such that $\phi(x_1)$ is greater than the value which $\phi(x)$ assumes for any other value of x in the interval from $x = x_1 - \epsilon$ to $x = x_1 + \epsilon'$. Similarly for a minimum.

Since the comparison is made with values of the function in the immediate neighbourhood only of x_1, a maximum is not necessarily the *greatest*, nor a minimum the *least*, of all the values of the function. See Fig. 33, p. 105.

We will limit ourselves for the present to the case, which includes all the more important applications, where $\phi(x)$ has a determinate and finite derivative at all points of the range considered. The argument of Art. 47 then shews that if $\phi(x_1)$ be a maximum or minimum, $\phi'(x_1)$ cannot differ from zero. For if it be either positive or negative, there will be points in the immediate neighbourhood of x_1 for which $\phi(x)$ will be greater, and others for which it will be less, than $\phi(x_1)$. Hence, in the case supposed, a first condition for a maximum or minimum value of $\phi(x)$ is that $\phi'(x)$ should vanish.

This condition is necessary, but it is not sufficient. To investigate the matter further, we will suppose that on each side of the point x_1 there is a certain interval throughout which $\phi'(x)$ is altogether positive or altogether negative*. Now if $\phi'(x)$ be positive for all values of x between $x_1 - \epsilon$ and x_1, $\phi(x)$ will (Art. 47) steadily increase throughout the interval thus defined; and if $\phi'(x)$ be negative for all values of x between x_1 and $x_1 + \epsilon'$, $\phi(x)$ will steadily decrease throughout the corresponding interval. Hence if both these conditions hold, $\phi(x_1)$ is a maximum. And it is evident that if the signs be otherwise, $\phi(x_1)$ cannot be the greatest value which the function assumes within the interval extending from $x_1 - \epsilon$ to $x_1 + \epsilon'$.

We may express this shortly by saying that the necessary and sufficient condition in order that $\phi(x_1)$ may be a maximum value of $\phi(x)$ is that $\phi'(x)$ should change sign from + to − as x increases through the value x_1.

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* That is, we exclude cases where $\phi'(x)$ changes sign an infinite number of times within any interval including x_1, however short. The point $x=0$ in the function $x^2 \sin 1/x$ is an instance.

In the same way we find that the necessary and sufficient condition in order that $\phi(x_1)$ may be a minimum value of $\phi(x)$ is that $\phi'(x)$ should change sign from $-$ to $+$ as x increases through the value x_1.

In geometrical language, when the ordinate of a curve is a maximum the gradient must change from positive to negative; when the ordinate is a minimum the gradient must change from negative to positive. This is abundantly illustrated in our diagrams; see, for example, Figs. 13, 17, 18, 33.

Ex. 1. The distance (s), from an arbitrary origin, of a point moving in a straight line is a maximum when the velocity (ds/dt) changes from positive to negative, and is a minimum when the velocity changes from negative to positive.

Thus, in the case of a particle moving upwards under gravity, we have

$$s = ut - \tfrac{1}{2}gt^2, \quad \frac{ds}{dt} = u - gt.$$

Hence ds/dt changes from positive to negative as t increases through the value u/g. The altitude (s) is therefore then a maximum.

Ex. 2. To find the rectangle of greatest area having a given perimeter.

Denoting the perimeter by $2a$, the lengths of two adjacent sides may be taken to be x and $a - x$; hence we have to find the maximum value of the function

$$x(a - x) \quad \text{.....................} \quad (1).$$

The derivative of this is $a - 2x$, which changes sign from $+$ to $-$ as x increases through the value $\frac{1}{2}a$. The rectangle of greatest area is therefore a square.

The graph of the function (1) has been given in Fig. 13, p. 27.

Ex. 3. To find the maxima and minima of the function

$$\phi(x) \equiv 4x^3 - 21x^2 + 18x + 20 \quad \text{..............} \quad (2).$$

We have $$\phi'(x) = 12\left(x - \tfrac{1}{2}\right)(x - 3) \quad \text{....................} \quad (3).$$

This can only change sign when x passes through the values $\frac{1}{2}$ and 3. Now when x is a little less than $\frac{1}{2}$, the signs of the second and third factors are $-$, $-$; whilst when x is a little greater than $\frac{1}{2}$ they are $+$, $-$. Hence as x increases through the value $\frac{1}{2}$, $\phi'(x)$ changes sign from $+$ to $-$. In a similar manner we find that as x increases through the value 3, $\phi'(x)$ changes sign from – to +. Hence $\phi(x)$ is a maximum when $x=\frac{1}{2}$, and a minimum when $x=3$. If we substitute in (2) we find that the maximum value is $24\frac{1}{4}$, and the minimum value -7. See Fig. 33, p. 105.

Ex. 4. If

$$\phi(x)=\frac{2x}{1+x^2} \quad \text{......................} (4),$$

we find

$$\phi'(x)=\frac{2(1-x^2)}{(1+x^2)^2} \quad \text{......................} (5).$$

This can only change sign for $x=\pm 1$. As x increases (algebraically) through the value -1, $1-x^2$ changes sign from – to +. As x increases through $+1$, $1-x^2$ changes sign from + to –. Hence for $x=-1$ we have a minimum value -1 of $\phi(x)$, and for $x=1$ a maximum value 1. See Fig. 17, p. 31.

Ex. 5. If

$$\phi(x)=\frac{2x}{1-x^2} \quad \text{......................} (6),$$

we have

$$\phi'(x)=\frac{2(1+x^2)}{(1-x^2)^2} \quad \text{......................} (7).$$

Here $\phi'(x)$ is always positive, and the function $\phi(x)$ has no finite maxima or minima. See Fig. 16, p. 31.

Ex. 6. To find the right circular cylinder of least surface for a given volume.

If x denote the radius and y the altitude, the surface is

$$2\pi x^2+2\pi xy,$$

and if the given volume be $2\pi a^3$, we have

$$x^2y=2a^3.$$

Hence, eliminating y, the expression to be made a minimum is

$$x^2+\frac{2a^3}{x},$$

the derived function of which is

$$2\left(x-\frac{a^3}{x^2}\right).$$

This changes sign as x increases through the value a, and the change is from – to +. Hence $x=a$ makes the surface a minimum; and since y then $=2a$, the height of the cylinder is equal to its diameter.

The reader may verify that with these proportions the surface is 1·1447...of that of a sphere of equal volume.

Whenever the derived function $\phi'(x)$ vanishes, the rate of increase (Art. 33) of the original function $\phi(x)$ is momentarily zero, and the value of $\phi(x)$ is said to be 'stationary.' As already stated, a stationary value is not necessarily a maximum or minimum, for cases may of course occur in which $\phi'(x)$ vanishes without changing sign.

Ex. 7. The simplest instance of this is furnished by the function

$$\phi(x) \equiv x^3 \qquad \text{.........................(8).}$$

This makes $\phi'(x) = 3x^2$, which vanishes, but does not change sign, as x increases through the value 0. Hence $\phi(x)$, though 'stationary,' is not a maximum or minimum for $x = 0$. Fig. 34 shews the graph of the function x^3.

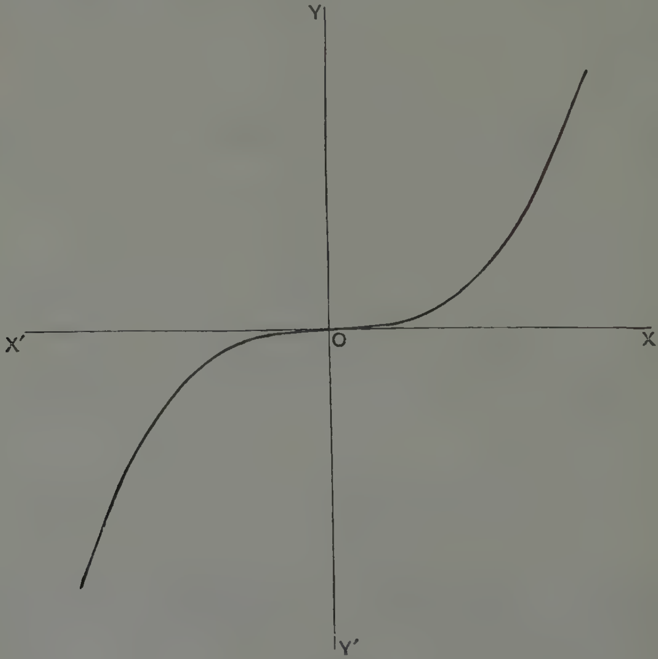

Fig. 34.

In most cases of interest, the derived function $\phi'(x)$ is continuous as well as determinate (and finite). It can then only change sign by passing through the value zero; and it is further evident from Art. 10 that the changes (if there are more than one) will take place from + to −, and from − to +, alternately. The maxima and minima will therefore occur alternately. See Fig. 18, p. 34.

51. Exceptional Cases.

It will, however, occasionally happen that $\phi'(x)$, though generally continuous, becomes discontinuous for some isolated value of x; and if the discontinuity be accompanied by a change of sign as x increases through the value in question, we shall have a maximum or minimum, by the same argument as in Art. 50.

Ex. If
$$\phi(x) = a^{\frac{1}{3}}x^{\frac{2}{3}} \quad\ldots\ldots\ldots\ldots(1),$$
we have
$$\phi'(x) = \frac{2}{3}\left(\frac{a}{x}\right)^{\frac{1}{3}} \quad\ldots\ldots\ldots\ldots(2).$$

As x increases through the value 0, this changes from $-\infty$ to $+\infty$. Hence $\phi(x)$ is a minimum for $x = 0$. See Fig. 35.

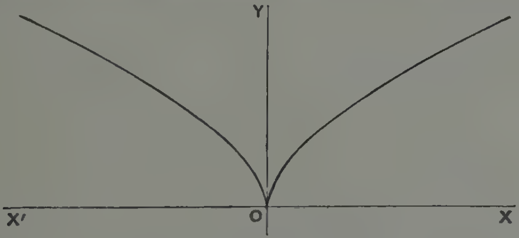

Fig. 35.

Again, in Fig. 32 there occurs a point where $\phi'(x)$ is discontinuous, passing abruptly from a finite positive to a finite negative value. The ordinate is then a maximum.

52. Algebraical Methods.

It is to be noticed that many important problems of maxima and minima can be solved by elementary algebraical methods, without recourse to the Calculus. This is especially the case with questions involving quadratic expressions. These are all easily treated by the method of 'completing the square.'

Ex. 1. Thus, in the problem of Ex. 2, Art. 50, we have

$$x(a-x)=\tfrac{1}{4}a^2-(x-\tfrac{1}{2}a)^2.$$

Since the last term cannot fall below zero, this expression has its greatest value ($\frac{1}{4}a^2$) when $x=\frac{1}{2}a$.

Ex. 2. The expression $2x^2-3x+2$, may be put in the form

$$2(x^2-\tfrac{3}{2}x+1)=2(x-\tfrac{3}{4})^2+\tfrac{7}{8}.$$

Hence the expression has the minimum value $\frac{7}{8}$, corresponding to $x=\frac{3}{4}$.

Again, the solution of many important problems comes at once from identities such as

$$xy=\tfrac{1}{4}\{(x+y)^2-(x-y)^2\} \quad\text{............(1)},$$

$$(x+y)^2=(x-y)^2+4xy \quad\text{......................(2)},$$

$$x^2+y^2=\tfrac{1}{2}\{(x+y)^2+(x-y)^2\} \quad\text{............(3)}.$$

Thus:

The product (xy) of two positive magnitudes, whose sum ($x+y$) is given, is greatest when they are equal;

The sum of two positive magnitudes whose product is given is least when they are equal;

The sum of the squares of two magnitudes whose sum is given is least when they are equal.

Ex. 3. To find the greatest rectangle which can be inscribed in a given circle.

If $2x$, $2y$ be the sides, we have to make xy a maximum subject to the condition that $x^2+y^2=a^2$, where a is the radius of the circle. Now

$$2xy=x^2+y^2-(x-y)^2=a^2-(x-y)^2 \quad\text{............(4)},$$

which is obviously greatest when $x=y$. Hence the greatest inscribed rectangle is a square.

Ex. 4. To find the minimum value of

$$a\cot\theta+b\tan\theta \quad\text{........................(5)},$$

for values of θ between 0 and $\frac{1}{2}\pi$.

The product of $a\cot\theta$ and $b\tan\theta$ is constant, hence their sum is least when they are equal, *i.e.* when

$$\tan\theta=(a/b)^{\frac{1}{2}} \quad\text{..........................(6)}.$$

The minimum value of the sum is therefore $2a^{\frac{1}{2}}b^{\frac{1}{2}}$.

Ex. 5. To find the greatest cylinder which can be inscribed in a frustum of a paraboloid of revolution cut off by a plane perpendicular to the axis.

Supposing the paraboloid to be generated by the revolution of the curve

$$y^2 = 4ax \text{..............................} (7),$$

about the axis of x, then if h be the length of the axis, and x the abscissa of the end of the cylinder nearest the origin, the volume of the cylinder is

$$\pi y^2 (h - x) = \pi y^2 \left(h - \frac{y^2}{4a}\right) \text{..................} (8).$$

Now the sum of the quantities y^2 and $4ah - y^2$ is constant; their product is therefore greatest when they are equal, *i.e.* when

$$y^2 = 2ah, \quad \text{or} \quad x = \tfrac{1}{2}h \text{.....................} (9).$$

The height of the cylinder is therefore one-half that of the frustum.

EXAMPLES. XIII.

1. Verify the theorem of Art. 48 in the following cases:

(1) $\phi(x) = (x-a)^m (x-b)^n$,

(2) $\phi(x) = \log \dfrac{x^2 + ab}{(a+b)x}$,

(3) $\phi(x) = \dfrac{(x-a)(x-b)}{x^2}$.

2. Prove that the curves

$$y = x^4 - 6x^3 + 9x^2 + 4x - 12,$$

and

$$y = x^4 - x^3 - 3x^2 + 5x - 2,$$

touch the axis of x, and find where they cut it. Trace the curves.

3. Prove that when x increases through a root of $\phi(x) = 0$, $\phi(x)$ and $\phi'(x)$ will have opposite signs just before, and the same sign just after the passage. Does this hold in the case of a double root?

4. If, for $a > x > 0$, $\phi(x) = \dfrac{x^2}{a} - a$,

and, for $x > a$, $$\phi(x) = a - \frac{a^3}{x^2},$$
whilst for $x = a$, $$\phi(x) = 0,$$
prove that $\phi(x)$ and $\phi'(x)$ are continuous from $x = 0$ to $x = \infty$. Trace the curve $y = \phi(x)$.

5. Prove that the expression
$$(x-1)e^x + 1,$$
is positive for all positive values of x.

6. Prove that in the rectilinear motion of a point, the velocity is a maximum or a minimum when the acceleration changes sign.

Illustrate this from the simple-harmonic motion
$$s = a \cos nt.$$

7. Find the maxima or minima of the function
$$x^4 - 8x^3 + 22x^2 - 24x + 12.$$

8. Prove that the function
$$2x^3 - 3x^2 - 36x + 10$$ ✓
is a maximum when $x = -2$, and a minimum when $x = 3$.

9. The function $4x^3 - 18x^2 + 27x - 7$
has no maxima or minima.

10. Find the stationary points of the function
$$x^5 - 5x^4 + 5x^3 + 1,$$ ✓
and examine for which of them the function is a maximum or minimum.

11. Prove that the function
$$10x^6 - 12x^5 + 15x^4 - 20x^3 + 20$$
has a minimum value when $x = 1$, and no other maxima or minima.

12. Prove that the function
$$x\sqrt{(ax - x^2)}$$ ✓
is a maximum when $x = \frac{3}{4}a$.

13. The function $$\frac{x}{(a+x)(b+x)}$$
is a maximum when $x = \sqrt{(ab)}$, and a minimum when $x = -\sqrt{(ab)}$.

14. Prove that the function

$$\frac{(x-1)^2}{(x+1)^3}$$

has a maximum value $\frac{2}{27}$, and a minimum value 0.

15. Prove that the expression

$$\frac{1+x+x^2}{1-x+x^2}$$

has a maximum value 3, and a minimum value $\frac{1}{3}$.

16. The function $$\frac{x(x^2+1)}{x^4-x^2+1}$$

has a maximum value 2, and a minimum value -2.

17. The function $$\frac{x(x^2-1)}{x^4-x^2+1}$$

has two maxima, each $=\frac{1}{2}$, and two minima, each $=-\frac{1}{2}$.

18. Prove that $\cos\theta+\sin\theta$

is a maximum when $\theta=\frac{1}{4}\pi$.

19. Prove that $\sin(\theta-\alpha)\cos(\theta-\beta)$

is a maximum or a minimum when

$$\theta=\tfrac{1}{2}(\alpha+\beta)+\tfrac{1}{4}\pi+\tfrac{1}{2}n\pi,$$

according as n is even or odd.

20. Find the maximum ordinate of the curve

$$y=xe^{-x}.$$

Trace the curve.

21. The curve $y=x\log x$

has a minimum ordinate $-\cdot3678\ldots.$

Trace the curve.

22. Prove that the ratio of the logarithm of a number (x) to the number itself is greatest when $x=e$.

23. Prove that the expression

$$a\cos\theta+b\sin\theta$$

has the maximum and minimum values $\pm\sqrt{(a^2+b^2)}$.

24. Prove that if $a > b$ the expression

$$a \cosh x + b \sinh x$$

has the minimum value $\sqrt{(a^2 - b^2)}$, but that if $a < b$ it has neither a maximum nor a minimum.

25. Prove that the function ✓

$$\cosh x + \cos x$$

has a minimum value when $x = 0$, but no other maxima or minima.

26. Prove that the function

$$\cosh x \cos x$$

has a maximum value when $x = 0$, a minimum value when $x = \frac{5}{4}\pi$ (nearly), and a series of alternate maxima and minima corresponding to $x = n\pi + \frac{1}{4}\pi$, approximately, where $n = 1, 2, 3....$

27. Prove that the function

$$m_1 (x - x_1)^2 + m_2 (x - x_2)^2 + \ldots + m_n (x - x_n)^2$$

is a minimum when

$$x = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}.$$

28. The velocity of waves of length λ on deep water is proportional to

$$\sqrt{\left(\frac{\lambda}{a} + \frac{a}{\lambda}\right)},$$ ✓

where a is a certain linear magnitude; prove that the velocity is a minimum when $\lambda = a$.

29. The inclination of a pendulum to the vertical, when the resistance of the air is taken into account, is given by the formula ✓

$$\theta = ae^{-kt} \cos (nt + \epsilon);$$

prove that the greatest elongations occur at equal intervals π/n of time, and that they form a series diminishing in geometrical progression.

30. The force exerted by a circular electric current of radius a on a small magnet whose axis coincides with the axis of the circle, varies as

$$\frac{x}{(a^2 + x^2)^{\frac{5}{2}}},$$

where x is the distance of the magnet from the plane of the circle. Hence prove that the force is a maximum when $x = \frac{1}{2}a$.

31. Find the straight line of quickest descent from a given point to a given vertical straight line, assuming that the time of sliding a distance s from rest at an inclination θ to the horizontal is

$$\left(\frac{2s}{g \sin \theta}\right)^{\frac{1}{2}}.$$

[The minimum time is $2\,(a/g)^{\frac{1}{2}}$, corresponding to $\theta = \frac{1}{4}\pi$, where a is the horizontal distance of the point from the given straight line.]

EXAMPLES. XIV.

1. The rectangle of least perimeter for a given area is a square.

2. The rectangle of given perimeter which has the shortest diagonal is a square.

3. The greatest rectangle which can be inscribed in a given triangle has one-half the area of the triangle.

4. A rectangle is inscribed in a right-angled triangle, so as to have one angle coincident with the right angle; prove that its area is a maximum when the opposite corner bisects the hypothenuse.

Shew also that under the same circumstances the perimeter of the rectangle has neither a maximum nor a minimum value.

5. Find the rectangle of greatest or least perimeter which can be inscribed in a given circle.

6. If through a given point A within a circle a chord PAQ be drawn, the sum of the squares of the segments PA, AQ is least when the chord is perpendicular to the diameter through A, and greatest when the chord coincides with the diameter.

7. Given a fixed straight line, and two fixed points A, B outside it, it is required to find a point P in the straight line such that $AP^2 + PB^2$ shall be a minimum.

8. Find the square of least area which can be inscribed in a given square; and the square of greatest area which can be circumscribed to a given square.

9. A straight line drawn through a point (a, b) meets the (rectangular) coordinate axes in P and Q, respectively; prove that the minimum value of $OP + OQ$ is

$$a + 2\sqrt{(ab)} + b.$$

10. A straight line is drawn through a fixed point (a, b); prove that the minimum length intercepted between the coordinate axes (supposed rectangular) is

$$(a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}.$$

11. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Shew that when the volume contained in the box is a maximum, the depth will be ✓

$$\tfrac{1}{6}\{a + b - \sqrt{(a^2 - ab + b^2)}\},$$

where a, b are the sides of the original rectangle.

12. At what distance from the wall of a house must a man whose eye is 5 feet from the ground station himself in order that a window 5 feet high, whose sill is 20 feet from the ground, may subtend the greatest vertical angle?

13. It is required to cut from a cylindrical tree-trunk a beam of rectangular section of maximum flexural rigidity; prove that the breadth of the section must be $\frac{1}{2}$ the diameter, and its depth ·866 of the diameter. (Assume that the flexural rigidity varies as the breadth and as the cube of the depth.) ✓

14. A straight road runs along the edge of a common, and a person on the common at a distance of one mile from the nearest point (A) of the road wishes to go to a distant place on the road in the least time possible. If his rates of walking on the common and on the road be 4 and 5 miles an hour, respectively, shew that he must strike the road at a point distant $1\frac{1}{3}$ miles from A.

15. Find at what height on the wall of a room a source of light must be placed in order to produce the greatest brightness at a point on the floor at a given distance a from the wall. (Assume that the brightness of a surface varies inversely as the square of the distance from the source, and directly as the cosine of the angle which the rays make with the normal to the surface.)

16. Two particles P, Q describe fixed straight lines intersecting in O, with constant velocities u, v. Prove that if A, B be simultaneous positions of the particles, and if $OA = a$, $OB = b$, $\angle AOB = \omega$, the distance PQ will be least after a time

$$\frac{au + bv - (av + bu)\cos\omega}{u^2 - 2uv\cos\omega + v^2},$$

and that the least distance will be

$$\frac{(av \sim bu)\sin\omega}{\sqrt{(u^2 - 2uv\cos\omega + v^2)}}.$$

✓

17. Prove that the greatest rectangle which can be inscribed in a segment of a parabola bounded by a chord perpendicular to the axis has a length equal to $\frac{2}{3}$ that of the segment.

18. The greatest rectangle which can be inscribed in a given ellipse has its diagonals along the equi-conjugate diameters.

19. If the length of a tangent to an ellipse intercepted between the axes be a minimum, the tangent is divided at the point of contact into two portions equal to the semi-axes of the ellipse, respectively.

20. If a tangent to an ellipse includes with the principal axes (produced) a triangle of minimum area, it is parallel to one of the equi-conjugate diameters.

21. A circular sector has a given perimeter; prove that the area is a maximum when the angle of the sector is 2 radians, and that the area is then equal to the square on the radius.

22. If a triangle have a given base, and if the sum of the other two sides be given, the area is greatest when these two sides are equal.

23. A quadrilateral has its four sides of given lengths, in a given order; prove that its area is greatest when it can be inscribed in a circle.

24. If the power required to propel a steamer through the water vary as the cube of the speed, the most economical rate of steaming against a current will be at a speed equal to $1\frac{1}{2}$ times that of the current. ✓

EXAMPLES. XV.

[The following results may be assumed:

(1) The curved surface of a right circular cylinder of height h and radius a is $2\pi ah$;

(2) The volume of the same cylinder is $\pi a^2 h$;

(3) The curved surface of a right circular cone of height h, base-radius a, and slant side l is πal;

(4) The volume of the same cone is $\frac{1}{3}\pi a^2 h$;

(5) The surface of a sphere of radius a is $4\pi a^2$;

(6) The volume of the same sphere is $\frac{4}{3}\pi a^3$.]

1. The cylinder of greatest volume which can be inscribed in a given sphere has a volume equal to ·5773 of that of the sphere.

2. The cylinder of greatest superficial area which can be inscribed in a given sphere has a surface equal to ·8090 of that of the sphere.

3. The cylinder of greatest volume for a given superficial area has its height equal to the diameter of the base, and its volume is ·8165 of that of a sphere having the given superficial area.

4. Find the cylinder of least surface for a given volume; and prove that the ratio of its surface to that of a sphere of equal volume is 1·145.

5. Find the proportions of a thin open cylindrical vessel in order that the amount of material required may be the least possible for a given volume.

[The height must equal the radius of the base.]

6. A cylinder is inscribed in a right circular cone; prove that its volume is a maximum when its altitude is $\frac{1}{3}$ that of the cone, and that its volume is then $\frac{4}{9}$ that of the cone. ✓

7. If a cylinder be inscribed in a right circular cone the *curved* surface is a maximum when the altitude of the cylinder is $\frac{1}{2}$ that of the cone. ✗

Shew also that the *total* surface of the cylinder cannot have a maximum value if the semi-angle of the cone exceeds 26° 34′ $[=\tan^{-1}\frac{1}{2}]$.

8. The cone of greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ the diameter of the sphere.

Prove, also, that the *curved* surface of the cone is a maximum for the same value of the altitude.

9. If a right circular cone be circumscribed to a given sphere, its volume will be a minimum when the altitude is double the diameter of the sphere. Shew also that the semi-vertical angle will be $19^\circ\ 28'$ $[=\sin^{-1}\frac{1}{3}]$.

10. The right circular cone of greatest surface for a given volume has an altitude equal to $\sqrt{2}$ times the diameter of the base.

11. From a given circular sheet of metal it is required to cut out a sector so that the remainder can be formed into a conical vessel of maximum capacity; prove that the angle of the sector removed must be about 66°. ✓

53. Geometrical Applications of the Derived Function. Cartesian Coordinates.

We have seen (Art. 32) that if ψ denotes the angle which the tangent, drawn to the right, at any point of the curve

$$y=\phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

makes with the positive direction of the axis of x, then

$$\frac{dy}{dx}=\tan\psi \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

With the help of this formula, several magnitudes connected with a curve may be expressed in terms of x, y, and dy/dx.

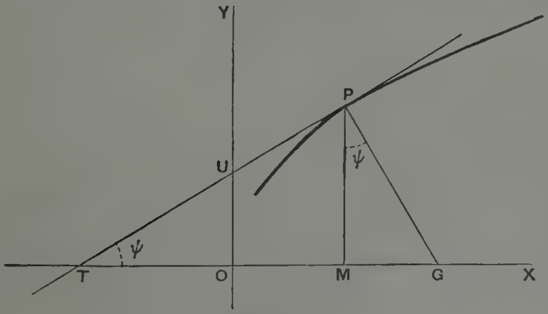

Fig. 36.

If the tangent and the normal at the point P meet the axis of x in T and G, respectively, and if M be the foot of the ordinate, then TM is called the 'subtangent' and MG the 'subnormal.' Hence we find

$$\text{subtangent} = TM = MP \cot\psi = y \div \frac{dy}{dx} \quad\ldots\ldots\ldots\ldots(3),$$

$$\text{subnormal} = MG = MP \tan\psi = y\frac{dy}{dx} \quad\ldots\ldots\ldots\ldots(4),$$

$$\text{tangent} = TP = MP \operatorname{cosec}\psi$$
$$= y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \div \frac{dy}{dx} \ldots(5),$$

$$\text{normal} = PG = MP \sec\psi = y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \ldots\ldots(6).$$

Again, the intercepts of the tangent on the coordinate axes are

$$\left.\begin{aligned} OT &= OM - TM = x - \frac{y}{\frac{dy}{dx}} \\ OU &= TO\tan\psi = y - x\frac{dy}{dx} \end{aligned}\right\} \ldots\ldots\ldots(7).$$

Ex. 1. In the parabola $y^2 = 4ax$ .... (8),
we have, differentiating both sides with respect to x, and omitting the factor 2,

$$y\frac{dy}{dx} = 2a \quad\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

which shews that the subnormal is constant and equal to $2a$.

Again, the subtangent is

$$y \div \frac{dy}{dx} = \frac{y^2}{2a} = 2x \quad\ldots\ldots\ldots\ldots\ldots(10),$$

and is therefore double the abscissa; in other words, the origin O bisects TM.

Ex. 2. In the hyperbola

$$xy = k^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

we have
$$x\frac{dy}{dx} + y = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Hence the formulæ (7) for the intercepts of the tangent on the coordinate axes give $2x$, $2y$ as the value of these intercepts, respectively. Hence M bisects OT, and therefore P bisects TU; *i.e.* the portion of the tangent included between the coordinate axes is bisected at the point of contact.

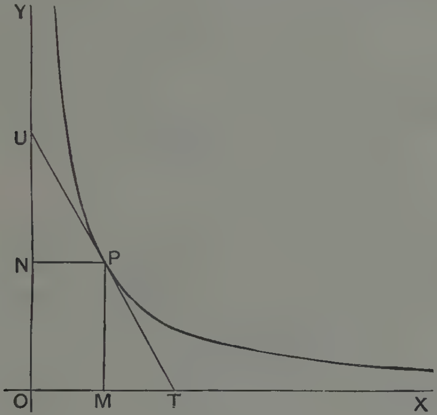

Fig. 37.

Again, the product of these intercepts is equal to $4xy$, or $4k^2$. Hence the area of the triangle OTU is constant and equal to $2k^2$.

Ex. 3. More generally, in the curve

$$x^m y^n = \text{const.} \quad \text{..........................(13)},$$

we find, taking logarithms of both sides and then differentiating,

$$\frac{m}{x} + \frac{n}{y}\frac{dy}{dx} = 0 \quad \text{..........................(14)}.$$

This makes $$OT = x - y \Big/ \frac{dy}{dx} = x + \frac{n}{m}x = \frac{m+n}{m}x.$$

Hence $$UP : PT = OM : MT = x : \frac{n}{m}x = m : n \quad \text{......(15)},$$

that is, the tangent UT is divided in a constant ratio at the point of contact.

This includes the two preceding cases. In the parabola (*Ex.* 1) we had $m = -1$, $n = 2$; in the hyperbola (*Ex.* 2) $m = 1$, $n = 1$.

An important physical example is that of the 'adiabatic' relation between the pressure and the volume of a gas, viz.

$$pv^\gamma = \text{const.} \quad \text{.......................} (16).$$

If a curve be constructed with v as abscissa and p as ordinate, the tangent is divided at the point of contact in the ratio $\gamma : 1$.

Ex. 4. In the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{.......................} (17)$$

we find, on differentiating,

$$\frac{x}{a^2} + \frac{y}{b^2}\frac{dy}{dx} = 0,$$

whence

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y} \quad \text{.......................} (18).$$

The intercept made by the tangent on the axis of x is

$$OT = x - y \Big/ \frac{dy}{dx} = x + \frac{a^2}{x}\frac{y^2}{b^2} = x + \frac{a^2}{x}\left(1 - \frac{x^2}{a^2}\right) = \frac{a^2}{x},$$

whence

$$OM \,.\, OT = a^2 \text{.......................} (19).$$

The intercept made by the normal is

$$OG = x + y\frac{dy}{dx} = \left(1 - \frac{b^2}{a^2}\right)x = e^2 \,.\, OM \quad \text{........} (20),$$

where e is the eccentricity.

54. Coordinates determined by a single variable

A curve is sometimes defined by means of two equations of the type

$$x = \phi(t), \quad y = \chi(t) \quad \text{..................} (1)$$

giving the coordinates in terms of a subsidiary variable t.

For example, in Dynamics, the coordinates of a moving particle may be given as functions of the time.

If we take any convenient series of values of t, we can calculate the corresponding values of x and y, and so plot out as many points as we please on the curve.

If δx, δy, δt be simultaneous increments of x, y, t, we have

$$\frac{\delta y}{\delta x} = \frac{\delta y}{\delta t} \div \frac{\delta x}{\delta t},$$

and therefore in the limit, when δt is indefinitely diminished,

$$\tan\psi = \frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt} \quad\text{.................(2).}$$

Ex. 1. In the ellipse

$$x = a\cos\phi, \quad y = b\sin\phi \quad\text{.....................(3),}$$

we have
$$\tan\psi = \frac{dy}{d\phi}\bigg/\frac{dx}{d\phi} = -\frac{b}{a}\cot\phi \quad\text{..................(4).}$$

Ex. 2. In the case of a projectile moving under gravity, we have

$$x = a + u_0 t, \quad y = b + v_0 t - \tfrac{1}{2}gt^2 \quad\text{...............(5),}$$

whence
$$\tan\psi = \frac{dy}{dt}\bigg/\frac{dx}{dt} = \frac{v - gt}{u_0}.$$

55. Polar Coordinates.

Let P, P' be two neighbouring points on a curve, and let r, θ be the polar coordinates of P, and $r + \delta r$, $\theta + \delta\theta$ those of P'. If we join PP', and draw PN perpendicular to OP', we have

$$PN = OP\sin PON = r\sin\delta\theta,$$
$$P'N = OP' - ON = r + \delta r - r\cos\delta\theta = \delta r + r(1 - \cos\delta\theta).$$

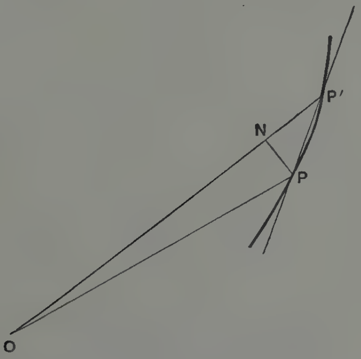

Fig. 38.

When $\delta\theta$ is indefinitely diminished, the ratio of $\sin\delta\theta$ to $\delta\theta$ tends to the limiting value unity, and $1 - \cos\delta\theta, = 2\sin^2\frac{1}{2}\delta\theta$, is a small quantity of the second order. Hence we may write

$$\tan PP'O = \frac{PN}{P'N} = \frac{r\delta\theta}{\delta r} + \sigma \quad \ldots\ldots\ldots\ldots\ldots(1),$$

where σ is a quantity whose limiting value is zero. Hence ultimately, when P' coincides with P, we have, if ϕ denotes the angle which the tangent to the curve at P, drawn on the side of θ increasing, makes with the positive direction of the radius vector,

$$\tan\phi = r\frac{d\theta}{dr} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)^*.$$

Here θ is regarded as a function of r. If r be regarded as a function of θ, the formula is

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 1. In the circle $r = 2a\sin\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$

we have $\log r = \log 2a + \log\sin\theta,$

and therefore $\frac{1}{r}\frac{dr}{d\theta} = \cot\theta,$

whence $\cot\phi = \cot\theta, \quad \text{or} \quad \phi = \theta \ldots\ldots\ldots\ldots\ldots\ldots(5).$

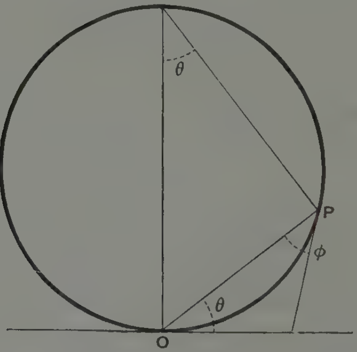

Fig. 39.

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[*] The argument, which is an application of a principle stated in Art. 30, may be amplified as follows. We have, exactly,

$$\tan PP'O = \frac{r\sin\delta\theta}{\delta r + 2r\sin^2\frac{1}{2}\delta\theta} = r\frac{\delta\theta}{\delta r}\cdot\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{1}{1 + \frac{r\delta\theta}{\delta r}\cdot\frac{\sin\frac{1}{2}\delta\theta}{\frac{1}{2}\delta\theta}\cdot\sin\frac{1}{2}\delta\theta},$$

and the limiting value of this is evidently $rd\theta/dr$.

Ex. 2. When the radius vector of a curve is a maximum or a minimum, it is in general normal to the curve.

For if $dr/d\theta = 0$, we have $\cot\phi = 0$, or $\phi = \frac{1}{2}\pi$.

EXAMPLES. XVI.

1 Prove that the condition that the tangent to a curve should pass through the origin is

$$\frac{y}{x} = \frac{dy}{dx}.$$

Prove that a pair of straight lines can be drawn through the origin, each of which touches all the curves obtained by giving c different values in the equation

$$y = c\cosh\frac{x}{c}.$$

2. Prove that the perpendicular drawn from the foot of the ordinate to the tangent of a curve is

$$y \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

Hence shew that in the catenary $y = c\cosh x/c$ this perpendicular is constant.

3. Prove that the perpendicular from the origin on the tangent is

$$\left(y - x\frac{dy}{dx}\right) \div \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}.$$

Verify that in the circle

$$y = \pm\sqrt{(a^2 - x^2)}$$

this perpendicular is constant, and that in the rectangular hyperbola

$$xy = k^2$$

it is equal to $$\frac{2k^2}{\sqrt{(x^2 + y^2)}}.$$

4. In the exponential curve (Fig. 20, p. 38)

$$y = be^{x/a},$$

the subtangent is constant, and the subnormal is y^2/a.

5. In the catenary $y = c \cosh x/c$,
the subtangent is $c \coth x/c$, the subnormal is $\frac{1}{2}c \sinh 2x/c$, and the normal is y^2/c.

6. The subtangent of the curve

$$y^n = a^{n-1} x$$

is nx.

7. Prove that the curve

$$\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$$

touches the straight line $\frac{x}{a} + \frac{y}{b} = 2$

at the point (a, b), whatever the value of n.

8. In the curve of sines

$$y = b \sin \frac{x}{a},$$

the subtangent is $a \tan x/a$, the subnormal is $\frac{1}{2}b^2/a \,.\, \sin 2x/a$, and the normal is

$$b \sin \frac{x}{a} \,.\, \sqrt{\left(1 + \frac{b^2}{a^2} \cos^2 \frac{x}{a}\right)}.$$

9. Prove that the curves

$$y = e^{-\alpha x} \sin \beta x, \quad y = e^{-\alpha x}$$

touch at the points for which $\beta x = 2n\pi + \frac{1}{2}\pi$, where n is integral. Sketch the curves.

10. Prove that in the parabola

$$r = \frac{a}{\sin^2 \frac{1}{2}\theta},$$

the focus being pole, $\phi = \pi - \frac{1}{2}\theta$,

and hence shew that the tangent makes equal angles with the focal distance and with the axis.

11. Two adjacent points P, P' on a curve being taken, straight lines PR, $P'R$ are drawn at right angles to the radii; prove that the limiting value of PR, when P' coincides with P, is $dr/d\theta$.

12. If ϕ be the angle which the tangent to a curve makes with the radius vector drawn from the origin, prove that

$$\tan\phi = \frac{x\dfrac{dy}{dx} - y}{x + y\dfrac{dy}{dx}}.$$

13. If a curve be constructed with the velocity (v) of a moving point as ordinate, and the space described (s) as abscissa, the acceleration will be represented by the subnormal.

14. If a curve be constructed with the kinetic energy ($\frac{1}{2}mv^2$) of a particle as ordinate, and the space s as abscissa, the force will be represented by the gradient of the curve.

56. Mean-Value Theorem. Consequences.

The following very important theorem is an extension of that given in Art. 48.

If a function $\phi(x)$ be continuous, and have a determinate derivative, throughout the interval from $x=a$ to $x=b$, then

$$\frac{\phi(b)-\phi(a)}{b-a} = \phi'(x_1) \quad \text{..................(1)}$$

where x_1 is some value of x *between* a and b.

Consider the function

$$\phi(x) - \phi(a) - \frac{\phi(b)-\phi(a)}{b-a}(x-a) \quad \text{.........(2).}$$

This is, under the conditions stated, continuous from $x=a$ to $x=b$, and it obviously vanishes for each of these values of x. Hence its derived function

$$\phi'(x) - \frac{\phi(b)-\phi(a)}{b-a} \quad \text{.................. (3)}$$

must vanish for some value (x_1, say,) of x between a and b. This proves the statement (1).

The meaning of this result, and the nature of the proof, should be carefully studied. The geometrical interpretation is as follows. In the annexed figure, we have

$$OA = a, \qquad OB = b,$$

$$PA = \phi(a), \qquad QB = \phi(b),$$

and

$$\tan QPN = \frac{QN}{PN} = \frac{\phi(b) - \phi(a)}{b - a} \quad \ldots\ldots\ \ldots\ldots (4).$$

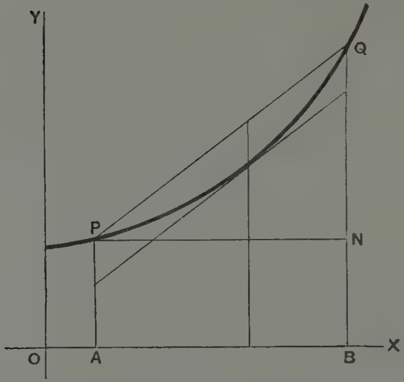

Fig. 40.

The theorem therefore asserts that (under the restrictions stated) there is some point between P and Q where the tangent to the curve $y = \phi(x)$ is parallel to the chord PQ.

The equation of the chord PQ is

$$y = \phi(a) + \frac{\phi(b) - \phi(a)}{b - a}(x - a) \ldots\ldots\ldots\ldots (5),$$

as is easily verified, and the expression (2) therefore measures the difference between the ordinate of the curve and that of the chord. This difference vanishes at P and Q, so that there must be one point at least between P and Q at which it is a maximum or a minimum.

Ex. 1. If

$$\phi(x) = x^2,$$

we have

$$\frac{\phi(b) - \phi(a)}{b - a} = b + a,$$

which is equal to the value of $\phi'(x)$ for $x = \frac{1}{2}(a + b)$.

This is equivalent to the statement that any chord of a parabola is parallel to the tangent at the extremity of that diameter which bisects the chord.

The fraction
$$\frac{\phi(b)-\phi(a)}{b-a} \qquad\ldots\ldots\ldots\ldots(6),$$
that is, the ratio of the increment of the function to that of the independent variable, measures what may be called the 'mean rate of increase' of the function in the interval $b-a$. Hence the theorem expresses that, under the conditions stated, the mean rate of increase in any interval is equal to the actual rate of increase at some point within the interval.

For instance, the mean velocity of a moving point in any interval of time is equal to the actual velocity at some instant within the interval.

Some other modes of stating the result (1) are to be noticed. The fact that x_1 lies between a and b may be expressed by putting
$$x_1 = a+\theta(b-a) \qquad\ldots\ldots\ldots\ldots(7),$$
where θ stands for 'some quantity between 0 and 1.' The precise value of θ will in general depend on the values of a and b. If we further write $a+h$ for b, we get the very useful form
$$\frac{\phi(a+h)-\phi(a)}{h} = \phi'(a+\theta h) \qquad\ldots\ldots\ldots\ldots(8)$$
or
$$\phi(a+h) = \phi(a)+h\phi'(a+\theta h) \qquad\ldots\ldots\ldots\ldots(9).$$
Again, if we write x for a, and δx for h, we have
$$\delta\phi(x) = \phi'(x+\theta\,\delta x)\,.\,\delta x \qquad\ldots\ldots\ldots\ldots(10).$$

A very important deduction from the preceding theorem is that if
$$\phi'(x) = 0 \qquad\ldots\ldots\ldots\ldots(11)$$
for *all* values of x within a certain range, then $\phi(x)$ must be constant throughout that range.

For if $\phi(x)$ vary, let a and b be two values of x for which it has unequal values. The fraction
$$\frac{\phi(b)-\phi(a)}{b-a} \qquad\ldots\ldots\ldots\ldots(12),$$
will then be different from zero, and there will therefore be some intermediate value of x for which $\phi'(x)$ will differ from zero, contrary to the hypothesis.

Moreover, if two functions $\phi(x)$ and $\psi(x)$ have equal derivatives for all values of x within a certain range, they can only differ by a constant. For, by hypothesis,

$$\phi'(x) - \psi'(x) = 0 \quad \text{.................(13)},$$

or

$$\frac{d}{dx}\{\phi(x) - \psi(x)\} = 0 \quad \text{..............(14)}.$$

Hence

$$\phi(x) - \psi(x) = \text{const.} \quad \text{..............(15)},$$

by the preceding case.

Ex. 2. If the normals to a curve all pass through a fixed point, the curve must be a circle.

For, by hypothesis, if the fixed point be taken as pole, we have, in the notation of Art. 55, $\phi = \frac{1}{2}\pi$, and therefore $dr/d\theta = 0$, for all values of θ. Hence $r = \text{const.}$

57. Notation of Differentials.

We return to the equation

$$\frac{\delta y}{\delta x} = \phi'(x) + \sigma \text{.......................(1)},$$

of Art. 47. This is equivalent to

$$\delta y = \phi'(x)\,\delta x + \sigma\delta x \text{.................(2)}.$$

As δx approaches the value 0, the second term on the right hand becomes more and more insignificant compared with the first, since the limiting value of σ is zero. Hence it becomes more and more nearly true that

$$\delta y = \phi'(x)\,\delta x \quad \text{........................ (3)},$$

not in the sense that both sides ultimately vanish, but in the sense that the *ratio* of the two sides approaches the value unity. In this artificial sense, the last equation is often written in the form

$$dy = \phi'(x)\,dx \quad \text{........................ (4)}.$$

The vanishing quantities dx, dy are called 'differentials.'*

The student need not take exception to the above mode of expression, which is purely conventional. Its use is simply to express the fact that in calculations involving the quantities δx and δy, which are afterwards made to approach the limit zero, we may at any stage replace δy by $\phi'(x)\,\delta x$, whenever it is plain that the omission of quantities of the second order will make no difference to the accuracy of the final result.

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* It is on account of the position which it occupies in this formula that $\phi'(x)$ is called the 'differential coefficient.'

58. Calculation of Small Corrections.

The equation

$$\delta y = \phi'(x)\,\delta x \quad\text{.....................}\quad (1)$$

may, moreover, be employed as an *approximate* formula to find the effect on the value of a function of a small change in the independent variable, since (as we have seen) the outstanding error will be merely a small fraction of $\phi'(x)\,\delta x$ provided δx be sufficiently small. An important practical application is to find the error, or the uncertainty, in a numerical result deduced from given data, owing to given errors or uncertainties in the data.

Ex. 1. To calculate the difference for one minute in a table of log sines.

If $y = \log_{10} \sin x$, we have $dy/dx = \mu \cot x$,

and

$$\delta y = \mu \cot x \delta x,$$

approximately, provided δx be expressed in circular measure. Putting

$$\delta x = \text{circular measure of } 1' = \frac{\pi}{10800} = \cdot 0002909,$$

we find

$$\delta y = \cdot 0001263 \times \cot x.$$

The numerical factor agrees with the difference for 1′, in the neighbourhood of 45°, given in the tables.

Ex. 2. Two sides a, b of a triangle and the included angle C are measured; to find the error in the computed length of the third side c due to a small error in the angle.

We have

$$c^2 = a^2 + b^2 - 2ab \cos C \quad\text{..................}\quad (2),$$

and therefore, supposing C and c alone to vary,

$$c\delta c = ab \sin C\delta C,$$

whence

$$\delta c = \frac{ab}{c} \sin C\delta C = a \sin B\delta C \quad\text{...........}\quad (3).$$

This result may also be obtained geometrically; thus, if in the figure $\angle BCB' = \delta C$, and BN be drawn perpendicular to AB', we have, ultimately,

$$\delta c = B'N = BB' \cos BB'N = a\delta C \,.\, \sin CB'A = a\delta C \,.\, \sin B,$$

neglecting small quantities of the second order.

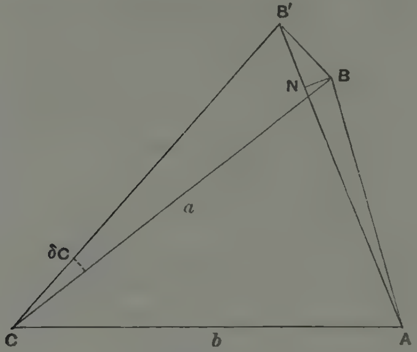

Fig. 41.

Again, to find the error in c due to a small error in the measured length of a, we have, on the hypothesis that a and c alone vary,

$$c\delta c = (a - b \cos C)\, \delta a = c \cos B \delta a,$$

or
$$\delta c = \cos B \delta a \text{ .... } (4),$$

a result which, like the former, admits of easy geometrical proof.

The above method is defective in one respect, in that there is no indication of the magnitude of the error involved in the approximation. This is supplied, however, by the theorem of Art. 56. It was there shewn that

$$\delta y = \phi'(x + \theta \delta x)\, \delta x \text{ .... } (5),$$

where θ is some quantity between 0 and 1. Hence if A and B be the greatest and least values which the derived function assumes in the interval from x to $x + \delta x$, the error committed in (1) cannot be greater than $|(A - B)\, \delta x|$.

59. Maxima and Minima of Functions of several Variables.

We close this chapter with a few indications concerning the extension of some of the preceding results to functions of two or more independent variables.

In the first place let us seek for the maxima and minima of a function

$$u = \phi(x, y) \qquad\qquad (1).$$

A first condition is that we must have simultaneously

$$\frac{\partial \phi}{\partial x} = 0, \qquad \frac{\partial \phi}{\partial y} = 0 \qquad\qquad (2),$$

where the differential coefficients are 'partial,' as in Art. 45. For if u be greater (or less) than any other value of the function obtained by varying x, y within certain limits, u will *à fortiori* be a maximum (or minimum) when y is kept constant and x alone is varied. This requires in general (Art. 50) that $\partial\phi/\partial x = 0$. Similarly, u must be a maximum (or minimum) when x is kept constant and y alone varies; this requires that $\partial\phi/\partial y = 0$.

As before, these conditions, though necessary, are not sufficient. The further examination of the question, in its general form, is deferred till Chapter XIV.; but it often happens that the existence of maxima and minima can be inferred, and the discrimination between them can be effected, by independent considerations. The conditions (2) then supply all that is analytically necessary.

Ex. To find the rectangular parallelepiped of least surface for a given volume.

Let x, y, z be the edges, and a^3 the given volume. Since

$$xyz = a^3 \qquad\qquad (3),$$

the function to be made a minimum is

$$u = xy + yz + zx = xy + \frac{a^3}{x} + \frac{a^3}{y} \qquad\qquad (4).$$

The conditions $\partial u/\partial x = 0$, $\partial u/\partial y = 0$ give

$$x^2 y = a^3, \quad xy^2 = a^3,$$

the only real solution of which is $x = y = a$, whence, also, $z = a$.

It appears from (4) that, x and y being essentially positive in this problem, there is a lower limit to the surface of the parallelepiped. And the above investigation shews that this limit is not attained unless the figure be a cube.

As in Art. 52, the solutions of various problems can be deduced from known algebraical identities, such as

$$x^2+y^2+z^2=\tfrac{1}{3}\{(x+y+z)^2+(y-z)^2+(z-x)^2 + (x-y)^2\}\text{.........}(5),$$

$$yz+zx+xy=x^2+y^2+z^2-\tfrac{1}{2}\{(y-z)^2+(z-x)^2 + (x-y)^2\}\text{.........}(6).$$

Thus:

If a straight line be divided into three segments, the sum of the squares on these is least when the segments are equal;

The surface of a parallelepiped inscribed in a given sphere ($x^2+y^2+z^2=a^2$) is greatest when the figure is a cube.

60. Total Variation of a Function of several Variables.

Let $$u=\phi(x, y)\text{......................}(1),$$
be a continuous function of x and y, and further let us suppose that the partial derivatives

$$\frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}\text{.....................}(2),$$

are also continuous functions of x and y.

Let δu be the increment of u due to increments δx and δy of the independent variables; *i.e.*

$$\delta u=\phi(x+\delta x, y+\delta y)-\phi(x, y)\text{..........}(3).$$

In the geometrical representation (Art. 45), δu is the difference of altitude of the two points of a surface which correspond to two points (x, y) and $(x+\delta x, y+\delta y)$ of the horizontal plane xy.

Now if x alone were varied, the corresponding increment of u would by Art. 56 (10), be of the form

$$P\delta x\text{................................}(4),$$

where P is a certain function of x, y, and δx. And it appears from the same Art., and from the meaning of a partial derivative, that the limiting value of P when δx is indefinitely diminished is

$$P_0 = \frac{\partial u}{\partial x} \quad \text{.........................(5).}$$

Similarly, if y alone were varied, the increment of u would be

$$Q\delta y \quad \text{..............................(6),}$$

where the limiting value of Q, when δy is indefinitely diminished, is

$$Q_0 = \frac{\partial u}{\partial y} \quad \text{.........................(7).}$$

Let us now suppose that the actual variation from x, y to $x + \delta x$, $y + \delta y$ is made in two successive steps, in the first of which x alone, and in the second of which y alone is varied. The total increment of u will then be

$$\delta u = P\delta x + Q'\delta y \quad \text{.................. (6)*,}$$

where Q' differs from Q owing to the fact that the starting point of the second variation is now $(x + \delta x, y)$ instead of (x, y).

To find the form which (6) assumes when δx and δy tend simultaneously to the value 0, preserving any assigned ratio to one another, we put

$$\delta x = \alpha\delta t, \qquad \delta y = \beta\delta t \quad \text{..................(7),}$$

when α, β are finite, and δt is made ultimately to vanish. We have, then

$$\frac{\delta u}{\delta t} = P\alpha + Q'\beta \quad \text{.....................(8).}$$

In virtue of the assumed continuity of the derivatives (2), the limiting value of the right-hand side, when $\delta t = 0$, is

$$P_0\alpha + Q_0\beta \quad \text{or} \quad \frac{\partial u}{\partial x}\alpha + \frac{\partial u}{\partial y}\beta \quad \text{............ (9).}$$

Hence, the smaller δx, δy are taken, provided they have some

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* If we write $\dfrac{\partial u}{\partial x} = \phi_x(x, y)$, $\dfrac{\partial u}{\partial y} = \phi_y(x, y)$,

we have $$P = \frac{\phi(x + \delta x, y) - \phi(x, y)}{\delta x} = \phi_x(x + \theta_1\delta x, y),$$

$$Q' = \frac{\phi(x + \delta x, y + \delta y) - \phi(x + \delta x, y)}{\delta y} = \phi_y(x + \delta x, y + \theta_2\delta y),$$

where θ , θ_2 lie between 0 and 1.

definite ratio to one another, the more nearly does it become true that

$$\delta u = \frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y \qquad\qquad (10),$$

in the sense that the *ratio* of the two sides is ultimately one of equality. This result is often expressed in the form

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy \qquad\qquad (11).$$

The symbols dx, dy, du are then called 'differentials,' and du is called the 'total differential' of u.

The equation (10) shews that in the neighbourhood of a maximum or minimum, the variation of u is of the second order of small quantities, since we then have

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0 \qquad\qquad (12),$$

by Art. 59. Thus, at a point of maximum or minimum altitude on a surface the tangent plane is in general horizontal. As already indicated, the converse is not necessarily true. See Art. 51.

The preceding theorem can be readily extended to the case of any number of independent variables x, y, z.... We have

$$\delta u = \frac{\partial u}{\partial x}\,\delta x + \frac{\partial u}{\partial y}\,\delta y + \frac{\partial u}{\partial z}\,\delta z + \ldots \qquad\qquad (13),$$

ultimately.

61. Application to Small Corrections.

The theorem of the preceding Art. can be applied after the manner of Art. 58 to the calculation of small corrections.

Ex. 1. In the case of Art. 58, Ex. 2, the total error in c, due to errors δa, δb, δC in the observed values of the two sides and the included angle, is to be found from

$$\delta(c^2) = \frac{\partial(c^2)}{\partial a}\,\delta a + \frac{\partial(c^2)}{\partial b}\,\delta b + \frac{\partial(c^2)}{\partial C}\,\delta C \qquad\qquad (1),$$

which gives

$$c\delta c = (a - b\cos C)\,\delta a + (b - a\cos C)\,\delta b + ab\sin C\delta C,$$

or

$$\delta c = \cos B\delta a + \cos A\delta b + a\sin B\delta C \qquad\qquad (2).$$

Ex. 2. If Δ be the area of a triangle, as determined from a measurement of two sides a, b, and the included angle C, we have

$$\Delta = \tfrac{1}{2}ab \sin C \quad\text{.......................}(3),$$

whence $$\log \Delta = \log \tfrac{1}{2} + \log a + \log b + \log \sin C \quad\text{.........}(4).$$

Hence, differentiating,

$$\frac{\delta\Delta}{\Delta} = \frac{\delta a}{a} + \frac{\delta b}{b} + \cot C \delta C \quad\text{..................}\ (5).$$

This gives the 'proportional error,' *i.e.* the ratio of the error ($\delta\Delta$) to the whole quantity (Δ) whose value is sought.

An important point brought out by the investigation of Art. 60 is that the small variations of a quantity due to independent causes are *superposed.* This follows from the linearity of the expression for δu in terms of δx, δy, δz, ...

Thus, in determining the weight of a body by the balance, the corrections for the buoyancy of the air, and for the inequality of the arms of the balance, may be calculated separately, and the (algebraic) sum of the results taken. The error involved in this process will be of the second order.

62. Differentiation of a Function of Functions, and of Implicit Functions.

Another important application of the formula (11) of Art. 60 is to the differentiation of a function of functions, and of implicit functions.

1°. Thus if $$u = \phi(x, y) \quad\text{.......................}(1),$$

where x, y are given functions of a variable t, we have, ultimately,

$$\frac{\delta u}{\delta t} = \frac{\partial \phi}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial \phi}{\partial y}\frac{\delta y}{\delta t} \quad\text{..................}\ (2),$$

or $$\frac{du}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} \quad\text{..................}\ (3).$$

This may be applied to reproduce various results obtained in Chap. II. To conform to previous notation we may write

$$y = \phi(u, v),$$

where u, v are given functions of x; the formula (3) then takes the shape

$$\frac{dy}{dx} = \frac{\partial\phi}{\partial u}\frac{du}{dx} + \frac{\partial\phi}{\partial v}\frac{dv}{dx} \quad \text{.................. (4).}$$

Thus, if
$$\phi(u, v) = uv \quad \text{..........................(5),}$$
we have
$$\partial\phi/\partial u = v, \quad \partial\phi/\partial v = u,$$
and therefore
$$\frac{d(uv)}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \quad \text{.....................(6),}$$
in agreement with Art. 37.

Again, if
$$\phi(u, v) = u^v \quad \text{...........................(7),}$$
we have
$$\partial\phi/\partial u = vu^{v-1}, \quad \partial\phi/\partial v = u^v . \log u,$$
by Art. 35. Hence
$$\frac{d}{dx}(u^v) = vu^{v-1}\frac{du}{dx} + u^v \log u \frac{dv}{dx} \quad \text{............ (8).}$$

2°. Again, if y be an implicit function of x, defined by the equation
$$\phi(x, y) = 0 \quad \text{........................(9),}$$
then differentiating this equation with respect to x, we have
$$\frac{\partial\phi}{\partial x}\frac{dx}{dx} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0,$$
or
$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \quad \text{.................. (10).}$$
This is an extension of a result given in Art. 46.

EXAMPLES. XVII.

1. Prove that in a table of logarithmic tangents to base 10 the difference for one minute in the neighbourhood of 60° will be ·00029, approximately.

2. The height h of a tower is deduced from an observation of the angular elevation (α) at a distance a from the foot; prove that the error due to an error $\delta\alpha$ in the observed elevation is
$$\delta h = a \sec^2 \alpha \delta\alpha.$$

If $a = 100$ feet, $\alpha = 30°$, and the error in the angle be $1'$, prove that $\delta h = {\cdot}47$ inch.

3. In a tangent galvanometer the tangent of the deflection of the needle is proportional to the current; prove that the proportional error in the inferred value of the current, due to a given error of reading, is least when the deflection is 45°.

4. The distances (x, x') of a point on the axis of a lens, and of its image, from the lens, are connected by the relation

$$\frac{1}{x} + \frac{1}{x'} = \frac{1}{f};$$

prove that the longitudinal magnification of a small object is $(x'/x)^2$.

5. Verify the theorem of Art. 56 in the case of

$$\phi(x) = x^3.$$

6. Prove that if $\phi(x)$ be continuous and differentiable, except for $x = x_1$, when it becomes infinite, then $\phi'(x_1)$ is also infinite.

7. The error in the area (S) of an ellipse due to small errors in the lengths of the semi-axes a, b is given by

$$\frac{\delta S}{S} = \frac{\delta a}{a} + \frac{\delta b}{b}.$$

8. If the three sides a, b, c of a triangle are measured, the error in the angle A, due to given small errors in the sides, is

$$\delta A = \frac{\sin A}{\sin B \sin C} \frac{\delta a}{a} - \cot C \frac{\delta b}{b} - \cot B \frac{\delta c}{c}.$$

9. If the area (Δ) of a triangle be computed from measurements of one side (a) and the adjacent angles (B, C), shew that the proportional error in the area, due to small errors in the measurements, is given by

$$\frac{\delta \Delta}{\Delta} = 2\frac{\delta a}{a} + \frac{c}{a}\frac{\delta B}{\sin B} + \frac{b}{a}\frac{\delta C}{\sin C}.$$

Also, verify this result geometrically.

10. If a triangle ABC be slightly varied, but so as to remain inscribed in the same circle, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$

11. If the density (s) of a body be inferred from its weights (W, W') in air and in water respectively, the proportional error due to errors δW, $\delta W'$ in these weighings is

$$\frac{\delta s}{s} = -\frac{W'}{W-W'} \cdot \frac{\delta W}{W} + \frac{\delta W'}{W-W'}.$$

12. A crank OP revolves about O with angular velocity ω, and a connecting rod PQ is hinged to it at P, whilst Q is constrained to move in a fixed groove OX. Prove that the velocity of Q is $\omega . OR$, where R is the point in which the line QP (produced if necessary) meets a perpendicular to OX drawn through O.

13. An open rectangular tank is to contain a given volume of water, find what must be its proportions in order that the cost of lining it with lead may be a minimum.

[The length and breadth must each be double the depth.]

14. Given the sum of three concurrent edges of a rectangular parallelepiped, find its form in order that the surface may be a maximum.

15. Prove that the parallelepiped of greatest volume which can be inscribed in a given sphere is a cube.

16. Prove that the rectangular parallelepiped of greatest volume for a given surface is a cube.

17. If a triangle of maximum area be inscribed in any closed oval curve the tangents at the vertices are respectively parallel to the opposite sides.

18. If a triangle of minimum area be circumscribed to a closed oval curve, the sides are bisected at the points of contact.

19. The triangle of maximum area inscribed in a given circle is equilateral; and the triangle of minimum area circumscribed to the circle is also equilateral.

20. A polygon of maximum area, and of a given number (n) of sides, inscribed in a given circle is regular; and a polygon of minimum area, of n sides, circumscribed to the circle is also regular.

21. Assuming that the rectangle of greatest area for a given perimeter is a square, explain how it follows immediately that the rectangle of least perimeter for a given area is a square.

What inferences can be drawn in like manner from the results of Examples 14 and 16, above?

22. The polygon of n sides, which has maximum area for a given perimeter, or minimum perimeter for a given area, is regular. (Assume the result of Example 23, p. 119.)

Hence shew that the figure of maximum area for a given perimeter, or of minimum perimeter for a given area, is a circle.

23. By the regulations of the parcel post, a parcel must not exceed six feet in length and girth combined; prove that the most voluminous parcel which can be sent is a cylinder 2 feet long and 4 feet in girth, and that its volume is 2·546 cubic feet.

CHAPTER IV.

DERIVATIVES OF HIGHER ORDERS.

63. Definition, and Notations.

If y be a function of x, the derived function dy/dx will in general be itself a differentiable function of x. The result of differentiating dy/dx is called the 'second differential coefficient,' or 'second derivative.' If this, again, admits of differentiation, the result is called the 'third differential coefficient,' or 'third derivative'; and so on.

If we look upon d/dx as a symbol of operation, the first, second, third, ... nth derivatives may be denoted by

$$\frac{d}{dx}\cdot y,\quad \left(\frac{d}{dx}\right)^2\cdot y,\quad \left(\frac{d}{dx}\right)^3\cdot y, \ldots \left(\frac{d}{dx}\right)^n\cdot y,$$

respectively. The more usual forms are

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3}, \ldots \frac{d^ny}{dx^n},$$

which may be regarded as contractions of the preceding, although (historically) they arose in a different manner.

Again, writing D for d/dx, as in Art. 31, we have the forms

$$Dy,\quad D^2y,\quad D^3y, \ldots D^ny.$$

If
$$y = \phi(x),$$
the successive derivatives are also denoted by

$$\phi'(x),\quad \phi''(x),\quad \phi'''(x), \ldots \phi^{(n)}(x).$$

Occasionally it is convenient to adopt the briefer notation

$$y',\quad y'',\quad y''', \ldots y^{(n)}.$$

There are a few cases in which simple expressions for the nth derivative of a function can be found. The more important of these are given in the following examples.

Ex. 1. If

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_mx^m, \quad \ldots\ldots(1),$$

we have

$$\left.\begin{array}{l} Dy = A_1 + 2A_2x + 3A_3x^2 + \ldots + mA_mx^{m-1}, \\ D^2y = 2.1A_2 + 3.2A_3 + \ldots + m(m-1)A_mx^{m-2}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^my = m(m-1)(m-2)\ldots 2.1A_m, \end{array}\right\}\ldots(2),$$

and therefore $D^{m+1}y = 0$, $D^{m+2}y = 0$, etc. ....(3).

Hence the mth derivative of a rational integral function of the mth degree is a constant, and all the higher derivatives vanish.

Ex. 2. If $y = e^{kx}$ ....(4),

we have $Dy = ke^{kx}$, $D^2y = k^2e^{kx}, \ldots,$

and, generally, $D^ny = k^ne^{kx}$ ....(5)

Hence, putting $k = \log a$, we have

$$D^na^x = (\log a)^n a^x \quad \ldots\ldots(6).$$

Ex. 3. If $y = \sin \beta x$ ....(7),

we have
$$\left.\begin{array}{ll} Dy = \beta\cos\beta x, & D^2y = -\beta^2\sin\beta x, \\ D^3y = -\beta^3\cos\beta x, & D^4y = \beta^4\sin\beta x, \end{array}\right\}\ldots\ldots(8),$$

and so on.

Otherwise, we have

$$Dy = \beta\sin(\beta x + \tfrac{1}{2}\pi),$$

and therefore $D^2y = \beta^2\sin(\beta x + \tfrac{1}{2}\pi + \tfrac{1}{2}\pi)$,

and, generally, $D^ny = \beta^n\sin(\beta x + \tfrac{1}{2}n\pi)$ ....(9).

Ex. 4. If $y = \cos\beta x$ ....(10),

we have
$$\left.\begin{array}{ll} Dy = -\beta\sin\beta x, & D^2y = -\beta^2\cos\beta x, \\ D^3y = \beta^3\sin\beta x, & D^4y = \beta^4\cos\beta x, \end{array}\right\}\ldots\ldots(11),$$

and so on.

Or, $Dy = \beta\cos(\beta x + \tfrac{1}{2}\pi)$,

whence $D^2y = \beta^2\cos(\beta x + \tfrac{1}{2}\pi + \tfrac{1}{2}\pi)$,

and, generally, $D^ny = \beta^n\cos(\beta x + \tfrac{1}{2}n\pi)$ ....(12).

Ex. 5. If
$$y = e^{\alpha x} \cos \beta x \text{ ....(13)},$$
we find
$$\left.\begin{aligned} Dy &= e^{\alpha x}(\alpha \cos \beta x - \beta \sin \beta x), \\ D^2 y &= e^{\alpha x}\{(\alpha^2 - \beta^2)\cos \beta x - 2\alpha\beta \sin \beta x\} \end{aligned}\right\} \text{ ....(14)}.$$

Similarly, if
$$y = e^{\alpha x} \sin \beta x \text{ ....(15)},$$
we have
$$\left.\begin{aligned} Dy &= e^{\alpha x}(\alpha \sin \beta x + \beta \cos \beta x) \\ D^2 y &= e^{\alpha x}\{(\alpha^2 - \beta^2)\sin \beta x + 2\alpha\beta \cos \beta x\} \end{aligned}\right\} \text{ ....(16)}.$$

General formulæ may be obtained, in these cases, by putting
$$\alpha = r\cos\theta, \qquad \beta = r\sin\theta \text{ ....(17)}.$$

This makes
$$\begin{aligned} D \,.\, e^{\alpha x}\cos\beta x &= e^{\alpha x}(\alpha\cos\beta x - \beta\sin\beta x) \\ &= re^{\alpha x}\cos(\beta x + \theta), \end{aligned}$$
and by repeated application of this result we find
$$D^n \,.\, e^{\alpha x}\cos\beta x = r^n e^{\alpha x}\cos(\beta x + n\theta) \text{ .. ..(18)}.$$

Similarly,
$$D^n \,.\, e^{\alpha x}\sin\beta x = r^n e^{\alpha x}\sin(\beta x + n\theta) \text{ ....(19)}.$$

Ex. 6. If
$$y = \log x \text{ ....(20)},$$
we have
$$Dy = x^{-1}, \qquad D^2 y = -x^{-2}, \qquad D^3 y = -1 . -2 . x^{-3}, \ldots,$$
and, generally,
$$\begin{aligned} D^n y &= -1 . -2 . -3 \ldots -(n-1)\, x^{-n} \\ &= (-)^{n-1}\frac{(n-1)!}{x^n} \text{ ....(21)}. \end{aligned}$$

64. Successive Derivatives of a Product. Leibnitz' Theorem.

If u, v be functions of x, we have by Art. 38 (20),
$$D(uv) = Du \,.\, v + u \,.\, Dv \text{ .... (1)}.$$

If we differentiate this again, we have
$$D^2(uv) = D(Du \,.\, v) + D(u \,.\, Dv).$$

Now, by the rule referred to, we have
$$\begin{aligned} D(Du \,.\, v) &= D^2 u \,.\, v + Du \,.\, Dv, \\ D(u \,.\, Dv) &= Du \,.\, Dv + u \,.\, D^2 v, \end{aligned}$$
whence
$$D^2(uv) = D^2 u \,.\, v + 2\, Du \,.\, Dv + u \,.\, D^2 v \text{.........(2)}.$$

The general formula for the nth derivative of a product is

$$D^n(uv) = D^n u \,.\, v + nD^{n-1}u \,.\, Dv + \frac{n(n-1)}{1 \,.\, 2} D^{n-2}u \,.\, D^2 v + \ldots$$
$$+ nDu \,.\, D^{n-1}v + u \,.\, D^n v \ldots\ldots(3),$$

the coefficients being the same as in the Binomial Theorem. This formula is due to Leibnitz.

To see the truth of (3), consider the process of formation of the first few derivatives of uv. Using the accent notation, we have

$$D(uv) = u'v + uv' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Differentiating this again,

$$\begin{aligned} D^2(uv) &= u''v + u'v' \\ &\qquad + u'v' + uv'' \\ &= u''v + 2u'v' + uv'' \ldots\ldots\ldots\ldots(5). \end{aligned}$$

The next differentiation gives

$$\begin{aligned} D^3(uv) &= u'''v + 2u''v' + u'v'' \\ &\qquad + u''v' + 2u'v'' + uv''' \ldots\ldots\ldots(6), \end{aligned}$$

where in the first line we have differentiated the first variable factor in each term of (5), and in the second line the second variable factor. The result is

$$D^3(uv) = u'''v + 3u''v' + 3u'v'' + uv''' \ldots\ldots\ldots\ldots (7)$$

It appears that the numerical coefficient of the rth term in (7) is the sum of the coefficients of the rth and $(r-1)$th terms in (5); and it is evident from the nature of the successive steps that this law will obtain for all the subsequent derivatives. Now this is precisely the law of formation of the coefficients in the expansions of the successive powers of $a+b$; and since the coefficients of $D(uv)$ are the same as those of the first power of $a+b$, it follows that the coefficients in the expanded form of $D^n(uv)$ will be the same as those of $(a+b)^n$.

Ex. 1. If

$$y = xu \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

we have

$$\begin{aligned} D^n y &= xD^n u + nDx \,.\, D^{n-1}u \\ &= xD^n u + nD^{n-1}u \ldots\ldots\ldots\ldots\ldots\ldots (9), \end{aligned}$$

since $D^2 x = 0$.

Thus if

$$y = x \sin \beta x \quad\ldots\ldots\ldots\ldots(10),$$

we have

$$D^2 y = x D^2 \sin \beta x + 2D \sin \beta x$$

$$= -\beta^2 x \sin \beta x + 2\beta \cos \beta x \quad\ldots\ldots\ldots\ldots(11).$$

Again, if

$$y = x \log x \quad\ldots\ldots\ldots\ldots(12),$$

we have

$$D^n y = x D^n \log x + n D^{n-1} \log x$$

$$= (-)^{n-1} \frac{(n-1)!}{x^{n-1}} + (-)^{n-2} \frac{n(n-2)!}{x^{n-1}},$$

by Art. 63 (20). Hence

$$D^n y = (-)^n \frac{(n-2)!}{x^{n-1}} \quad\ldots\ldots\ldots\ldots(13).$$

Ex. 2. If

$$y = e^{ax} u \quad\ldots\ldots\ldots\ldots(14),$$

we have

$$D^n y = e^{ax} . D^n u + n . De^{ax} . D^{n-1} u + \frac{n(n-1)}{1 . 2} D^2 e^{ax} . D^{n-2} u + \ldots$$

$$= e^{ax}\left(D^n u + naD^{n-1} u + \frac{n(n-1)}{1 . 2} a^2 D^{n-2} u + \ldots\right) \quad\ldots\ldots\ldots\ldots(15).$$

Thus, if

$$y = e^{ax} \sin \beta x \quad\ldots\ldots\ldots\ldots(16),$$

we have

$$D^2 y = e^{ax} \left(D^2 \sin \beta x + 2aD \sin \beta x + a^2 \sin \beta x\right)$$

$$= e^{ax}\{(a^2 - \beta^2) \sin \beta x + 2a\beta \cos \beta x\} \quad\ldots\ldots\ldots\ldots(17),$$

in agreement with Art. 63 (16).

65. Dynamical Illustrations.

The *second* derivative is especially predominant in the dynamical applications of the Calculus.

Thus, in the case of rectilinear motion, if s be the distance from a fixed origin, we have seen (Art. 33) that the velocity (v) and the acceleration (α) are given by the formulæ

$$v = \frac{ds}{dt}, \qquad \alpha = \frac{dv}{dt} \quad\ldots\ldots\ldots\ldots(1).$$

Hence, in the present notation, we have

$$\alpha = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2 s}{dt^2} \quad\ldots\ldots\ldots\ldots(2),$$

i.e. the second derivative of s (with respect to the time) measures the *acceleration.*

So also the angular acceleration of a body about a fixed axis is given, in the notation of Art. 33, by

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \text{ ....(3).}$$

Ex. 1. If s be a quadratic function of t, say

$$s = At^2 + Bt + C \text{ ....(4),}$$

we have

$$\frac{ds}{dt} = 2At + B,$$

$$\frac{d^2s}{dt^2} = 2A \text{ ....(5),}$$

i.e. the acceleration is constant.

Ex. 2. In 'simple-harmonic' motion we have

$$s = a \cos (nt + \epsilon) \text{ ....(6),}$$

whence

$$\frac{ds}{dt} = -na \sin (nt + \epsilon),$$

$$\frac{d^2s}{dt^2} = -n^2a \cos (nt + \epsilon) = -n^2s \text{ ....(7),}$$

i.e. the acceleration is directed always towards a fixed point (the origin of s) and varies as the distance from that point.

Ex. 3. If

$$s = A \cosh nt + B \sinh nt \text{ ....(8),}$$

we have

$$\frac{ds}{dt} = nA \sinh nt + nB \cosh nt,$$

$$\frac{d^2s}{dt^2} = n^2A \cosh nt + n^2B \sinh nt = n^2s \text{........(9),}$$

i.e. the acceleration is *from* a fixed point, and varies as the distance.

EXAMPLES. XVIII.

Verify the following differentiations:

1. $y = x^2 (1 - x)^2$, $D^2y = 2 - 12x + 12x^2$.
2. $y = \frac{1}{2}\mu x^2 (l - \frac{1}{3}x)$, $D^2y = \mu (l - x)$.
3. $y = \frac{1}{24}\mu x(l-x)(l^2+lx-x^2)$, $D^2y = \frac{1}{2}\mu x (x - l)$.
4. $y = \frac{1}{48}\mu x^2 (3l^2 - 4lx + 2x^2)$, $D^2y = \frac{1}{2}\mu (x - \frac{1}{2}l)^2$.

5. $y = \dfrac{1}{x^2 - x - 2}$, $\qquad D^2y = \frac{2}{3}\left\{\dfrac{1}{(x-2)^3} - \dfrac{1}{(x+1)^3}\right\}$.

6. $y = \dfrac{1}{1+x^2}$, $\qquad D^4y = \dfrac{24\,(1 - 10x^2 + 5x^4)}{(1+x^2)^5}$.

7. $y = \dfrac{2x}{1-x^2}$, $\qquad D^2y = \dfrac{4x\,(x^2+3)}{(1-x^2)^3}$.

8. $y = (1-x)^{-m}$, $\qquad D^ny = m(m+1)\ldots(m+n-1)\,(1-x)^{-m-n}$.

9. $y = \dfrac{1+x}{1-x}$, $\qquad D^ny = \dfrac{2\,.\,n!}{(1-x)^{n+1}}$.

10. $y = \sin^2 x$, $\qquad D^2y = 2\cos 2x$.

11. $y = \cos^2 x$, $\qquad D^ny = 2^{n-1}\cos\left(2x + \frac{1}{2}n\pi\right)$.

12. $y = \sec x$, $\qquad D^2y = 2\sec^3 x - \sec x$.

13. $y = x^2 \sin x$, $\qquad D^2y = 4x\cos x - (x^2-2)\sin x$.

14. $y = \sin^3 x \cos x$, $\qquad D^3y = 6 - 60\sin^2 x + 64\sin^4 x$.

15. $y = \sin x \sinh x$, $\qquad D^2y = 2\cos x \cosh x$.

16. $y = \cos x \cosh x$, $\qquad D^2y = -2\sin x \sinh x$.

17. $y = \sin x \cosh x$, $\qquad D^2y = 2\cos x \sinh x$.

18. $y = \cos x \sinh x$, $\qquad D^2y = -2\sin x \cosh x$.

19. $y = \sin^{-1} x$, $\qquad D^2y = \dfrac{x}{(1-x^2)^{\frac{3}{2}}}$.

20. The first five derivatives of $\tan x$ are

$$1+t^2, \quad 2t\,(1+t^2), \quad 2\,(1+3t^2)(1+t^2), \quad 8t\,(2+3t^2)(1+t^2),$$
$$8\,(2+15t^2+15t^4)(1+t^2),$$

where $t = \tan x$.

21. $y = x^3 \log \dfrac{x}{a}$, $\qquad D^4y = \dfrac{6}{x}$.

22. $y = x^2 \log x$, $\qquad D^ny = (-)^{n-1}\dfrac{2\,(n-3)\,!}{x^{n-2}}$.

23. $y = xe^x$, $\qquad D^ny = (x+n)\,e^x$.

24. $y = x^2e^x$, $\qquad D^ny = \{x^2 + 2nx + n\,(n-1)\}\,e^x$.

25. By applying Leibnitz' Theorem to the differentiation of the identity

$$x^m \cdot x^n = x^{m+n},$$

prove that

$$m_r + \frac{r}{1} m_{r-1} n_1 + \frac{r(r-1)}{1 . 2} m_{r-2} n_2 + \frac{r(r-1)(r-2)}{1 . 2 . 3} m_{r-3} n_3 + \dots + n_r = (m+n)_r,$$

where $m_r = m(m-1)(m-2) \dots (m-r+1)$.

26. The equation $$\frac{d^2s}{dt^2} + k\frac{ds}{dt} + n^2 s = 0$$

is satisfied by $$s = Ae^{-\frac{1}{2}kt} \cos(\sigma t + \epsilon),$$

for all values of A and ϵ, provided

$$\sigma^2 = n^2 - \tfrac{1}{4}k^2.$$

27. The equation $$\frac{d^2s}{dt^2} + 2n\frac{ds}{dt} + n^2 s = 0$$

is satisfied by $$s = (A + Bt)e^{-nt}.$$

28. If $$x = \phi(t), \qquad y = \chi(t),$$

prove that $$\frac{d^2y}{dx^2} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

66*. Geometrical Interpretations of the Second Derivative.

In Art. 56 an important property of the derived function was obtained by a process which consisted virtually in a comparison of the curve

$$y = \phi(x) \quad \dots\dots\dots\dots\dots\dots\dots\dots (1)$$

with a straight line $$y = A + Bx \dots\dots\dots\dots\dots\dots\dots\dots (2),$$

the constants A, B being determined so as to make (1) and (2) intersect for two given values of x.

We proceed, in a somewhat similar manner, to compare the curve (1) with a parabola

$$y = A + Bx + Cx^2 \quad \dots\dots\dots\dots\dots\dots (3),$$

---

* Arts. 66, 67 can be postponed.

where the constants A, B, C are determined so as to make (1) and (3) intersect for three given values of x.

1°. We will first suppose these values of x to be equidistant; let them be $a-h$, a, $a+h$. The equations to determine the constants are then

$$\left.\begin{aligned} A+B(a-h)+C(a-h)^2 &= \phi(a-h), \\ A+Ba \qquad\quad +Ca^2 \qquad &= \phi(a), \\ A+B(a+h)+C(a+h)^2 &= \phi(a+h) \end{aligned}\right\} \text{......} (4).$$

Let us now write

$$F(x)=\phi(x)-(A+Bx+Cx^2) \text{............} (5),$$

i.e. $F(x)$ denotes the difference of the ordinates of the curves (1) and (3). By hypothesis, $F(x)$ vanishes for $x=a-h$, and for $x=a$; hence, by Art. 48, the derived function $F'(x)$ will vanish for some intermediate value of x, that is

$$F'(a-\theta_1 h)=0 \text{.....................} (6),$$

where $1>\theta_1>0$. Again, since $F(x)$ vanishes for $x=a$, and for $x=a+h$, we shall have

$$F'(a+\theta_2 h)=0 \text{.....................} (7),$$

where $1>\theta_2>0$.

By a further application of the same argument, since the function $F'(x)$ vanishes for $x=a-\theta_1 h$ and for $x=a+\theta_2 h$, its derived function $F''(x)$ will vanish for some intermediate value of x; we have therefore

$$F''(a+\theta h)=0 \text{.....................} (8),$$

where θ is some quantity lying between $-\theta_1$ and θ_2, and *à fortiori* between ± 1. Since, by (5),

$$F''(x)=\phi''(x)-2C \text{..................} (9),$$

it follows that, for some value of θ between ± 1,

$$\phi''(a+\theta h)=2C \text{..................} (10).$$

Now from (4) we find

$$\phi(a+h)-2\phi(a)+\phi(a-h)=2Ch^2 \text{......} (11),$$

and therefore

$$\frac{\phi(a+h)-2\phi(a)+\phi(a-h)}{h^2}=\phi''(a+\theta h) \text{...} (12).$$

Hence

$$\lim_{h=0} \frac{\phi(a+h) - 2\phi(a) + \phi(a-h)}{h^2} = \phi''(a) \ldots (13).$$

In the same way we could prove that

$$\lim_{h=0} \frac{\phi(a+2h) - 2\phi(a+h) + \phi(a)}{h^2} = \phi''(a) \ldots\ldots (14).$$

If the difference

$$\phi(a+h) - \phi(a)$$

be denoted by δy, the expression

$$\{\phi(a+2h) - \phi(a+h)\} - \{\phi(a+h) - \phi(a)\}$$

may be denoted by $\delta(\delta y)$ or $\delta^2 y$. Hence the formula (14) is equivalent to

$$\lim_{\delta x=0} \frac{\delta^2 y}{(\delta x)^2} = \frac{d^2 y}{dx^2} \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

To interpret the theorem (13) geometrically, let, in Fig. 42,

$$OA = a, \quad OH = a - h, \quad OH' = a + h,$$

and let AQ, HP, $H'P'$ be the corresponding ordinates of the curve (1). Join PP', and let AQ meet PP' in V. Then

$$VA = \tfrac{1}{2}(P'H + PH)$$
$$= \tfrac{1}{2}\{\phi(a+h) + \phi(a-h)\},$$

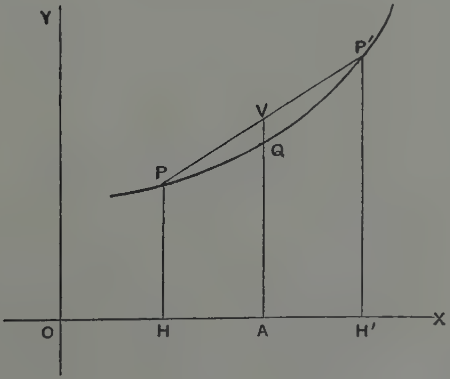

Fig. 42.

and therefore

$$VQ = VA - QA$$
$$= \tfrac{1}{2}\{\phi(a+h) - 2\phi(a) + \phi(a-h)\}.$$

Hence the theorem (13) asserts that

$$VQ = \tfrac{1}{2}HA^2 \,.\, \phi''(a) \qquad\qquad (16),$$

ultimately.

It appears that the chord is above or below the arc according as $\phi''(a)$ is positive or negative.

2°. We will next suppose that two of the three points at which the curves (1) and (3) intersect are coincident. More precisely, we suppose that for $x=a$ the curves not only intersect but touch, and that they intersect again for $x=a+h$. The conditions that, for $x=a$, y and dy/dx should have the same values in the two curves, are

$$\left.\begin{aligned} A + Ba + Ca^2 &= \phi(a), \\ B + 2Ca &= \phi'(a) \end{aligned}\right\} \qquad\qquad (17),$$

while the third condition gives

$$A + B(a+h) + C(a+h)^2 = \phi(a+h) \qquad\qquad (18).$$

With the same definition of $F(x)$ as before, we have

$$F(a) = 0, \quad F(a+h) = 0 \qquad\qquad (19),$$

and therefore
$$F'(a+\theta_1 h) = 0 \qquad\qquad (20),$$

where $1 > \theta_1 > 0$. Again, since $F'(x)$ vanishes for $x=a$, and for $x = a + \theta_1 h$, we have

$$F''(a+\theta h) = 0 \qquad\qquad (21),$$

where $\theta_1 > \theta > 0$.

Now from (17) and (18) we find

$$\phi(a+h) - \phi(a) - h\phi'(a) = Ch^2 \qquad\qquad (22).$$

Hence, by (9) and (21),

$$\phi(a+h) = \phi(a) + h\phi'(a) + \tfrac{1}{2}h^2\phi''(a+\theta h) \qquad (23).$$

This very important result will be recognised, later, as a particular case of Lagrange's form of Taylor's Theorem (see Chap. XIV). It includes as much of this theorem as is ordinarily required in the dynamical and physical applications of the subject.

From (23) we deduce

$$\lim_{h=0} \frac{\phi(a+h) - \phi(a) - h\phi'(a)}{h^2} = \tfrac{1}{2}\phi''(a) \quad .. \quad (24).$$

In Fig. 43, let $OA = a$, $AH = h$, and let AP, HQ be the corresponding ordinates of the curve (1). If QH meet the tangent at P in V, we have

$$QH = \phi(a+h), \quad VH = \phi(a) + h\phi'(a).$$

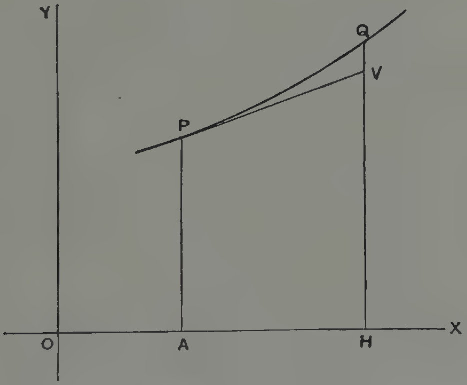

Fig. 43.

Hence (24) asserts that

$$QV = \tfrac{1}{2}AH^2 . \phi''(a) \ldots\ldots\ldots\ldots\ldots\ldots (25),$$

ultimately.

Hence, ultimately, the deviation of a curve from a tangent, in the neighbourhood of the point of contact, is in general a small quantity of the second order.

If $\phi''(a) \neq 0$, QV does not change sign with h, and the curve in the immediate neighbourhood of P lies altogether above, or altogether below, the tangent line, according as $\phi''(a)$ is positive or negative.

The formulæ (16) and (25) have an interesting application in the theory of Curvature. See Chap. X.

67. Theory of Proportional Parts.

Let us make the curves

$$y = \phi(x) \quad \text{........................(1)},$$

and

$$y = A + Bx + Cx^2 \quad \text{......................(2)},$$

intersect for $x = a, \quad x = a + zh, \quad x = a + h$,

where $1 > z > 0$.

We find

$$(1 - z)\,\phi(a) + z\phi(a + h) - \phi(a + zh) = z(1 - z)\,h^2C \text{...(3)};$$

and consequently, by the method of the preceding Art.,

$$(1 - z)\,\phi(a) + z\phi(a + h) - \phi(a + zh) = \tfrac{1}{2}z(1 - z)\,h^2\phi''(a + \theta h) \quad \text{......(4)},$$

where $1 > \theta > 0$.

This result, which includes the theorems of Art. 66 as particular cases, is here introduced for the sake of its bearing on the theory of 'proportional parts.' Suppose that $\phi(x)$ is a function which has been tabulated for a series of values of x at equal intervals h. Let a be one of these values, and suppose that $\phi(x)$ is required for some value of x between this and the next tabular value $a + h$; say for $a + zh$, where $1 > z > 0$. In the method of 'proportional parts,' the interpolation is made as if the function increased *uniformly* from $x = a$ to $x = a + h$, *i.e.* we assume

$$\frac{\phi(a + zh) - \phi(a)}{\phi(a + h) - \phi(a)} = \frac{z}{1} \quad \text{...............(5)},$$

or

$$\phi(a + zh) = (1 - z)\,\phi(a) + z\phi(a + h) \text{......(6)}.$$

The formula (4) gives the error involved in this process, which is equivalent to assuming that the arc of the curve (1) between $x = a$ and $x = a + h$ may be replaced without sensible error by its chord.

The maximum value of $z(1 - z)$ is $\frac{1}{4}$, by Art. 50, Ex. 2. Hence if R denote the greatest value which $\phi''(x)$ assumes in the interval from $x = a$ to $x = a + h$, the formula (4) shews that the error

$$\ngtr \tfrac{1}{8}h^2R \quad \text{........................(7)}.$$

Ex. 1. In a seven-figure logarithmic table, the logarithms of all numbers from 10000 to 100000 are given at intervals of unity. Now if

$$\phi(x) = \log_{10} x \quad \text{.......................(8)},$$

we have

$$\phi''(x) = -\frac{\mu}{x^2} \quad \text{.......................(9)}.$$

Hence, putting $h=1$, in (7), we find that in the interpolation between $\log_{10} n$ and $\log_{10}(n+1)$ the error involved in the method of proportional parts is not greater than

$$\cdot 05429 \div n^2 \quad \text{.......................(10)}.$$

Thus for $n = 10000$, where it is greatest, the error does not exceed

$$\cdot 00000000543,$$

and is therefore quite insensible from the stand-point of a seven-figure table.

It appears from (4) that the method may be expected to fail whenever $\phi''(x)$ is large. The differences are then said to be 'irregular.'

Ex. 2. If

$$\phi(x) = \log_{10} \sin x \quad \text{....................(11)},$$

we have

$$\phi''(x) = -\mu \operatorname{cosec}^2 x \quad \text{.................(12)}.$$

Hence, putting

$$h = \frac{\pi}{10800} = \cdot 000291,$$

we find

$$\tfrac{1}{8} h^2 \phi''(x) = -\cdot 0000000460 \operatorname{cosec}^2 x \quad \text{........(13)}.$$

Since $\operatorname{cosec}^2 18° = 10{\cdot}47$, it appears that in a table of log sines at intervals of 1′ the error of interpolation may amount to half a unit in the seventh place when the angle falls below 18°.

68. Concavity and Convexity. Points of Inflexion.

Just as $\phi'(x)$ measures (Art. 33) the rate of increase of $\phi(x)$, so $\phi''(x)$ measures the rate of increase of $\phi'(x)$. Hence if $\phi''(x)$ be positive the gradient of the curve

$$y = \phi(x) \quad \text{.......................(1)}$$

increases with x; whilst if $\phi''(x)$ be negative the gradient decreases as x increases.

If $\phi''(x) = 0$, the rate of change of the gradient is momentarily zero, and we have a 'stationary tangent.' The simplest case of this is at a 'point of inflexion,' *i.e.* a point at which the curve crosses its tangent; see p. 159.

A curve is said to be concave upwards at a point P when in the immediate neighbourhood of P it lies wholly above the tangent at P. Similarly, it is said to be convex upwards when in the immediate neighbourhood of P it lies wholly below the tangent at P.

If the curve, to the right of P, lie above the tangent at P, it is easily seen from Art. 56 that within any range (however short) extending to the right of P there will be points at which $\phi'(x)$ is greater than at P. Hence, by Art. 47, the value of $\phi''(x)$ at P cannot be negative. The same conclusion holds if the curve, to the left of P, lie above the tangent at P.

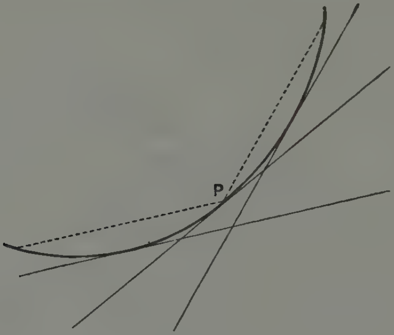

Fig. 44.

Similarly, if the curve, either to the right or left of P, lie below the tangent at P, the value of $\phi''(x)$ at P cannot be positive.

It follows that the curve is concave upwards when $\phi''(x)$ is positive, and convex upwards when $\phi''(x)$ is negative. This result may be inferred also from Art. 66 (25), which shews that QV has the same sign as $\phi''(a)$.

It appears, moreover, that at a point of inflexion, where the curve crosses its tangent, $\phi''(x)$ cannot be either positive or negative, and therefore (since it is assumed to be finite) must vanish. This condition, though essential, is not sufficient. It is further necessary that $\phi''(x)$ should change sign as x increases through the value in question. Suppose, for instance, that to the left of P the curve lies below the tangent at P, and that to the right of P it lies above it. It appears then from Art. 56 that there will be points of the curve both to the right and to the left, in the immediate neighbourhood of P, at which the gradient is greater than at P, *i.e.* the gradient is a minimum at P, and $\phi''(x)$ must therefore change (Art. 50) from negative to positive.

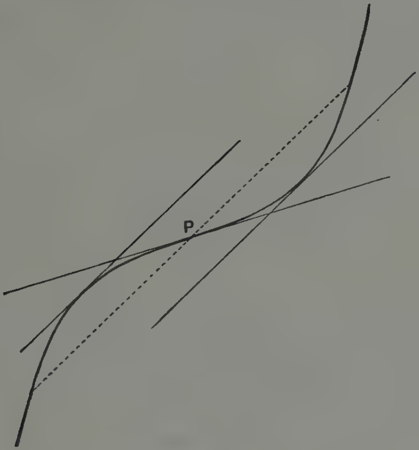

Fig. 45.

If the crossing is in the opposite direction, the gradient is a maximum at P, and $\phi''(x)$ changes from positive to negative.

Ex. 1. If $$y = x^3 \quad \text{..........................} (2),$$

we have $$y'' = 6x.$$

This changes from – to + as x increases through 0. Hence we have a point of inflexion; see Fig. 34, p. 110.

Ex. 2. $$y = \frac{2x}{1+x^2} \quad \text{..........................} (3).$$

This makes $$y'' = \frac{4x(x^2-3)}{(1+x^2)^3}.$$

Hence there are three points of inflexion, viz. when $x=0$ and when $x=\pm\sqrt{3}$. See Fig. 17, p. 31.

Ex. 3. In the curve of sines

$$y=b\sin\frac{x}{a}\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

we have
$$y''=-\frac{b}{a^2}\sin\frac{x}{a}=-\frac{y}{a^2}.$$

Hence y'' changes sign, and there is a point of inflexion, whenever the curve crosses the axis of x. See Fig. 18, p. 34.

Ex. 4. In the curve

$$y=x^4\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

we have
$$y''=12x^2.$$

This vanishes, but does not change sign, when $x=0$. Hence we have a stationary tangent, but not a point of inflexion in the strict sense. It is in fact obvious, since x^4 is essentially positive, that the curve lies wholly on one side of the tangent at the origin.

69. Application to Maxima and Minima.

The criterion of Art. 50 for distinguishing maxima and minima values of a function $\phi(x)$ can also be expressed in general in terms of the second derivative $\phi''(x)$.

Since $\phi''(x)$ is the derivative of $\phi'(x)$, it appears that if, as x increases through a root of $\phi'(x)=0$, $\phi''(x)$ is *positive*, $\phi'(x)$ must be increasing, and therefore changing sign from $-$ to $+$. Hence $\phi(x)$ is a *minimum*.

Similarly, if $\phi''(x)$ is *negative* when $\phi'(x)=0$, $\phi'(x)$ must be decreasing, and therefore changing sign from $+$ to $-$. Hence $\phi(x)$ is a *maximum*.

The connection of these results with the criterion of concavity and convexity (Art. 68) is obvious.

Ex. 1. In rectilinear motion, the distance (s) from the origin, is a maximum or minimum when the velocity (ds/dt) vanishes, according as the acceleration (d^2s/dt^2) is then negative or positive.

Ex. 2. Let
$$\phi(x)=\frac{2x}{1+x^2}.$$

We have seen, Art. 50, Ex. 4, that $\phi'(x)$ vanishes for $x=1$ and $x=-1$. Also from the value of $\phi''(x)$ given in Art. 68, Ex. 2, it appears that

$$\phi''(1)=-1, \quad \phi''(-1)=1.$$

Hence the former value of x gives a maximum, and the latter a minimum, value of $\phi(x)$. See Fig. 17, p. 31.

It may happen, however, that a value of x which makes $\phi'(x)=0$ also makes $\phi''(x)=0$. It is easily shewn that in this case $\phi(x)$ is in general neither a maximum nor a minimum (cf. Fig. 34, p. 110), but it is hardly worth while to continue the discussion here. The complete rule will be given later (Chap. XIV.) as a deduction from Taylor's Theorem.

70. Successive Derivatives in the Theory of Equations.

The successive derived functions play a great part in the Theory of Equations.

We have seen (Art. 49) that, if $\phi(x)$ be a rational integral function, at least one root of $\phi'(x)=0$ will occur between any two roots of $\phi(x)=0$. Similarly, at least one root of $\phi''(x)=0$ will occur between any two roots of $\phi'(x)=0$, and so on.

Moreover, since an r-fold root of $\phi(x)=0$ is an $(r-1)$-fold root of $\phi'(x)=0$, it will be an $(r-2)$-fold root of $\phi''(x)=0, \ldots$, and finally a simple root of $\phi^{(r-1)}(x)=0$. Hence the necessary and sufficient conditions for an r-fold root of $\phi(x)=0$ are that the functions

$$\phi(x), \quad \phi'(x), \quad \phi''(x), \quad \ldots, \quad \phi^{(r-1)}(x) \ldots\ldots\ldots(1)$$

should simultaneously vanish.

Ex. If $\phi(x)=2x^5+5x^4+4x^3+2x^2+2x+1$,

we have $\phi'(x)=10x^4+20x^3+12x^2+4x+2$,

$$\phi''(x)=4(10x^3+15x^2+6x+1).$$

These all vanish for $x=-1$, which is therefore a triple root of $\phi(x)=0$. We find, in fact, that

$$\phi(x)=(x+1)^3(2x^2-x+1).$$

EXAMPLES. XIX.

Prove that, in a table of natural sines at intervals of 1′, the error of proportional parts never exceeds

$$\cdot0000000106.$$

2. Shew that in a table of natural tangents the method of proportional parts fails for angles near 90°.

Also prove that the limit of error for angles near 45°, when the tangents are given at intervals of 1′, is

$$\cdot0000000423.$$

3. Shew that in a table of log tangents the method of proportional parts fails both for angles near 0° and for angles near 90°.

Shew also that the maximum error involved in the method is least for angles near 45°.

4. Prove that the curve

$$y = \log x$$

is everywhere convex upwards.

5. Prove that the curve

$$y = x \log x$$

is everywhere concave upwards. Trace the curve.

6. Find the maximum ordinate, and the point of inflexion, of the curve

$$y = xe^{-x}.$$

Trace the curve.

[The maximum ordinate corresponds to $x = 1$; the inflexion to $x = 2$.]

7. Shew that the curve $y = e^{-x^2}$ has inflexions at the points for which $x = \pm \dfrac{1}{\surd 2}$; and trace it.

8. Find the maximum and minimum ordinates, and the inflexions, of the curve

$$y = xe^{-x^2}.$$

Trace the curve.

[The maximum and minimum ordinates are given by $x = \pm \surd\frac{1}{2}$; the inflexions by $x = 0, \pm \surd\frac{3}{2}$.]

9. A certain function $\phi(x)$ is constant and

$$= \tfrac{1}{2}(b^2 - a^2)$$

for $0 < x < a$; it

$$= \tfrac{1}{2}b^2 - \tfrac{1}{6}x^2 - \tfrac{1}{3}\frac{a^3}{x}$$

for $a < x < b$; and

$$= \tfrac{1}{3}\frac{b^3 - a^3}{x}$$

for $x > b$. Prove that $\phi(x)$ and $\phi'(x)$ are continuous, but that $\phi''(x)$ is discontinuous.

Trace the curve $y = \phi(x)$.

10. Shew that

$$y = x^2(3 - x)$$

has an inflexion at the point (1, 2). Trace the curve.

11. Shew that

$$y = x^2(1 - x^2)$$

has inflexions at the points $\left(\pm\frac{1}{\sqrt{6}}, \frac{5}{36}\right)$. Trace the curve.

12. Find the points of inflexion of the curve

$$y = \frac{b^3}{a^2 + x^2}. \qquad [x = \pm a/\sqrt{3}.]$$

13. Shew that

$$y = \frac{a^2 x}{(x - a)^2}$$

has a point of inflexion at $(-2a, -\tfrac{2}{9}a)$. Trace the curve.

14. Find the points of inflexion of the curve

$$y = \frac{x^3}{a^2 + x^2},$$

and trace the curve. $[x = 0, \pm a\sqrt{3}.]$

15. Shew that the curve

$$y = \frac{1 - x}{1 + x^2},$$

has three points of inflexion, and that they lie in a straight line. Trace the curve.

16. Prove that the equation

$$x^5 - 10x^2 + 15x - 6 = 0$$

has a triple root.

17. Prove that the equation

$$x^6 - 5x^5 + 5x^4 + 9x^3 - 14x^2 - 4x + 8 = 0$$

has a triple root; and find all the roots.

18. If PN, $P'N'$ be two neighbouring ordinates of a curve $y = \phi(x)$, and if QH, any intermediate ordinate, meet the chord PP' in V, prove that

$$QV = \tfrac{1}{2} NH \, . \, HN' \, . \, \phi''(c),$$

ultimately, where c is the abscissa of some point between N and N'.

19. Shew that in the formula

$$\phi(a+h) = \phi(a) + h\phi'(a+\theta h)$$

of Art. 56, the limiting value of θ, when h is infinitely small, is in general $\frac{1}{2}$.

What is the geometrical meaning of this result?

20. Shew that the variation in the value of a function, in the neighbourhood of a maximum or minimum, is in general of the *second order* of small quantities.

21. Explain why the rate of a compensated chronometer, at any particular temperature, differs from the rate at the temperature of exact compensation by an amount proportional to the square of the difference of temperature.

22. Shew that, in a mathematical table calculated for equal intervals of the variable, the maximum error of interpolation by proportional parts, in any part of the table, is one eighth of the 'second difference' (*i.e.* of the difference of the differences of successive entries).

CHAPTER V.

INTEGRATION.

71. Nature of the Problem.

In the preceding chapters we have been occupied with the rate of variation of functions given *à priori.* The Integral Calculus, to which we now turn, is concerned with the inverse problem; viz. the rate of variation of a function being given, and the value of the function for some particular value of the independent variable being assigned, it is required to find the value of the function for any other assigned value of the independent variable. In symbols, it is required to solve the equation

$$\frac{dy}{dx} = \phi(x) \quad \text{........................(1)},$$

where $\phi(x)$ is a given function of x, subject to the condition that for some specified value (a, say) of x, y shall have a given value (b).

For example, the law of velocity of a moving point being given, and the position of the point at the time t_0, it is required to find its position at any other time t. This is equivalent to solving the equation

$$\frac{ds}{dt} = \phi(t) \quad \text{..........................(2)},$$

where $\phi(t)$ is a given function of t, subject to the condition that $s = s_0$ (say) for $t = t_0$.

If we can discover a continuous function $\psi(x)$ such that

$$\psi'(x) = \phi(x),$$

the equation (1) becomes

$$\frac{dy}{dx}=\frac{d}{dx}\psi(x) \quad\text{.....................(3).}$$

Hence if, as is the case in most practical applications of the subject, y be restricted to be continuous, we have, by Art. 56,

$$y=\psi(x)+C \quad\text{.....................(4),}$$

where C is a constant. The precise value of C is indeterminate, so far as the equation (1) is concerned; C is therefore called an 'arbitrary constant.' Its use is that it enables us to satisfy the remaining condition of the problem as above stated.

Thus if $y=b$ for $x=a$, we must have

$$b=\psi(a)+C,$$

whence

$$y-b=\psi(x)-\psi(a) \quad\text{..................(5).}$$

Ex. Given that the velocity of a moving point is $u+gt$, we have

$$\frac{ds}{dt}=u+gt=\frac{d}{dt}(ut+\tfrac{1}{2}gt^2) \quad\text{............ (6),}$$

whence

$$s=ut+\tfrac{1}{2}gt^2+C \quad\text{........................ (7).}$$

Determining C so that $s=s_0$ for $t=t_0$, we have

$$s-s_0=u(t-t_0)+\tfrac{1}{2}g(t^2-t_0^2) \quad\text{...............(8).}$$

If, as in Art. 31, we use the symbol D for the operator d/dx, the equation (1) may be written

$$Dy=\phi(x) \quad\text{.......................(9),}$$

and its solution may, consistently with the principles of algebraic notation, be written

$$y=D^{-1}\phi(x) \quad\text{.....................(10),}$$

the definition of the 'inverse' operator D^{-1} being that

$$D\{D^{-1}\phi(x)\}=\phi(x) \quad\text{.................(11).}$$

The function

$$D^{-1}\phi(x) \quad\text{.......................(12),}$$

when it exists, is called the 'indefinite integral' of $\phi(x)$ with respect to x. It is more usually denoted by

$$\int\phi(x)\,dx \quad\text{.........................(13).}$$

The origin of this notation will be explained in the next chapter; in the meantime (13) is to be regarded as merely another way of writing (12).

The distinction between 'direct' and 'inverse' operations is one that occurs in many branches of Mathematics. A direct operation is one which can always be performed on any given function, according to definite rules, with an unambiguous result. An inverse operation is of the nature of a question: what function, operated on in a certain way, will produce an assigned result? To this question there may or may not be an answer, or there may be more than one answer (cf. Art. 20). In the case of the operator D^{-1}, we have seen that if there is one answer, there are an infinite number, owing to the indeterminateness of the additive constant C. Whether there is, in every case, an answer is a matter yet to be investigated; but we may state, although this is rather more than we shall have occasion formally to prove, that every *continuous* function has an indefinite integral. In the rest of this chapter we shall be occupied with the problem of actually discovering indefinite integrals of various classes of mathematical functions.

72. Standard Forms.

There are no infallible rules by which we can ascertain the indefinite integral

$$D^{-1}\phi(x) \text{ or } \int \phi(x)\,dx$$

of any given continuous function $\phi(x)$. As above stated, integration is an inverse process, in which we can only be guided by our recollections of the results of previous direct processes.

The integral, moreover, although in a certain sense it always exists, may not admit of being expressed (in a finite form) in terms of the functions, whether algebraic or transcendental, which are ordinarily employed in mathematics. The following are instances:

$$\int e^{x^2}dx, \quad \int \frac{\sin x}{x}\,dx, \quad \int \frac{dx}{\sqrt{(1+x^3)}},$$

and the list might easily be extended indefinitely.

The first step towards making a more or less systematic record of achieved integrations is to write down a list of differentiations of various simple functions; each of these will, on inversion, furnish us with a result in indefinite integration. The arbitrary additive constant which always attaches to an indefinite integral need not be explicitly introduced, but its existence will occasionally be forced on the attention of the student by the fact of integrals of the same expression, arrived at in different ways, differing by a constant.

The student should make himself thoroughly familiar with the following results, which are fundamental:

$$\frac{d}{dx}\cdot x^n = nx^{n-1}, \qquad \int x^n\,dx = \frac{1}{n+1}x^{n+1}, \qquad (A)$$ ✓

$$[\text{except for } n=-1],$$

$$\frac{d}{dx}\cdot \log x = \frac{1}{x}, \qquad \int \frac{dx}{x} = \log x, \qquad (B)$$ ✓

$$\frac{d}{dx}\cdot e^{kx} = ke^{kx}, \qquad \int e^{kx}\,dx = \frac{1}{k}e^{kx}, \qquad (C)$$

$$\frac{d}{dx}\cdot \sin x = \cos x, \qquad \int \cos x\,dx = \sin x, \qquad (D)$$ ✓

$$\frac{d}{dx}\cdot \cos x = -\sin x, \qquad \int \sin x\,dx = -\cos x, \qquad (E)$$ ✓

$$\frac{d}{dx}\cdot \tan x = \sec^2 x, \qquad \int \sec^2 x\,dx = \tan x, \qquad (F)$$

$$\frac{d}{dx}\cdot \cot x = -\operatorname{cosec}^2 x, \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x, \qquad (G)$$

$$\frac{d}{dx}\cdot \sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{(a^2-x^2)}}, \qquad \int \frac{dx}{\sqrt{(a^2-x^2)}} = \sin^{-1}\frac{x}{a},^{*} \qquad (H)$$ ✓

$$\frac{d}{dx}\cdot \tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}, \qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}, \qquad (I)$$ ✓

$$\frac{d}{dx}\cdot \sinh x = \cosh x, \qquad \int \cosh x\,dx = \sinh x, \qquad (J)$$

$$\frac{d}{dx}\cdot \cosh x = \sinh x, \qquad \int \sinh x\,dx = \cosh x, \qquad (K)$$

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* As to the question of sign, see Art. 41.

$$\frac{d}{dx}\cdot\tanh x=\operatorname{sech}^2 x,\qquad \int \operatorname{sech}^2 x\,dx=\tanh x, \tag{L}$$ ✓

$$\frac{d}{dx}\cdot\coth x=-\operatorname{cosech}^2 x,\qquad \int \operatorname{cosech}^2 x\,dx=-\coth x, \tag{M}$$ ✓

$$\frac{d}{dx}\cdot\sinh^{-1}\frac{x}{a}=\frac{1}{\sqrt{(x^2+a^2)}},\qquad \int\frac{dx}{\sqrt{(x^2+a^2)}}=\sinh^{-1}\frac{x}{a}$$

$$=\log\frac{x+\sqrt{(x^2+a^2)}}{a},^{*} \tag{N}$$ ✓

$$\frac{d}{dx}\cdot\cosh^{-1}\frac{x}{a}=\frac{1}{\sqrt{(x^2-a^2)}},\qquad \int\frac{dx}{\sqrt{(x^2-a^2)}}=\cosh^{-1}\frac{x}{a},$$

$$=\log\frac{x+\sqrt{(x^2-a^2)}}{a},^{*} \tag{O}$$ ✓

$$\frac{d}{dx}\cdot\tanh^{-1}\frac{x}{a}=\frac{a}{a^2-x^2},\qquad \int\frac{dx}{a^2-x^2}=\frac{1}{a}\tanh^{-1}\frac{x}{a}$$

$$[x^2<a^2]\qquad =\frac{1}{2a}\log\frac{a+x}{a-x}, \tag{P}$$ ✓

$$\frac{d}{dx}\cdot\coth^{-1}\frac{x}{a}=-\frac{a}{x^2-a^2},\qquad \int\frac{dx}{x^2-a^2}=-\frac{1}{a}\coth^{-1}\frac{x}{a}$$

$$[x^2>a^2]\qquad =\frac{1}{2a}\log\frac{x-a}{x+a}. \tag{Q}$$ ✓

73. Simple Extensions.

To extend the above results, we first notice that the addition of a constant to x makes no essential difference in the form of the result (cf. Art. 39, 1°).

Thus, obviously,

$$\int (x+a)^m dx=\frac{1}{m+1}(x+a)^{m+1} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

$$\int\frac{dx}{x+a}=\log(x+a)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

$$\int\frac{dx}{\sqrt{(2ax-x^2)}}=\int\frac{dx}{\sqrt{\{a^2-(x-a)^2\}}}=\sin^{-1}\frac{x-a}{a}\quad\ldots\ldots(3),$$

and so on. Some further illustrations occur in Arts. 74, 75.

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* As to the sign, see Art. 44.

Again, if x be multiplied by a factor k, the integral has the same form as before, except that it is divided by this factor (see Art. 39, 2°).

Thus
$$\int \sin kx\,dx = -\frac{1}{k}\cos kx \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$
$$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b) \quad\ldots\ldots\ldots\ldots\ldots(5),$$
and so on.

Again, we have the theorems
$$\int Cu\,dx = C\int u\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$
$$\int (u+v+w+\ldots)\,dx = \int u\,dx + \int v\,dx + \int w\,dx + \ldots \ldots(7);$$
since, if we perform the operation d/dx on both sides we get in each case an identity, by Arts. 36, 37.

Thus the indefinite integral of a rational integral function
$$A_0x^m + A_1x^{m-1} + \ldots + A_{m-1}x + A_m \ldots\ldots\ldots(8),$$
is
$$\frac{1}{m+1}A_0x^{m+1} + \frac{1}{m}A_1x^m + \ldots + \tfrac{1}{2}A_{m-1}x^2 + A_mx \ldots\ldots(9).$$

Again, suppose we have a rational fraction of the form
$$\frac{F(x)}{x+a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

By division this can be reduced to the sum of a rational integral function and a fraction
$$\frac{A}{x+a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The former part can be integrated as above, and the integral of (11) is
$$A\log(x+a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Ex. 1. $\int(x-1)^{\frac{3}{2}}\,dx = \dfrac{1}{\frac{3}{2}+1}(x-1)^{\frac{3}{2}+1} = \frac{2}{5}(x-1)^{\frac{5}{2}}.$

Ex. 2. $\displaystyle\int\frac{dx}{2x-1} = \tfrac{1}{2}\log(2x-1).$

Ex. 3. $\int\sin^2 x\,dx = \frac{1}{2}\int(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x.$

Ex. 4. $\int\tan^2 x\,dx = \int(\sec^2 x - 1)\,dx = \tan x - x.$

Ex. 5.
$$\int \frac{x^3}{2x-1}\,dx = \int \left\{\tfrac{1}{2}x^2 + \tfrac{1}{4}x + \tfrac{1}{8} + \frac{1}{8(2x-1)}\right\} dx$$
$$= \tfrac{1}{6}x^3 + \tfrac{1}{8}x^2 + \tfrac{1}{8}x + \tfrac{1}{16}\log(2x-1).$$

EXAMPLES. XX.

Find the indefinite integrals of the following expressions*:

1. $\dfrac{1}{1-x}$, $\dfrac{1}{(1-x)^2}$.
2. $\dfrac{1}{2x-1}$, $\dfrac{1}{(2x-1)^2}$.
3. $(x-1)^3$, $\dfrac{1}{(x-1)^3}$.
4. $\sqrt{x}$, $\dfrac{1}{\sqrt{x}}$, $\dfrac{1+x}{\sqrt{x}}$.
5. $\dfrac{1}{\sqrt{(2+x)}}$, $\dfrac{1}{\sqrt{(3-2x)}}$.
6. $\dfrac{1+x}{x}$, $\dfrac{1+x}{x^2}$.
7. $\dfrac{1-2x}{3+x}$, $\dfrac{2+x}{3-x}$.
8. $\left(x+\dfrac{1}{x}\right)^2$, $\left(x+\dfrac{1}{x}\right)^3$.
9. $\dfrac{1+x}{1-x}$, $\dfrac{1-x}{1+x}$.
10. $\dfrac{x^2}{1+x}$, $\dfrac{x^2}{1-x}$.
11. $\dfrac{1-x^2}{1+x^2}$, $\dfrac{1+x^2}{1-x^2}$.
12. $\cos^2 x$, $\cot^2 x$.
13. $(\cos x - \sin x)^2$.
14. $\cosh^2 x$, $\sinh^2 x$.
15. $\tanh^2 x$, $\coth^2 x$.
16. $\dfrac{1-x^m}{1-x}$, $\dfrac{1+x^{2m+1}}{1+x}$.

74. Rational Fractions with a Quadratic Denominator.

We next shew how to integrate any expression of the form
$$\frac{F(x)}{x^2+px+q} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$
where $F(x)$ is rational and integral. If necessary, we first divide the numerator by the denominator until the remainder is of the form $ax+b$. We thus get the function (1) expressed as the sum of a rational integral function and a fraction
$$\frac{ax+b}{x^2+px+q} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

---

* The student should test the accuracy of his results by differentiation.

The former part can be integrated as in Art. 73; it remains only to consider the form (2).

We take first the case

$$\frac{1}{x^2+px+q} \quad \text{.......................} (3).$$

The form of the result will depend on whether $p^2 \lesseqgtr 4q$.

If $p^2 < 4q$, we have

$$x^2+px+q=(x+\tfrac{1}{2}p)^2+(q-\tfrac{1}{4}p^2)=(x-\alpha)^2+\beta^2,$$

where α, β are real. Now

$$\int\frac{dx}{(x-\alpha)^2+\beta^2}=\frac{1}{\beta}\tan^{-1}\frac{x-\alpha}{\beta} \quad \text{.............} (4),$$

by an obvious extension of Art. 72 (*I*).

If $p^2=4q$, we have

$$x^2+px+q=(x+\tfrac{1}{2}p)^2,$$

and

$$\int\frac{dx}{(x+\frac{1}{2}p)^2}=-\frac{1}{x+\frac{1}{2}p} \quad \text{.................} (5).$$

If $p^2>4q$, we have a choice of methods. In the first place, writing

$$x^2+px+q=(x+\tfrac{1}{2}p)^2-(\tfrac{1}{4}p^2-q)=(x-\alpha)^2-\beta^2,$$

where α, β are real, we have by Art. 72 (*Q*)

$$\int\frac{dx}{(x-\alpha)^2-\beta^2}=-\frac{1}{\beta}\coth^{-1}\frac{x-\alpha}{\beta}$$

$$=\frac{1}{2\beta}\log\frac{x-\alpha-\beta}{x-\alpha+\beta} \quad \text{...............} (7).^*$$

The more usual method of treating this case depends on the fact that when $p^2>4q$ the quadratic expression can be resolved into real factors; thus

$$x^2+px+q=(x-\alpha')(x-\beta'),$$

where

$$\alpha'=\alpha+\beta, \quad \beta'=\alpha-\beta.$$

With a proper choice of the constants A, B we may then put

$$\frac{1}{(x-\alpha')(x-\beta')}=\frac{A}{x-\alpha'}+\frac{B}{x-\beta'} \quad \text{..........} (8),$$

---

[*] It is assumed that $x>\alpha+\beta$. The modifications necessary in other cases may be easily supplied.

viz. this will be an identity, provided

$$1 = A(x-\beta') + B(x-\alpha') \text{ .... } (9),$$

i.e. provided

$$A + B = 0, \quad A\beta' + B\alpha' = -1 \text{ .... } (10),$$

or

$$A = \frac{1}{\alpha'-\beta'}, \qquad B = -\frac{1}{\alpha'-\beta'} \text{ .... } (11).$$

Hence

$$\int \frac{dx}{(x-\alpha')(x-\beta')} = \frac{1}{\alpha'-\beta'}\left\{\int \frac{dx}{x-\alpha'} - \int \frac{dx}{x-\beta'}\right\}$$

$$= \frac{1}{\alpha'-\beta'}\{\log(x-\alpha') - \log(x-\beta')\}$$

$$= \frac{1}{\alpha'-\beta} \log \frac{x-\alpha'}{x-\beta'} \text{ .... } (12),$$

which agrees with the last form in (7).

When we have once learned that the two sides of (8) can be made identical, the proper values of A, B are most easily found as follows. We first multiply both sides of the identity by $x-\alpha'$, and *afterwards* put $x=\alpha'$. This gives the value of A. Again, multiplying both sides by $x-\beta'$, and afterwards putting $x=\beta'$, we find B*.

Ex. 1. $$\int \frac{dx}{1-x+x^2} = \int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \frac{2}{\sqrt{3}} \tan^{-1} \frac{x-\frac{1}{2}}{\frac{1}{2}\sqrt{3}}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x-1}{\sqrt{3}}.$$

Ex. 2. $$\int \frac{dx}{1-4x+4x^2} = \int \frac{dx}{(2x-1)^2} = -\frac{1}{2}\,\frac{1}{2x-1}.$$

Ex. 3. $$\int \frac{dx}{2-x-x^2} = \int \frac{dx}{\frac{9}{4}-(x+\frac{1}{2})^2} = \frac{2}{3} \tanh^{-1} \frac{x+\frac{1}{2}}{\frac{3}{2}}.$$

$$= \frac{2}{3} \tanh^{-1} \frac{2x+1}{3}.$$

Otherwise, assuming

$$\frac{1}{(1-x)(x+2)} = \frac{A}{1-x} + \frac{B}{x+2},$$

---

* Hence the simple rule: To find A omit the corresponding factor in t denominator of the expression which is to be resolved into partial fraction and substitute α' for x in the expression as thus modified. Similarly for

we find, by the method just indicated,

$$A = \tfrac{1}{3}, \quad B = \tfrac{1}{3}.$$

Hence
$$\int \frac{dx}{2 - x - x^2} = -\tfrac{1}{3} \log (1 - x) + \tfrac{1}{3} \log (x + 2)$$
$$= \tfrac{1}{3} \log \frac{x + 2}{1 - x}.$$

Proceeding to the more general case (2), we observe that, by a proper choice of the constants λ, μ, we can make

$$ax + b = \lambda (2x + p) + \mu \quad \text{..............} (13),$$

viz. we must have

$$\lambda = \tfrac{1}{2} a, \quad \mu = b - \tfrac{1}{2} pa. \quad \text{..............} (14).$$

Hence

$$\int \frac{ax + b}{x^2 + px + q} dx = \lambda \int \frac{2x + p}{x^2 + px + q} dx + \mu \int \frac{dx}{x^2 + px + q} \quad \text{...} (15).$$

Of the two integrals on the right hand, the former is obviously equal to

$$\log (x^2 + px + q) \quad \text{..................} (16),$$

and the latter has been dealt with above.

Ex. 4.
$$\int \frac{1 - x}{1 - x + x^2} dx = \int \frac{\frac{1}{2} - \frac{1}{2}(2x - 1)}{1 - x + x^2} dx$$
$$= \tfrac{1}{2} \int \frac{dx}{1 - x + x^2} - \tfrac{1}{2} \int \frac{2x - 1}{1 - x + x^2} dx$$
$$= \frac{1}{\sqrt{3}} \tan^{-1} \frac{2x - 1}{\sqrt{3}} - \tfrac{1}{2} \log (1 - x + x^2).$$

When the denominator can be resolved into real factors the integral on the left-hand side of (15) can be treated more simply by the method of 'partial fractions.' Thus, we have

$$\frac{ax + b}{(x - \alpha')(x - \beta')} = \frac{A}{x - \alpha'} + \frac{B}{x - \beta'} \quad \text{........} (17),$$

ovided
$$ax + b = A (x - \beta') + B (x - \alpha'),$$

e. provided
$$A + B = a, \quad A\beta' + B\alpha' = -b \quad \text{..........} (18),$$

$$A = \frac{a\alpha' + b}{\alpha' - \beta'}, \qquad B = \frac{a\beta' + b}{\beta' - \alpha'} \quad \text{...........} (19).$$

It is unnecessary, however, to go through this work in every case, as the values of A, B can be found more simply by the artifice explained on p. 173.

The integration of (17) then gives

$$\int \frac{ax+b}{(x-\alpha')(x-\beta')}\,dx = A\log(x-\alpha') + B\log(x-\beta') \ldots (20).$$

Ex. 5. To integrate $\dfrac{2x-3}{(x-2)(x+1)}$.

Assuming that this $= \dfrac{A}{x-2} + \dfrac{B}{x+1}$,

we find $A = \frac{1}{3}$, $B = \frac{5}{3}$.

The required integral is therefore

$$\tfrac{1}{3}\log(x-2) + \tfrac{5}{3}\log(x+1).$$

EXAMPLES. XXI.

1. $\displaystyle\int \frac{1+x}{1+x^2}\,dx = \tan^{-1}x + \log\sqrt{(1+x^2)}$.

2. $\displaystyle\int \frac{x^3}{1+x^2}\,dx = \tfrac{1}{2}x^2 - \log\sqrt{(1+x^2)}$.

3. $\displaystyle\int \frac{x^3}{1-x^2}\,dx = -\tfrac{1}{2}x^2 - \log\sqrt{(1-x^2)}$.

4. $\displaystyle\int \frac{dx}{1-2x+2x^2} = \tan^{-1}(2x-1)$.

5. $\displaystyle\int \frac{dx}{1+3x+2x^2} = \log\frac{2x+1}{x+1}$.

6. $\displaystyle\int \frac{3x+7}{2x^2-3x+5}\,dx = \tfrac{3}{4}\log(2x^2-3x+5) + \frac{37}{2\sqrt{31}}\tan^{-1}\frac{4x-3}{\sqrt{31}}$.

7. $\displaystyle\int \frac{2x-1}{x^2+2x+3} = \log(x^2+2x+3) - \frac{3}{\sqrt{2}}\tan^{-1}\frac{x+1}{\sqrt{2}}$.

8. $\displaystyle\int \frac{x^2+x+1}{x^2-x+1}\,dx = x + \log(x^2-x+1) + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

9. $\displaystyle\int \frac{x+1}{(x-1)^2}\,dx = \log(x-1) - \frac{2}{x-1}$.

10. $\int \frac{x dx}{x^2 + 6x + 8} = \log \frac{(x+4)^2}{x+2}.$

11. $\int \frac{x^2 - 1}{(x-2)(x-3)} dx = x - 3 \log (x-2) + 8 \log (x-3).$

12. $\int \frac{x^2 + x + 1}{x^2 - x - 1} dx = x + (\frac{3}{5}\sqrt{5} + 1) \log (2x - \sqrt{5} - 1)$
$- (\frac{3}{5}\sqrt{5} - 1) \log (2x + \sqrt{5} - 1).$

13. $\int \frac{1 - x^{2m}}{1 - x^2} dx = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \dots + \frac{1}{2m-1} x^{2m-1}.$

14. $\int \frac{1 - (-)^m x^{2m}}{1 + x^2} dx = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \dots + (-)^{m-1} \frac{1}{2m-1} x^{2m-1}.$

75. **Form** $\frac{ax + b}{\sqrt{Ax^2 + Bx + C}}$.

A somewhat similar treatment can be applied to functions of the above type.

1°. If A be positive, the form is equivalent to

$$\frac{ax + b}{\sqrt{(x^2 + px + q)}} \quad\text{.......................(1).}$$

Consider, in the first place, the form

$$\frac{1}{\sqrt{(x^2 + px + q)}} \quad\text{....................... (2).}$$

By completing the square, the expression under the root-sign may be put in one or other of the shapes

$$(x - \alpha)^2 \pm \beta^2.$$

Now, by Art. 72, (N), (O),

$$\int \frac{dx}{\sqrt{\{(x-\alpha)^2 + \beta^2\}}} = \sinh^{-1} \frac{x - \alpha}{\beta} \quad\text{...........(3),}$$

and
$$\int \frac{dx}{\sqrt{\{(x-\alpha)^2 - \beta^2\}}} = \cosh^{-1} \frac{x - \alpha}{\beta} \quad\text{...........(4).}$$

These functions have the alternative forms,

$$\log \frac{x - \alpha + \sqrt{\{(x-\alpha)^2 \pm \beta^2\}}}{\beta},$$

or $$\log \frac{x-\alpha+\sqrt{(x^2+px+q)}}{\beta} \quad\text{.................}\quad (5);$$
cf. Art. 23.

In the more general case (1), we assume
$$ax+b=\lambda\,(x+\tfrac{1}{2}p)+\mu \quad\text{...............}\quad (6),$$
which is satisfied by
$$\lambda=a, \quad \mu=b-\tfrac{1}{2}pa \quad\text{.................}\quad (7).$$
Hence
$$\int\frac{ax+b}{\sqrt{(x^2+px+q)}}\,dx=\lambda\int\frac{x+\frac{1}{2}p}{\sqrt{(x^2+px+q)}}\,dx+\mu\int\frac{dx}{\sqrt{(x^2+px+q)}} \quad\text{.........}\quad (8).$$
The former of these two integrals is obviously equal to
$$\sqrt{(x^2+px+q)},$$
and the latter has been dealt with above.

2°. We will next suppose that, in the form placed at the head of this Art., the coefficient A is negative. Without loss of generality we may put it $=-1$.

Consider, first, the function
$$\frac{1}{\sqrt{(q+px-x^2)}} \quad\text{....................}\quad (9).$$
Unless the quadratic expression be essentially negative, in which case the function would be imaginary for *all* real values of x, it can be put in the shape
$$\beta^2-(x-\alpha)^2.$$
Now $$\int\frac{dx}{\sqrt{\{\beta^2-(x-\alpha)^2\}}}=\sin^{-1}\frac{x-\alpha}{\beta} \quad\text{..........}\quad (10).$$

In the more general case of the function
$$\frac{ax+b}{\sqrt{(q+px-x^2)}} \quad\text{..................}\quad (11),$$
we assume $$ax+b=\lambda\,(\tfrac{1}{2}p-x)+\mu \quad\text{..............}\quad (12),$$
or $$\lambda=-a, \quad \mu=b+\tfrac{1}{2}pa \quad\text{..............}\quad (13).$$

Hence

$$\int\frac{ax+b}{\sqrt{(q+px-x^2)}}\,dx=\lambda\int\frac{\frac{1}{2}p-x}{\sqrt{(q+px-x^2)}}\,dx+\mu\int\frac{dx}{\sqrt{(q+px-x^2)}} \quad\ldots\ldots\ldots (14).$$

The former of these two integrals is equal to

$$\sqrt{(q+px-x^2)},$$

and the latter has been treated above.

Ex. 1.

$$\begin{aligned}\int\frac{1+x}{\sqrt{(1-x-x^2)}}\,dx&=\int\frac{(x+\frac{1}{2})+\frac{1}{2}}{\sqrt{(1-x-x^2)}}\,dx\\&=\int\frac{x+\frac{1}{2}}{\sqrt{(1-x-x^2)}}\,dx+\frac{1}{2}\int\frac{dx}{\sqrt{\{\frac{5}{4}-(x+\frac{1}{2})^2\}}}\\&=-\sqrt{(1-x-x^2)}+\frac{1}{2}\sin^{-1}\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{5}}\\&=-\sqrt{(1-x-x^2)}+\frac{1}{2}\sin^{-1}\frac{2x+1}{\sqrt{5}}.\end{aligned}$$

Ex. 2

$$\begin{aligned}\int\frac{x+1}{\sqrt{(x^2+x+1)}}\,dx&=\int\frac{(x+\frac{1}{2})+\frac{1}{2}}{\sqrt{(x^2+x+1)}}\,dx\\&=\int\frac{x+\frac{1}{2}}{\sqrt{(x^2+x+1)}}\,dx+\frac{1}{2}\int\frac{dx}{\sqrt{\{(x+\frac{1}{2})^2+\frac{3}{4}\}}}\\&=\sqrt{(x^2+x+1)}+\frac{1}{2}\sinh^{-1}\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\\&=\sqrt{(x^2+x+1)}+\frac{1}{2}\sinh^{-1}\frac{2x+1}{\sqrt{3}}.\end{aligned}$$

Ex. 3.

$$\begin{aligned}\int\sqrt{\left(\frac{1-x}{1+x}\right)}\,dx&=\int\frac{1-x}{\sqrt{(1-x^2)}}\,dx\\&=\int\frac{dx}{\sqrt{(1-x^2)}}-\int\frac{x\,dx}{\sqrt{(1-x^2)}}\\&=\sin^{-1}x+\sqrt{(1-x^2)}.\end{aligned}$$

EXAMPLES. XXII.

1. $\int \frac{dx}{\sqrt{(2-3x^2)}} = \frac{1}{\sqrt{3}} \sin^{-1}(\sqrt{\tfrac{3}{2}}x)$.

2. $\int \frac{dx}{\sqrt{(2+3x^2)}} = \frac{1}{\sqrt{3}} \sinh^{-1}(\sqrt{\tfrac{3}{2}}x)$.

3. $\int \frac{dx}{\sqrt{\{x(1-x)\}}} = \sin^{-1}(2x-1)$.

4. $\int \frac{dx}{\sqrt{\{x(1+x)\}}} = \cosh^{-1}(2x+1)$

5. $\int \frac{dx}{\sqrt{\{x(x-1)\}}} = \cosh^{-1}(2x-1)$.

6. $\int \frac{dx}{\sqrt{(1+2x+3x^2)}} = \frac{1}{\sqrt{3}} \sinh^{-1} \frac{3x+1}{\sqrt{2}}$.

7. $\int \frac{dx}{\sqrt{(1+2x-3x^2)}} = \frac{1}{\sqrt{3}} \sin^{-1} \frac{3x-1}{2}$.

8. $\int \frac{dx}{\sqrt{(2ax-x^2)}} = \cos^{-1}\left(1-\frac{x}{a}\right)$.

9. $\int \sqrt{\left(\frac{a-x}{x}\right)}\, dx = \sqrt{\{x(a-x)\}} + \tfrac{1}{2}a \sin^{-1} \frac{2x-a}{a}$.

10. $\int \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx = \sin^{-1} x - \sqrt{(1-x^2)}$.

11. $\int \sqrt{\left(\frac{x+1}{x-1}\right)}\, dx = \sqrt{(x^2-1)} + \cosh^{-1} x$.

76. Change of Variable.

There are two artifices of special use in integration; viz. the choice of a new independent variable, and the method of integration 'by parts.'

To change the variable in the integral

$$u = \int \phi(x)\, dx \quad\text{.........................(1)}$$

from x to t, where x is a given function of t, we have, by Art. 39,

$$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = \phi(x)\frac{dx}{dt} \quad\text{.................. (2)},$$

and therefore, by the definition of the inverse symbol $\int$,

$$u = \int \phi(x) \frac{dx}{dt} dt.$$

Hence

$$\int \phi(x)\, dx = \int \phi(x) \frac{dx}{dt} dt \text{ .... } (3)^*.$$

Conversely, whenever a proposed integral is recognized to be of the form

$$\int \phi(u) \frac{du}{dx} dx \text{......................} (4),$$

we may replace it by

$$\int \phi(u)\, du \text{ .... } (5),$$

which is often easier to find.

The following are important cases:

1°. $$\int \phi(x+a)\, dx = \int \phi(u)\, du \text{ .... } (6),$$

where $u = x + a$.

2°. $$\int \phi(kx)\, dx = \frac{1}{k} \int \phi(u)\, du \text{..............} (7),$$

where $u = kx$.

These results have already been employed in Art. 73.

3°. $$\int \phi(x^2)\, x dx = \tfrac{1}{2} \int \phi(u)\, du \text{ .... } (8),$$

where $u = x^2$.

The following are examples of (8).

Ex. 1. $$\int \frac{dx}{x(1+x^2)} = \int \frac{x dx}{x^2(1+x^2)}$$
$$= \tfrac{1}{2} \int \frac{du}{u(1+u)} = \tfrac{1}{2} \int \left(\frac{1}{u} - \frac{1}{1+u}\right) du$$
$$= \tfrac{1}{2} \log \frac{u}{1+u} = \tfrac{1}{2} \log \frac{x^2}{1+x^2}$$
$$= \log \frac{x}{\sqrt{(1+x^2)}}.$$

---

* Hence the rule: After the sign $\int$ replace dx by $\frac{dx}{dt} dt$.

Ex. 2.
$$\int\frac{xdx}{x^4-1}=\tfrac{1}{2}\int\frac{du}{u^2-1}$$
$$=\tfrac{1}{4}\int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$
$$=\tfrac{1}{4}\log\frac{u-1}{u+1}$$
$$=\tfrac{1}{4}\log\frac{x^2-1}{x^2+1}.$$

Ex. 3.
$$\int\frac{xdx}{1+x^4}=\tfrac{1}{2}\int\frac{du}{1+u^2}=\tfrac{1}{2}\tan^{-1}u$$
$$=\tfrac{1}{2}\tan^{-1}x^2.$$

Ex. 4.
$$\int\frac{xdx}{\sqrt{(a^4-x^4)}}=\tfrac{1}{2}\int\frac{du}{\sqrt{(a^4-u^2)}}=\tfrac{1}{2}\sin^{-1}\frac{u}{a^2}$$
$$=\tfrac{1}{2}\sin^{-1}\frac{x^2}{a^2}.$$

The student will, after a little practice, find it easy to make such simple substitutions as the above mentally.

4°. Occasionally the integration of an algebraical function is facilitated by the substitution
$$x=1/t,\qquad dx/dt=-1/t^2.*$$

Thus
$$\int\frac{dx}{x\sqrt{(a^2+x^2)}}=-\int\frac{dt}{t\sqrt{(a^2+t^{-2})}}=-\frac{1}{a}\int\frac{dt}{\sqrt{(t^2+a^{-2})}}$$
$$=-\frac{1}{a}\sinh^{-1}at$$
$$=-\frac{1}{a}\sinh^{-1}\frac{a}{x}$$
$$=\frac{1}{a}\log\frac{x}{a+\sqrt{(a^2+x^2)}}\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Similarly,
$$\int\frac{dx}{x\sqrt{(a^2-x^2)}}=-\frac{1}{a}\cosh^{-1}\frac{a}{x}$$
$$=\frac{1}{a}\log\frac{x}{a+\sqrt{(a^2-x^2)}}\quad\ldots\ldots(10);$$

---

* The substitution is equivalent to writing
$$-\frac{dt}{t}\text{ for }\frac{dx}{x}.$$

and $$\int \frac{dx}{x\sqrt{(x^2-a^2)}} = -\frac{1}{a}\sin^{-1}\frac{a}{x} \quad \text{............ (11).}$$

More generally, the integral

$$\int \frac{dx}{(x+a)\sqrt{(Ax^2+Bx+C)}} \quad \text{............ (12)}$$

is reduced by the substitution

$$x + a = 1/t$$

to one or other of the forms discussed in Art. 75.

Again, the substitution $x = 1/t$ gives

$$\int \frac{dx}{(a^2+x^2)^{\frac{3}{2}}} = -\int \frac{dt}{t^2(a^2+t^{-2})^{\frac{3}{2}}} = -\int \frac{t\,dt}{(1+a^2t^2)^{\frac{3}{2}}}$$

$$= \frac{1}{a^2}\frac{1}{(1+a^2t^2)^{\frac{1}{2}}}$$

$$= \frac{1}{a^2}\frac{x}{\sqrt{(a^2+x^2)}} \quad \text{........................ (13).}$$

Similarly $$\int \frac{dx}{(a^2-x^2)^{\frac{3}{2}}} = \frac{1}{a^2}\frac{x}{\sqrt{(a^2-x^2)}} \quad \text{.............. (14),}$$

and $$\int \frac{dx}{(x^2-a^2)^{\frac{3}{2}}} = -\frac{1}{a^2}\frac{x}{\sqrt{(x^2-a^2)}} \quad \text{............ (15).}$$

The form $$\int \frac{dx}{(Ax^2+Bx+C)^{\frac{3}{2}}} \quad \text{..................... (16)}$$

can, by 'completing the square,' be brought under one or other of the preceding cases.

77. Integration of Trigonometrical Functions.

1°. $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

$$= -\int \frac{d(\cos x)}{\cos x} = -\log\cos x$$

$$= \log\sec x \quad \text{........................ (1).}$$

Similarly $$\int \cot x\,dx = \log\sin x \quad \text{.................(2).}$$

Again, by the same artifice,

$$\int \frac{\sin x}{\cos^2 x}\,dx = -\int \frac{d(\cos x)}{\cos^2 x}$$

$$= \frac{1}{\cos x} = \sec x \text{ .... } (3).$$

In a similar manner

$$\int \frac{\cos x}{\sin^2 x}\,dx = -\operatorname{cosec} x \text{ .... } (4).$$

Cf. Art. 38, 2°.

2°.
$$\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{1}{2}x\cos\frac{1}{2}x}$$

$$= \tfrac{1}{2}\int \frac{\sec^2\frac{1}{2}x\,dx}{\tan\frac{1}{2}x} = \int \frac{d(\tan\frac{1}{2}x)}{\tan\frac{1}{2}x}$$

$$= \log\tan\tfrac{1}{2}x \text{ .... } (5).$$

From this we deduce

$$\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(\frac{1}{2}\pi + x)} = \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}x) \text{ ...... } (6).$$

The formulæ (1) to (6) rank almost as standard results, and should be remembered.

3°.
$$\int \frac{dx}{a + b\cos x} = \int \frac{dx}{(a+b)\cos^2\frac{1}{2}x + (a-b)\sin^2\frac{1}{2}x}$$

$$= \int \frac{\sec^2\frac{1}{2}x\,dx}{(a+b) + (a-b)\tan^2\frac{1}{2}x} \text{ .... } (7).$$

If we put $\tan\frac{1}{2}x = u$, this takes the shape

$$2\int \frac{du}{(a+b) + (a-b)u^2} \text{ .... } (8),$$

and so comes under one or other of the standard forms (*I*), (*P*), (*Q*) of Art. 72.

Similarly, with the same substitution,

$$\int \frac{dx}{a + b\sin x} = 2\int \frac{du}{a + 2bu + au^2} \text{ .... } (9).$$

4°. $$\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x} = \int \frac{\sec^2 x\, dx}{a^2 + b^2 \tan^2 x} \text{......} (10).$$

If we put $$\tan x = u,$$

we get $$\int \frac{du}{a^2 + b^2 u^2} = \frac{1}{b} \int \frac{d\,(bu)}{a^2 + (bu)^2} = \frac{1}{ab} \tan^{-1} \frac{bu}{a}$$

$$= \frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \tan x \right) \text{..................} (11).$$

The analogous results involving hyperbolic functions may be noted. We easily find

$$\int \tanh x\, dx = \log \cosh x, \quad \int \coth x\, dx = \log \sinh x \text{ ...} (12),$$

$$\int \frac{\sinh x}{\cosh^2 x}\, dx = -\operatorname{sech} x, \quad \int \frac{\cosh x}{\sinh^2 x}\, dx = -\operatorname{cosech} x \text{...} (13),$$

$$\int \frac{dx}{\sinh x} = \log \tanh \tfrac{1}{2} x \text{ ....} (14),$$

$$\int \frac{dx}{\cosh x} = 2 \int \frac{e^x\, dx}{e^{2x} + 1} = 2 \tan^{-1} e^x \text{ ....} (15).$$

Similarly the forms

$$\int \frac{dx}{a + b \cosh x}, \quad \int \frac{dx}{a + b \sinh x}$$

can be integrated by the substitution $\tanh \frac{1}{2} x = u$.

78. Trigonometrical Substitutions.

The integration of an algebraic function involving the square root of a quadratic expression is often facilitated by the substitution of a trigonometrical or a hyperbolic function for the independent variable.

Thus: the occurrence of $\sqrt{(a^2 - x^2)}$ suggests the substitution $$x = a \sin \theta, \text{ or } x = a \tanh u;$$

that of $\sqrt{(x^2 - a^2)}$ suggests

$$x = a \sec \theta, \text{ or } x = a \cosh u;$$

that of $\sqrt{(x^2 + a^2)}$ suggests

$$x = a \tan \theta, \text{ or } x = a \sinh u.$$

Ex. 1. To find $\int \surd(a^2 - x^2)\,dx$ ……………………(1).

Putting $x = a \sin\theta,\quad dx = a\cos\theta\,d\theta,$

we find

$$\int \surd(a^2 - x^2)\,dx = a^2\int \cos^2\theta\,d\theta$$
$$= \tfrac{1}{2}a^2\int(1 + \cos 2\theta)\,d\theta$$
$$= \tfrac{1}{2}a^2(\theta + \tfrac{1}{2}\sin 2\theta)$$
$$= \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a} + \tfrac{1}{2}x\surd(a^2 - x^2) \;\ldots\ldots(2).$$

Ex. 2. To find $\int \frac{\surd(x^2 + a^2)\,dx}{x^2}$ ……………………(3).

Putting $x = a\sinh u,\quad dx = a\cosh u\,du,$

we obtain the form $\int \coth^2 u\,du,$

which

$$= \int(1 + \operatorname{cosech}^2 u)\,du = u - \coth u$$
$$= \sinh^{-1}\frac{x}{a} - \frac{\surd(x^2 + a^2)}{x} \;\ldots\ldots\ldots\ldots(4).$$

Ex. 3. To find $\int \frac{dx}{(1 - x)\surd(1 - x^2)}.$

If we put $x = \cos\theta,\quad dx = -\sin\theta\,d\theta,$

the integral becomes

$$-\int\frac{d\theta}{1 - \cos\theta} = -\tfrac{1}{2}\int\frac{d\theta}{\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta$$
$$= \sqrt{\left(\frac{1 + x}{1 - x}\right)}.$$

EXAMPLES. XXIII.

1. $\int\frac{x^2\,dx}{1 - x^3} = \frac{1}{3}\log\frac{1}{1 - x^3}.$

2. $\int\frac{x^2\,dx}{1 - x^6} = \frac{1}{6}\log\frac{1 + x^3}{1 - x^3},\qquad \int\frac{x^2\,dx}{1 + x^6} = \frac{1}{3}\tan^{-1}x^3.$

3. $\int\frac{\log x}{x}\,dx = \frac{1}{2}(\log x)^2.$ 4. $\int\sin x\cos x\,dx = \frac{1}{2}\sin^2 x$

5. $\int\frac{\sin^{-1}x}{\surd(1 - x^2)}\,dx = \frac{1}{2}(\sin^{-1}x)^2.$

6. $\int \frac{\cos x}{1+\sin x}\,dx = \log(1+\sin x)$,

$\int \frac{\sin x}{a+b\cos x}\,dx = -\frac{1}{b}\log(a+b\cos x)$.

7. $\int \sin x \cos^3 x\,dx = -\frac{1}{4}\cos^4 x$.

8. $\int \frac{\sin x\cos x}{a\cos^2 x + b\sin^2 x}\,dx = \frac{1}{2(b-a)}\log(a\cos^2 x + b\sin^2 x)$.

9. $\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \log\cos x$.

10. $\int \frac{\sin x}{\cos^5 x}\,dx = \frac{1}{4}\sec^4 x$. 11. $\int \sec^4 x\,dx = \tan x + \frac{1}{3}\tan^3 x$.

12. $\int (\sec x + \tan x)\,dx = \log\frac{1}{1-\sin x}$,

$\int (\sec x - \tan x)\,dx = \log(1+\sin x)$.

13. $\int \frac{dx}{1+\cos x} = \operatorname{cosec} x - \cot x$, $\int \frac{dx}{1-\cos x} = -\cot x - \operatorname{cosec} x$.

14. $\int \frac{dx}{1+\sin x} = \tan x - \sec x$, $\int \frac{dx}{1-\sin x} = \tan x + \sec x$.

15. $\int \frac{dx}{\sin^2 x\cos^2 x} = \tan x - \cot x$.

16. $\int \frac{dx}{\sin x\cos^2 x} = \sec x + \log\tan\frac{1}{2}x$.

17. $\int \frac{dx}{\sin x\cos^3 x} = \frac{1}{2}\sec^2 x + \log\tan x$.

18. $\int \frac{dx}{1+\tan x} = \frac{1}{2}x + \frac{1}{2}\log(\cos x + \sin x)$.

19. $\int \frac{dx}{1+\cos^2 x} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{1}{\sqrt{2}}\tan x\right)$.

20. $\int x\sqrt{(a^2+x^2)}\,dx = \frac{1}{3}(a^2+x^2)^{\frac{3}{2}}$.

21. $\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\log\frac{x^n}{x^n+1}$.

22. Evaluate $\int \sqrt{(x^2+a^2)}\,dx$ and $\int \sqrt{(x^2-a^2)}\,dx$ by hyperbolic substitutions.

23. $\int \frac{dx}{x^2\sqrt{(1+x^2)}} = -\frac{\sqrt{(1+x^2)}}{x}$.

24. $\displaystyle\int\frac{dx}{x^2\sqrt{(a^2-x^2)}}=-\frac{\sqrt{(a^2-x^2)}}{a^2x}.$

25. $\displaystyle\int\frac{x^3\,dx}{\sqrt{(a^2+x^2)}}=\tfrac{1}{3}(x^2-2a^2)\sqrt{(a^2+x^2)}.$

79. Integration by Parts.

The second method referred to in Art. 76, viz. that of 'integration by parts,' consists in an inversion of the formula

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}\quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

given in Art. 37. Integrating both sides, we find

$$uv=\int u\frac{dv}{dx}\,dx+\int v\frac{du}{dx}\,dx,$$

whence

$$\int u\frac{dv}{dx}\,dx=uv-\int v\frac{du}{dx}\,dx\quad\ldots\ldots\ldots\ldots\ldots(2).*$$

This gives the following rule:

If the expression to be integrated consists of two factors, one of which (dv/dx) is by itself immediately integrable, we may integrate as if the remaining factor (u) were constant, provided we subtract the integral of the product of the integrated factor (v) into the derivative (du/dx) of the other factor.

A very useful particular case is obtained by putting $v=x$, in (2). Thus

$$\int u\,dx=xu-\int x\frac{du}{dx}\,dx\quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

The following are important applications of the method.

1°. $$\int\log x\,dx=x\log x-\int x\,.\,\frac{1}{x}\,dx$$
$$=x\log x-x\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

2°. To find $\int\sqrt{(a^2-x^2)}\,dx.$

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* If we write v for dv/dx, and therefore $D^{-1}v$ for v, this takes the form
$$D^{-1}(uv)=uD^{-1}v-D^{-1}(Du\,.\,D^{-1}v).$$

Putting $u=\sqrt{(a^2-x^2)}$ in (3), we have

$$\int\sqrt{(a^2-x^2)}\,dx=x\sqrt{(a^2-x^2)}+\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}\ldots\ldots(5).$$

But
$$\int\sqrt{(a^2-x^2)}\,dx=\int\frac{a^2-x^2}{\sqrt{(a^2-x^2)}}\,dx$$
$$=a^2\int\frac{dx}{\sqrt{(a^2-x^2)}}-\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}$$
$$=a^2\sin^{-1}\frac{x}{a}-\int\frac{x^2dx}{\sqrt{(a^2-x^2)}}\ \ldots\ldots\ldots(6).$$

Adding to the former result, and dividing by 2, we find

$$\int\sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}x\sqrt{(a^2-x^2)}+\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}\ \ldots(7);$$

cf. Art. 78, Ex. 1.

In exactly the same way we should find

$$\int\sqrt{(a^2+x^2)}\,dx=\tfrac{1}{2}x\sqrt{(a^2+x^2)}+\tfrac{1}{2}a^2\sinh^{-1}\frac{x}{a}\ \ldots.(8),$$

$$\int\sqrt{(x^2-a^2)}\,dx=\tfrac{1}{2}x\sqrt{(x^2-a^2)}-\tfrac{1}{2}a^2\cosh^{-1}\frac{x}{a}\ldots.(9).$$

3°. To find the integrals

$$P=\int e^{\alpha x}\cos\beta x\,dx,\quad Q=\int e^{\alpha x}\sin\beta x\,dx\ldots..(10).$$

Putting
$$u=\cos\beta x,\quad v=\frac{1}{\alpha}e^{\alpha x}$$
in (2), we find

$$P=\frac{1}{\alpha}e^{\alpha x}\cos\beta x-\int\frac{1}{\alpha}e^{\alpha x}.(-\beta\sin\beta x)\,dx$$
$$=\frac{1}{\alpha}e^{\alpha x}\cos\beta x+\frac{\beta}{\alpha}Q\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Similarly
$$Q=\frac{1}{\alpha}e^{\alpha x}\sin\beta x-\int\frac{1}{\alpha}e^{\alpha x}.\beta\cos\beta x\,dx$$
$$=\frac{1}{\alpha}e^{\alpha x}\sin\beta x-\frac{\beta}{\alpha}P\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Hence
$$\left.\begin{aligned}\alpha P-\beta Q&=e^{\alpha x}\cos\beta x,\\ \beta P+\alpha Q&=e^{\alpha x}\sin\beta x\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(13),$$
and therefore
$$\left.\begin{aligned}\int e^{\alpha x}\cos\beta x\,dx=P&=\frac{\beta\sin\beta x+\alpha\cos\beta x}{\alpha^2+\beta^2}\,e^{\alpha x},\\ \int e^{\alpha x}\sin\beta x\,dx=Q&=\frac{\alpha\sin\beta x-\beta\cos\beta x}{\alpha^2+\beta^2}\,e^{\alpha x}\end{aligned}\right\}\ldots(14).$$

80. Integration by Successive Reduction.

Sometimes, by an integration 'by parts,' or otherwise, one integral can be made to depend on another of simpler form.

1°. Let
$$u_n=\int x^n e^{\alpha x}\,dx\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$
We have
$$u_n=\frac{1}{\alpha}e^{\alpha x}\,.\,x^n-\int\frac{1}{\alpha}e^{\alpha x}\,.\,nx^{n-1}dx$$
$$=\frac{1}{\alpha}x^ne^{\alpha x}-\frac{n}{\alpha}u_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$
If n be a positive integer, we can by successive applications of this formula obtain u_n in terms of
$$u_0,=\int e^{\alpha x}dx,=\frac{1}{\alpha}e^{\alpha x}\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 1. Thus, if
$$u_n=\int x^n e^{-x}\,dx\ldots\ldots\ldots\ldots\ldots\ldots(4),$$
we have
$$u_n=-x^ne^{-x}+nu_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots(5).$$
For example,
$$u_3=-x^3e^{-x}+3u_2=-x^3e^{-x}+3\,(-x^2e^{-x}+2u_1)$$
$$=-x^3e^{-x}-3x^2e^{-x}+6\,(-xe^{-x}+u_0),$$
or
$$\int x^3e^{-x}dx=-(x^3+3x^2+6x+6)\,e^{-x}.$$

2°. Let
$$\left.\begin{aligned}u_n&=\int x^n\cos\beta x\,dx,\\ v_n&=\int x^n\sin\beta x\,dx\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\ldots(6).$$
We find
$$u_n=\frac{1}{\beta}\sin\beta x\,.\,x^n-\int\frac{1}{\beta}\sin\beta x\,.\,nx^{n-1}dx$$
$$=\frac{1}{\beta}\sin\beta x\,.\,x^n-\frac{n}{\beta}v_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

and
$$v_n = -\frac{1}{\beta}\cos\beta x \,.\, x^n - \int\left(-\frac{1}{\beta}\cos\beta x\right).\, nx^{n-1}dx$$
$$= -\frac{1}{\beta}\cos\beta x \,.\, x^n + \frac{n}{\beta}u_{n-1} \quad\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

If n is a positive integer, these formulæ enable us to express u_n and v_n in terms of either u_0 or v_0, which are known.

Ex. 2. Thus, if $\beta = 1$, we have
$$u_n = x^n \sin x - nv_{n-1}, \quad v_n = -x^n \cos x + nu_{n-1} \quad\ldots\ldots (9).$$

For example,
$$u_3 = x^3 \sin x - 3v_2 = x^3 \sin x - 3(-x^2 \cos x + 2u_1)$$
$$= x^3 \sin x + 3x^2 \cos x - 6(x \sin x - v_0),$$
or
$$\int x^3 \cos x\, dx = (x^3 - 6x) \sin x + (3x^2 - 6) \cos x.$$

3°. If
$$u_n = \int \tan^n \theta\, d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$
$$= \int \tan^{n-2}\theta\,(\sec^2\theta - 1)\, d\theta$$
$$= \int \tan^{n-2}\theta\, d(\tan\theta) - \int \tan^{n-2}\theta\, d\theta,$$
we have
$$u_n = \frac{1}{n-1}\tan^{n-1}\theta - u_{n-2} \quad\ldots\ldots\ldots\ldots (11).$$

Hence if n be a positive integer, u_n can be made to depend

either on $\quad u_1, = \int \tan\theta\, d\theta, = \log\sec\theta \quad\ldots\ldots\ldots\ldots (12),$

or on $\quad u_0, = \int d\theta, = \theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$

according as n is odd or even.

Similarly, if
$$v_n = \int \cot^n \theta d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (14),$$
we find
$$v_n = -\frac{1}{n-1}\cot^{n-1}\theta - v_{n-2} \quad\ldots\ldots\ldots\ldots (15).$$

81. Reduction Formulæ, continued.

1°. Let
$$u_n = \int \cos^n \theta d\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

We have
$$u_n = \int \cos^{n-1}\theta\, d(\sin\theta).$$
$$= \sin\theta\cos^{n-1}\theta - \int\sin\theta.(n-1)\cos^{n-2}\theta.(-\sin\theta)d\theta$$
$$= \sin\theta\cos^{n-1}\theta + (n-1)\int(1-\cos^2\theta)\cos^{n-2}\theta d\theta$$
$$= \sin\theta\cos^{n-1}\theta + (n-1)(u_{n-2} - u_n).$$

Transposing, and dividing by n, we find

$$u_n = \frac{1}{n} \sin\theta \cos^{n-1}\theta + \frac{n-1}{n} u_{n-2} \text{ ....} (2).$$

By successive applications of this formula we reduce the index by 2 at each step; until finally, if n be a positive integer, the integral u_n is made to depend upon either

$$u_1, = \int \cos\theta\, d\theta, = \sin\theta \text{ ....} (3),$$

or on

$$u_0, = \int d\theta, = \theta \text{ ....} (4),$$

according as n is odd or even.

2°. By a similar process, if

$$v_n = \int \sin^n\theta\, d\theta \text{ ....} (5),$$

we find

$$v_n = -\frac{1}{n} \cos\theta \sin^{n-1}\theta + \frac{n-1}{n} v_{n-2} \text{ ....} (6).$$

In this way v_n, when n is a positive integer, is made to depend either on

$$v_1, = \int \sin\theta\, d\theta, = -\cos\theta \text{ ....} (7),$$

or on

$$v_0, = \int d\theta, = \theta \text{ ....} (8).$$

3°. The same method can be applied to the more general form

$$u_{m,n} = \int \sin^m\theta \cos^n\theta\, d\theta \text{ ....} (9).$$

We have

$$\begin{aligned} u_{m,n} &= \int \sin^m\theta \cos^{n-1}\theta\, d(\sin\theta) \\ &= \frac{1}{m+1} \sin^{m+1}\theta \cos^{n-1}\theta \\ &\qquad - \frac{1}{m+1} \int \sin^{m+1}\theta \,.\, (n-1) \cos^{n-2}\theta \,.\, (-\sin\theta)\, d\theta \\ &= \frac{1}{m+1} \sin^{m+1}\theta \cos^{n-1}\theta \\ &\qquad + \frac{n-1}{m+1} \int \sin^m\theta \cos^{n-2}\theta\, (1-\cos^2\theta)\, d\theta \\ &= \frac{1}{m+1} \sin^{m+1} \cos^{n-1}\theta + \frac{n-1}{m+1} (u_{m,n-2} - u_{m,n}). \end{aligned}$$

Clearing of fractions, transposing, and dividing by $m+n$, we obtain

$$u_{m,n} = \frac{1}{m+n}\sin^{m+1}\theta\cos^{n-1}\theta + \frac{n-1}{m+n}u_{m,\,n-2}\ldots(10).$$

In a similar manner we should find

$$u_{m,n} = -\frac{1}{m+n}\sin^{m-1}\theta\cos^{n+1}\theta + \frac{m-1}{m+n}u_{m-2,\,n}\ldots(11).$$

By successive applications of (10) and (11) we can reduce either index by 2 at each step, so that finally, if m, n are positive integers, the integral $u_{m,n}$ is made to depend on one or other of the following forms:

$$u_{1,1}, = \int \sin\theta\cos\theta\,d\theta, = \tfrac{1}{2}\sin^2\theta \ldots\ldots\ldots\ldots (12),$$

$$u_{0,0}, = \int d\theta, = \theta \ldots\ldots\ldots\ldots (13),$$

$$u_{0,1}, = \int \cos\theta\,d\theta, = \sin\theta \ldots\ldots\ldots\ldots (14),$$

$$u_{1,0}, = \int \sin\theta\,d\theta, = -\cos\theta \ldots\ldots\ldots\ldots (15).$$

The investigations of this section are important as leading to some simple and practically very useful results in definite integrals. See Art. 95.

EXAMPLES. XXIV.

1. $\int x e^{x/a}\,dx = a(x-a)\,e^{x/a}$.

2. $\int x\log x\,dx = \frac{1}{2}x^2\left(\log x - \frac{1}{2}\right)$.

3. $\int x^m \log x\,dx = \dfrac{x^{m+1}}{m+1}\left(\log x - \dfrac{1}{m+1}\right)$.

4. $\int x\sin x\,dx = -x\cos x + \sin x$.

5. $\int x\cos x\,dx = x\sin x + \cos x$.

6. $\int x\sin x\cos x\,dx = -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x$.

7. $\int \cos mx\cos nx\,dx = \dfrac{m\sin mx\cos nx - n\cos mx\sin nx}{m^2-n^2}$.

8. $\int \sin mx\sin nx\,dx = \dfrac{n\sin mx\cos nx - m\cos mx\sin nx}{m^2-n^2}$.

9. $\int \sin mx\cos nx\,dx = -\dfrac{m\cos mx\cos nx + n\sin mx\sin nx}{m^2-n^2}$.

10. $\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{(1-x^2)}$.

11. $\int \tan^{-1} x\,dx = x \tan^{-1} x - \log \sqrt{(1+x^2)}$.

12. $\int \sec^{-1} x\,dx = x \sec^{-1} x - \cosh^{-1} x$.

13. $\int x \tan^{-1} x\,dx = \frac{1}{2}(1+x^2)\tan^{-1} x - \frac{1}{2}x$.

14. $\int x \sec^2 x\,dx = x \tan x + \log \cos x$.

15. $\int \frac{x + \sin x}{1 + \cos x}\,dx = x \tan \frac{1}{2}x$.

16. $\int \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx = -\sqrt{(1-x^2)} \sin^{-1} x + x$.

17. $\int \cosh x \cos x\,dx = \frac{1}{2}(\sinh x \cos x + \cosh x \sin x)$.

18. $\int \sinh x \sin x\,dx = \frac{1}{2}(\cosh x \sin x - \sinh x \cos x)$.

19. $\int \cosh x \sin x\,dx = \frac{1}{2}(\sinh x \sin x - \cosh x \cos x)$.

20. $\int \sinh x \cos x\,dx = \frac{1}{2}(\cosh x \cos x + \sinh x \sin x)$.

21. $\int e^x \sin x \cos x\,dx = \frac{1}{10}(\sin 2x - 2 \cos 2x)\,e^x$.

22. $\int x^5 e^{-x}\,dx = -(x^5 + 5x^4 + 20x^3 + 60x^2 + 120x + 120)\,e^{-x}$.

23. $\int x^4 \sin x\,dx = -(x^4 - 12x^2 + 24)\cos x + (4x^3 - 24x)\sin x$.

24. If $\quad u_n = \int x^n \cosh x\,dx, \quad v_n = \int x^n \sinh x\,dx,$

prove that $\quad u_n = x^n \sinh x - n v_{n-1}, \quad v_n = x^n \cosh x - n u_{n-1}$.

Deduce the values of u_4 and v_4.

$$\left[u_4 = (x^4 + 12x^2 + 24)\sinh x - (4x^3 + 24x)\cosh x,\right.$$
$$\left.v_4 = (x^4 + 12x^2 + 24)\cosh x - (4x^3 + 24x)\sinh x.\right]$$

25. If u be a rational integral function of x, prove that

$$\int e^{ax} u\,dx = \frac{e^{ax}}{a}\left(1 - \frac{D}{a} + \frac{D^2}{a^2} - \dots\right)u,$$

where $D = d/dx$.

82. Integration of Rational Fractions.

We return to the integration of algebraic functions. There are certain classes of such functions which can be treated by general methods.

We begin with the case of *rational* functions. A rational fraction

$$\frac{F(x)}{f(x)} \quad \dots\dots\dots\dots\dots\dots\dots\dots(1),$$

in which the numerator is of lower dimensions than the denominator, is called a 'proper' fraction. Any rational fraction in which this condition is not fulfilled can (by division) be reduced to the sum of an integral function and a proper fraction; it will therefore be sufficient for us to consider the integration of proper fractions. Accordingly, if $f(x)$ be a polynomial of degree n, say

$$f(x) \equiv x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n \quad \ldots(2),$$

we shall suppose that $F(x)$ is at most of degree $n-1$.

To facilitate the integration, we resolve (1) into the sum of a series of 'partial fractions.' The possibility of this resolution depends on certain general theorems of Algebra, for the proof of which reference may be made to the special treatises on that subject*.

The student will find, however, that for such comparatively simple cases as are usually met with in practice, a mastery of the algebraical theory is not essential; since the results obtained by the rules to be given may be easily verified *à posteriori*.

We will first suppose that the roots of the equation

$$f(x) = 0 \quad \ldots\ldots\ldots\ldots(3)$$

are all real and distinct, say they are $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$. The polynomial $f(x)$ then resolves into n distinct factors of the first degree, thus

$$f(x) \equiv (x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n) \quad \ldots\ldots\ldots(4).$$

It may be shewn that in this case the fraction (1) can be resolved into the sum of n partial fractions whose denominators are the several factors of $f(x)$; thus

$$\frac{F(x)}{f(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \ldots + \frac{A_n}{x-\alpha_n} \quad \ldots\ldots\ldots(5),$$

where $A_1, A_2, \ldots, A_n$ are certain constants. If we clear (5) of fractions, and then equate coefficients of $x^{n-1}, x^{n-2}, \ldots, x^1, x^0$ on the two sides, we get n conditions to be fulfilled by the n constants. This constitutes in fact one way (not the easiest) of determining the constants†.

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* For the complete theory of the matter, see Chrystal, *Algebra*, c. viii.

† To complete the demonstration of the possibility of the resolution (5) it would be necessary to shew that the n equations are consistent. In the

We then have

$$\int \frac{F(x)}{f(x)}\,dx = A_1 \log(x-\alpha_1) + A_2 \log(x-\alpha_2) + \ldots + A_n \log(x-\alpha_n) \quad \ldots\ldots\ldots\ldots(6).$$

Ex. 1. To find $$\int \frac{x^5\,dx}{x^4-5x^2+4} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

We write

$$\frac{x^5}{x^4-5x^2+4} = x + \frac{5x^3-4x}{x^4-5x^2+4} = x + \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x-2} + \frac{D}{x+2} \quad \ldots\ldots(8).$$

If we clear of fractions and then equate coefficients, we get four linear equations to determine A, B, C, D. It is simpler, however, to use the artifice explained on p. 173. If we multiply the assumed identity by $x-1$, and afterwards put $x=1$, we obtain the value of A; and similarly for the other coefficients*. We thus find

$$\frac{x^5}{x^4-5x^2+4} = x - \frac{1}{6}\frac{1}{x-1} - \frac{1}{6}\frac{1}{x+1} + \frac{8}{3}\frac{1}{x-2} + \frac{8}{3}\frac{1}{x+2} \ldots (9),$$

a result which is easily verified. Hence the required integral is

$$\tfrac{1}{2}x^2 - \tfrac{1}{6}\log(x-1) - \tfrac{1}{6}\log(x+1) + \tfrac{8}{3}\log(x-2) + \tfrac{8}{3}\log(x+2) \quad \ldots\ldots(10).$$

Ex. 2. To find $$\int \frac{dx}{x^2(1+x^2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

Treating $1/x^2(1+x^2)$ as a function of x^2, we have

$$\frac{1}{x^2(1+x^2)} = \frac{1}{x^2} - \frac{1}{1+x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12),$$

whence $$\int \frac{dx}{x^2(1+x^2)} = -\frac{1}{x} - \tan^{-1} x \quad \ldots\ldots\ldots\ldots(13).$$

83. Case of Equal Roots.

If the roots of the equation $f(x)=0$ are real but not all distinct, then, corresponding to an r-fold root β, we have a factor $(x-\beta)^r$ in $f(x)$. It may be shewn that the

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present case it is sufficient to notice that by the method illustrated under Ex. 1 we can make the two functions (of degree $n-1$), which are to be identified, equal for n distinct values of x, and so ensure their equality for *all* values of x.

* See the footnote on p. 173.

corresponding series of partial fractions, in the expansion of Art. 82 (1), is now

$$\frac{B_1}{x-\beta}+\frac{B_2}{(x-\beta)^2}+\ldots+\frac{B_r}{(x-\beta)^r}\ldots\ldots\ldots\ldots(1),$$

where B_1, B_2, … B_r are r constants, to be determined by the method of equating coefficients, or otherwise.

The indefinite integral of the expression (1) is

$$B_1\log(x-\beta)-\frac{B_2}{x-\beta}-\frac{1}{2}\frac{B_3}{(x-\beta)^2}-\ldots-\frac{1}{r-1}\frac{B_r}{(x-\beta)^{r-1}}\ldots(2).$$

Ex. 1. To find $$\int\frac{dx}{x^2(1-x)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

We assume $$\frac{1}{x^2(1-x)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{1-x}\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

If we multiply both sides by $1-x$, and then put $x=1$, we get $C=1$. Again, multiplying by x^2, and then putting $x=0$, we find $B=1$. The constant A remains to be found in some other way. If we multiply both sides of (4) by x, and then put $x=\infty$, we find $A-C=0$, whence $A=1$. An equivalent method is to clear of fractions and equate the coefficients of x^2. Again, we might assign some other special value to x; for example, putting $x=-1$, we find

$$-A+B+\tfrac{1}{2}C=\tfrac{1}{2},$$

which, combined with the previous results, gives $A=1$.

Hence $$\int\frac{dx}{x^2(1-x)}=\int\left(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{1-x}\right)dx$$

$$=\log x-\frac{1}{x}-\log(1-x)\ldots\ldots\ldots\ldots(5).$$

Ex. 2. To find $$\int\frac{2x+1}{(x+2)(x-3)^2}\,dx\ \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Assume $$\frac{2x+1}{(x+2)(x-3)^2}=\frac{A}{x+2}+\frac{B}{x-3}+\frac{C}{(x-3)^2}\ \ldots(7).$$

The short method of determining coefficients gives

$$A=\frac{-4+1}{(-2-3)^2}=-\frac{3}{25},\quad C=\frac{6+1}{3+2}=\frac{7}{5}.$$

Also, multiplying by x, and then putting $x=\infty$, we find

$$A+B=0,\quad\text{or}\quad B=\tfrac{3}{25}.$$

The integral is therefore

$$-\tfrac{3}{25}\log(x+2)+\tfrac{3}{25}\log(x-3)-\frac{7}{5(x-3)}\quad\text{.........(8).}$$

Ex. 3. To find

$$\int\frac{x^3\,dx}{(x^2+1)^2}\quad\text{.........................(9).}$$

We recall Art. 76, 3°. Regarding $x^2/(x^2+1)^2$ as a function of x^2, we find (by inspection)

$$\frac{x^2}{(x^2+1)^2}=\frac{(x^2+1)-1}{(x^2+1)^2}=\frac{1}{x^2+1}-\frac{1}{(x^2+1)^2}.$$

Hence

$$\int\frac{x^3\,dx}{(x^2+1)^2}=\int\frac{x\,dx}{x^2+1}-\int\frac{x\,dx}{(x^2+1)^2}$$

$$=\tfrac{1}{2}\log(x^2+1)+\frac{1}{2(x^2+1)}\quad\text{......(10).}$$

84. Case of Quadratic Factors.

The preceding methods are always applicable, but if some of the roots of $f(x)=0$ are imaginary, the integral is obtained in the first instance in an imaginary form. If we wish to avoid altogether the consideration of imaginary expressions, we may proceed as follows.

It is known from the Theory of Equations that a polynomial $f(x)$ whose coefficients are all real can be resolved into real factors of the first and second degrees. Then, in the resolution of the function

$$\frac{F(x)}{f(x)}\quad\text{.........................(1),}$$

into partial fractions, we have

(*a*) for each simple factor $x-\alpha$ which does not recur, a fraction of the form

$$\frac{A}{x-\alpha}\quad\text{.........................(2);}$$

(*b*) for a simple factor $x-\beta$ which occurs r times, a series of r fractions, of the form

$$\frac{B_1}{x-\beta}+\frac{B_2}{(x-\beta)^2}+\ldots+\frac{B_r}{(x-\beta)^r}\quad\text{.........(3);}$$

(*c*) for each quadratic factor x^2+px+q which does not recur, a fraction of the form

$$\frac{Cx+D}{x^2+px+q} \quad \text{.....................(4);}$$

(*d*) for a quadratic factor x^2+px+q which occurs r times, a series of partial fractions, of the form

$$\frac{C_1x+D_1}{x^2+px+q}+\frac{C_2x+D_2}{(x^2+px+q)^2}+\ldots+\frac{C_rx+D_r}{(x^2+px+q)^r} \text{....(5).}$$

It is easily seen that in this way we have altogether just sufficient constants at our disposal to effect the identification of the function (1) with the complete system of partial fractions, by the method of equating coefficients.

*It only remains to shew how the indefinite integral of the partial fraction

$$\frac{C_sx+D_s}{(x^2+px+q)^s} \quad \text{..........................(6)}$$

can be found. The case $s=1$ has been treated in Art. 74, and the general case can be reduced to this by a formula of reduction.

In the first place, we can find λ, μ so that

$$\frac{C_sx+D_s}{(x^2+px+q)^s}=\lambda\frac{2x+p}{(x^2+px+q)^s}+\frac{\mu}{(x^2+px+q)^s} \quad \text{.........(7).}$$

viz. we have $\lambda=\frac{1}{2}C_s$, $\mu=D_s-\frac{1}{2}pC_s$..................(8).

The integral of the first term on the right-hand of (7) is

$$-\frac{\lambda}{s-1}\cdot\frac{1}{(x^2+px+q)^{s-1}} \quad \text{..................(9),}$$

and it remains only to find

$$\int\frac{dx}{(x^2+px+q)^s}, \text{ or } \int\frac{dt}{(t^2+c)^s} \quad \text{..............(10),}$$

where

$$t=x+\tfrac{1}{2}p, \quad c=q-\tfrac{1}{4}p^2 \quad \text{.................(11).}$$

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* The investigation which follows is given for the sake of completeness, but it is seldom required in practice. The student will lose little by postponing it. Another method of integrating expressions of the type (10) is indicated in Ex. 2, below.

Now, by differentiation, we find

$$\begin{aligned}\frac{d}{dt}\frac{t}{(t^2+c)^{s-1}} &= \frac{1}{(t^2+c)^{s-1}} - (2s-2)\frac{t^2}{(t^2+c)^s}\\ &= \frac{1}{(t^2+c)^{s-1}} - (2s-2)\frac{(t^2+c)-c}{(t^2+c)^s}\\ &= -(2s-3)\frac{1}{(t^2+c)^{s-1}} + (2s-2)\frac{c}{(t^2+c)^s} \quad \text{......(12).}\end{aligned}$$

Hence, integrating,

$$\frac{t}{(t^2+c)^{s-1}} = -(2s-3)\int\frac{dt}{(t^2+c)^{s-1}} + (2s-2)\,c\int\frac{dt}{(t^2+c)^s},$$

or

$$\int\frac{dt}{(t^2+c)^s} = \frac{1}{(2s-2)c}\,\frac{t}{(t^2+c)^{s-1}} + \frac{2s-3}{2s-2}\cdot\frac{1}{c}\int\frac{dt}{(t^2+c)^{s-1}} \quad \text{......(13).}$$

Returning to our previous notation, we have

$$\begin{aligned}\int\frac{dx}{(x^2+px+q)^s} &= \frac{1}{2(s-1)(q-\frac{1}{4}p^2)}\,\frac{x+\frac{1}{2}p}{(x^2+px+q)^{s-1}}\\ &\quad + \frac{2s-3}{2(s-1)(q-\frac{1}{4}p^2)}\int\frac{dx}{(x^2+px+q)^{s-1}} \text{...(14),}\end{aligned}$$

which is the formula of reduction required. By successive applications of this result, the integral (10) is made to depend ultimately on

$$\int\frac{dx}{x^2+px+q} \text{ ....(15),}$$

which is a known form (Art. 74).

Ex. 1. To find

$$\int\frac{dx}{x^4+x^2+1} \text{ ....(16).}$$

The denominator has here two quadratic factors, x^2+x+1 and x^2-x+1, which are not further resolvable. We therefore assume, in conformity with the above rule,

$$\frac{1}{x^4+x^2+1} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{x^2-x+1} \text{ ....(17),}$$

or

$$1 = (Ax+B)(x^2-x+1) + (Cx+D)(x^2+x+1).$$

Equating coefficients of the several powers of x, we have

$$A+C=0, \quad -A+C+B+D=0,$$
$$A+C-B+D=0, \qquad B+D=1.$$

Hence $$A = -C = \tfrac{1}{2}, \quad B = D = \tfrac{1}{2} \quad \ldots\ldots\ldots\ldots\ldots(18).$$

The integration can now be effected by the method of Art. 74. We have

$$\begin{aligned}
\int \frac{dx}{x^4 + x^2 + 1} &= \tfrac{1}{2} \int \frac{x+1}{x^2 + x + 1}\, dx - \tfrac{1}{2} \int \frac{x-1}{x^2 - x + 1}\, dx \\
&= \tfrac{1}{4} \int \frac{(2x+1)+1}{x^2+x+1}\, dx - \tfrac{1}{4} \int \frac{(2x-1)-1}{x^2-x+1}\, dx \\
&= \tfrac{1}{4} \log (x^2+x+1) - \tfrac{1}{4} \log (x^2 - x + 1) \\
&\quad + \tfrac{1}{4} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} + \tfrac{1}{4} \int \frac{dx}{(x-\frac{1}{2})^2 + \frac{3}{4}} \\
&= \tfrac{1}{4} \log \frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}} \left(\tan^{-1} \frac{2x+1}{\sqrt{3}} + \tan^{-1} \frac{2x-1}{\sqrt{3}} \right) \\
&= \tfrac{1}{4} \log \frac{x^2+x+1}{x^2-x+1} + \frac{1}{2\sqrt{3}} \tan^{-1} \frac{\sqrt{3}x}{1-x^2} \quad \ldots\ldots\ldots\ldots\ldots(19).
\end{aligned}$$

Ex. 2. To find $$\int \frac{dx}{(1+x^2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(20).$$

This comes under (14), but may be treated more simply as follows. If we put

$$x = \tan\theta,$$

we get

$$\begin{aligned}
\int \frac{dx}{(1+x^2)^2} &= \int \cos^2\theta\, d\theta \\
&= \tfrac{1}{2} \int (1 + \cos 2\theta)\, d\theta \\
&= \tfrac{1}{2}\theta + \tfrac{1}{4} \sin 2\theta \\
&= \tfrac{1}{2} \tan^{-1} x + \tfrac{1}{2} \frac{x}{1+x^2} \quad \ldots\ldots\ldots\ldots(21).
\end{aligned}$$

EXAMPLES. XXV.

1. $$\int \frac{dx}{x(1-x^2)} = \log \frac{x}{\sqrt{(1-x^2)}}.$$

2. $$\int \frac{2x+3}{x(x-1)(x+2)}\, dx = \log \frac{(x-1)^{\frac{5}{3}}}{x^{\frac{3}{2}} (x+2)^{\frac{1}{6}}}.$$

3. $$\int \frac{x^2\, dx}{(x-1)(x-2)(x-3)}$$
$$= \tfrac{1}{2} \log (x-1) - 4 \log (x-2) + \tfrac{9}{2} \log (x-3).$$

4. $$\int \frac{2x-3}{(x^2-1)(2x+3)}\,dx$$
$$= \tfrac{5}{2}\log(x+1) - \tfrac{1}{10}\log(x-1) - \tfrac{12}{5}\log(2x+3).$$

5. $$\int \frac{x\,dx}{x^4-x^2-2} = \tfrac{1}{6}\log\frac{x^2-2}{x^2+1}.$$

6. $$\int \frac{dx}{x^3+3x^2-4} = \frac{1}{3(x+2)} + \tfrac{1}{9}\log\frac{x-1}{x+2}.$$

7. $$\int \frac{x^2\,dx}{(x-a)(x-b)(x-c)} = \frac{a^2}{(a-b)(a-c)}\log(x-a)$$
$$+ \frac{b^2}{(b-c)(b-a)}\log(x-b) + \frac{c^2}{(c-a)(c-b)}\log(x-c).$$

8. $$\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{b^2-a^2}\left(\frac{1}{a}\tan^{-1}\frac{x}{a} - \frac{1}{b}\tan^{-1}\frac{x}{b}\right).$$

9. $$\int \frac{x\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2(b^2-a^2)}\log\frac{x^2+a^2}{x^2+b^2}.$$

10. $$\int \frac{x^2\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left(a\tan^{-1}\frac{x}{a} - b\tan^{-1}\frac{x}{b}\right).$$

11. $$\int \frac{x^3\,dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{2(a^2-b^2)}\{a^2\log(x^2+a^2) - b^2\log(x^2+b^2)\}.$$

12. $$\int \frac{x\,dx}{(x+1)(x+2)^2} = -\frac{2}{x+2} + \log\frac{x+2}{x+1}.$$

13. $$\int \frac{x^2\,dx}{(x+1)(x+2)^2} = \frac{4}{x+2} + \log(x+1).$$

14. $$\int \frac{dx}{x^3-x^2-x+1} = \tfrac{1}{4}\log\frac{x+1}{x-1} - \frac{1}{2}\frac{1}{x-1}.$$

15. $$\int \frac{dx}{(x^2-1)^2} = -\frac{1}{2}\frac{x}{x^2-1} + \tfrac{1}{4}\log\frac{x+1}{x-1}.$$

16. $$\int \frac{dx}{x^2(1-x)^2} = \frac{2x-1}{x(1-x)} + 2\log\frac{x}{1-x}.$$

17. $$\int \frac{x^2\,dx}{(x^2-1)^2} = -\frac{1}{2}\frac{x}{x^2-1} - \tfrac{1}{4}\log\frac{x+1}{x-1}.$$

18. $$\int \frac{dx}{x^2(x^2-1)^2} = -\tfrac{3}{4}\log\frac{x-1}{x+1} - \frac{1}{x} - \frac{1}{2}\frac{x}{x^2-1}.$$

19. $\displaystyle\int \frac{dx}{x^3(x+1)} = -\frac{1}{2x^2} + \frac{1}{x} - \log(x+1) + \log x.$

20. $\displaystyle\int \frac{3x+2}{x(x+1)^3}\,dx = 2\log\frac{x}{x+1} + \frac{2}{x+1} - \frac{1}{2(x+1)^2}.$

21. $\displaystyle\int \frac{dx}{(x^2-1)^3} = \tfrac{3}{16}\log\frac{x-1}{x+1} + \frac{3}{8}\frac{x}{x^2-1} - \frac{1}{4}\frac{x}{(x^2-1)^2}.$

22. $\displaystyle\int \frac{1+x}{x(1+x^2)}\,dx = \log\frac{x}{\sqrt{(1+x^2)}} + \tan^{-1}x.$

23. $\displaystyle\int \frac{dx}{1-x^4} = \tfrac{1}{4}\log\frac{1+x}{1-x} + \tfrac{1}{2}\tan^{-1}x.$

24. $\displaystyle\int \frac{x^2\,dx}{1-x^4} = \tfrac{1}{4}\log\frac{1+x}{1-x} - \tfrac{1}{2}\tan^{-1}x.$

25. $\displaystyle\int \frac{dx}{1+x^3} = \tfrac{1}{6}\log\frac{(1+x)^2}{1-x+x^2} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$

26. $\displaystyle\int \frac{x\,dx}{1+x^3} = \tfrac{1}{6}\log\frac{1-x+x^2}{(1+x)^2} + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}.$

27. $\displaystyle\int \frac{dx}{(1+x)(1+x^2)} = \tfrac{1}{2}\log(1+x) - \tfrac{1}{4}\log(1+x^2) + \tfrac{1}{2}\tan^{-1}x.$

28. $\displaystyle\int \frac{x^2\,dx}{(1+x)(1+x^2)} = \tfrac{1}{2}\log(1+x) + \tfrac{1}{4}\log(1+x^2) - \tfrac{1}{2}\tan^{-1}x.$

29. $\displaystyle\int \frac{x^2\,dx}{x^4+x^2-2} = \tfrac{1}{6}\log\frac{x-1}{x+1} + \frac{\sqrt{2}}{3}\tan^{-1}\frac{x}{\sqrt{2}}.$

30. $\displaystyle\int \frac{x^3\,dx}{x^4+3x^2+2} = \log(x^2+2) - \tfrac{1}{2}\log(x^2+1).$

31. $\displaystyle\int \frac{x^2\,dx}{(x-1)^2(x^2+1)} = \tfrac{1}{2}\log(x-1) - \tfrac{1}{4}\log(x^2+1) - \frac{1}{2(x-1)}.$

32. $\displaystyle\int \frac{x^2-1}{x^4+x^2+1}\,dx = \tfrac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}.$

33. $\displaystyle\int \frac{dx}{x^4(1+x^2)} = \tan^{-1}x + \frac{1}{x} - \frac{1}{3x^3}.$

34. $\displaystyle\int \frac{x^3\,dx}{(1+x^2)^3} = -\frac{1+2x^2}{4(1+x^2)^2}.$

35. $$\int \frac{dx}{x^2(1+x^2)^2} = -\frac{2+3x^2}{2x(1+x^2)} - \tfrac{3}{2}\tan^{-1} x.$$

36. $$\int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}} \log \frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1-x^2}.$$

37. $$\int \frac{x^2 dx}{1+x^4} = \frac{1}{4\sqrt{2}} \log \frac{1-x\sqrt{2}+x^2}{1+x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}} \tan^{-1} \frac{x\sqrt{2}}{1-x^2}.$$

85. **Integration of Irrational Functions.**

The following are the leading results in this connection.

1°. In the case of an algebraic function involving no irrationalities except fractional powers of the variable, we may put

$$x = t^m, \quad dx/dt = mt^{m-1} \quad \text{.................(1)},$$

where m is the least common multiple of the denominators of the various fractional indices. The problem is thus reduced to the integration of a rational function of t.

2°. Any rational function of x and X, where

$$X = \sqrt{(a+bx)} \quad \text{.....................(2)},$$

can be integrated by the substitution

$$a + bx = t^2, \quad dx/dt = 2t/b \quad \text{..............(3)}.$$

Thus $$\int F(x, X)\, dx = \int F\left(\frac{t^2-a}{b},\, t\right) . \frac{2t\, dt}{b} \quad \text{......(4)},$$

and the function of t which follows the integral-sign is now rational.

Ex. 1. To find $$\int \frac{x\, dx}{\sqrt{x}+1}.$$

If we put $x = t^2$, this becomes

$$= 2\int \frac{t^3 dt}{t+1} = 2\int (t^2 - t + 1)\, dt - 2\int \frac{dt}{t+1}$$
$$= \tfrac{2}{3}t^3 - t^2 + 2t - 2\log(t+1)$$
$$= \tfrac{2}{3}x^{\frac{3}{2}} - x + 2x^{\frac{1}{2}} - 2\log(x^{\frac{1}{2}}+1).$$

Ex. 2. To find $$\int \frac{dx}{(2+x)\sqrt{(1+x)}},$$

put $$1 + x = t^2, \quad dx/dt = 2t.$$

We obtain
$$\int \frac{2t\,dt}{(1+t^2)\,t} = 2\int \frac{dt}{1+t^2}$$
$$= 2\tan^{-1} t = 2\tan^{-1}\sqrt{(1+x)}.$$

3°. If X stand for the square root of a quadratic expression, say
$$X = \sqrt{(ax^2+bx+c)},$$
the problem of finding
$$\int F(x, X)\,dx \quad\text{..........................(5)},$$
where $F(x, X)$ is a rational function of x and X, can also be reduced to the integration of a rational function.

If a be positive, we may write
$$X = \sqrt{a}\,.\,\sqrt{(x^2+px+q)} \quad\text{..................(6)},$$
where $p = b/a$, $q = c/a$. Now assume
$$\sqrt{(x^2+px+q)} = t - x,$$
whence
$$x = \frac{t^2-q}{2t+p}, \quad \frac{dx}{dt} = \frac{2(t^2+pt+q)}{(2t+p)^2} \quad\text{............(7)},$$
and
$$\sqrt{(x^2+px+q)} = \frac{t^2+pt+q}{2t+p} \quad\text{...............(8)}.$$
It is evident that by these substitutions the problem is reduced to the integration of a rational function of t.

If the factors of x^2+px+q are real, say
$$x^2+px+q = (x-\alpha)(x-\beta) \quad\text{..................(9)}$$
we may also make use of the substitution
$$x-\beta = (x-\alpha)\,t^2 \quad\text{........................(10)},$$
whence
$$x = \alpha + \frac{\beta-\alpha}{1-t^2}, \quad \frac{dx}{dt} = \frac{2(\beta-\alpha)\,t}{(1-t^2)^2} \quad\text{............(11)}$$
and
$$\sqrt{(x^2+px+q)} = (x-\alpha)\,t = \frac{(\beta-\alpha)\,t}{1-t^2} \quad\text{.........(12)}.$$

If a be negative, we may write
$$X = \sqrt{(-a)}\,.\,\sqrt{(q+px-x^2)} \quad\text{...............(13)},$$
where $p = -b/a$, $q = -c/a$. If the radical is to be real, the factors of $q+px-x^2$ must be real, for otherwise this expression would have the same sign for all values of x, and since it is obviously negative for sufficiently large values of x, it would be always negative. We have, then,
$$q+px-x^2 = (x-\alpha)(\beta-x) \quad\text{...............(14)},$$

where α, β are real. If we assume

$$\beta - x = (x - \alpha)\, t^2 \quad \text{......................(15)},$$

we find
$$x = \alpha + \frac{\beta - \alpha}{1 + t^2}, \quad \frac{dx}{dt} = -\frac{2(\beta - \alpha)\, t}{(1 + t^2)^2} \quad \text{.........(16)},$$

and
$$\sqrt{(q + px - x^2)} = (x - \alpha)\, t = \frac{(\beta - \alpha)\, t}{1 + t^2} \quad \text{.........(17)}.$$

These substitutions evidently render

$$F(x, X)\frac{dx}{dt}$$

a rational function of t.

The above investigations are of some importance, as shewing that functions of the given forms can be integrated, and that the results will be of certain mathematical types; but the actual integration, in particular cases, can often be effected much more easily in other ways*. We have had instances of this fact in the course of the Chapter; and we add one or two further illustrations.

Ex. 3. By rationalizing the denominator, we have

$$\int \frac{dx}{\sqrt{(1 + x)} + \sqrt{x}} = \int \{\sqrt{(1 + x)} - \sqrt{x}\}\, dx$$
$$= \tfrac{2}{3}(1 + x)^{\frac{3}{2}} - \tfrac{2}{3}x^{\frac{3}{2}}.$$

Ex. 4.
$$\int \frac{dx}{x + \sqrt{(x^2 - 1)}} = \int \{x - \sqrt{(x^2 - 1)}\}\, dx$$
$$= \tfrac{1}{2}x^2 - \int \sqrt{(x^2 - 1)}\, dx$$
$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sqrt{(x^2 - 1)} + \tfrac{1}{2}\cosh^{-1} x.$$

Otherwise, putting $x = \cosh u$, the integral takes the form

$$\int \frac{\sinh u\, du}{\cosh u + \sinh u} = \int e^{-u} \sinh u\, du$$
$$= \tfrac{1}{2}\int (1 - e^{-2u})\, du = \tfrac{1}{2}u + \tfrac{1}{4}e^{-2u},$$

which may be easily shewn to differ from the former result only by an additive constant.

---

* See especially the methods of Arts. 75, 76, 78.

EXAMPLES. XXVI.

1. $\int x\sqrt{(1+x)}\,dx = \frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}}.$

2. $\displaystyle\int\frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}} = \frac{2}{3(a-b)}\{(x+a)^{\frac{3}{2}}-(x+b)^{\frac{3}{2}}\}.$

3. $\displaystyle\int\frac{dx}{\sqrt{x}-1} = 2\sqrt{x} + 2\log(1-\sqrt{x}).$

4. $\displaystyle\int\frac{dx}{(1-x)\sqrt{x}} = \log\frac{1+\sqrt{x}}{1-\sqrt{x}}.$

5. $\displaystyle\int\frac{dx}{(1-x)\sqrt{(1+x)}} = \sqrt{2}\tanh^{-1}\sqrt{\left(\frac{1+x}{2}\right)}.$

6 $\displaystyle\int\frac{dx}{(1+x)\sqrt{(1-x)}} = -\sqrt{2}\tanh^{-1}\sqrt{\left(\frac{1-x}{2}\right)}.$

7. $\displaystyle\int\frac{dx}{x+\sqrt{(x-1)}} = \log\{x+\sqrt{(x-1)}\} - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{(x-1)}+1}{\sqrt{3}}.$

8. $\displaystyle\int\frac{\sqrt{x}}{x-1}\,dx = 2\sqrt{x} + \log\frac{\sqrt{x}-1}{\sqrt{x}+1}.$

9. $\displaystyle\int\frac{dx}{x\sqrt{(1+x)}} = \log\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}+1}.$

10. $\displaystyle\int\frac{dx}{x^2\sqrt{(1+x)}} = -\frac{\sqrt{(1+x)}}{x} - \frac{1}{2}\log\frac{\sqrt{(1+x)}-1}{\sqrt{(1+x)}+1}.$

11. $\displaystyle\int\frac{x^2dx}{\sqrt{(x^2+1)}} = \frac{1}{2}x\sqrt{(x^2+1)} - \frac{1}{2}\sinh^{-1}x.$

12. $\displaystyle\int\frac{x^2dx}{(1+x^2)^{\frac{3}{2}}} = -\frac{x}{\sqrt{(1+x^2)}} + \sinh^{-1}x.$

13. $\displaystyle\int\frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{\sqrt{(1-x^2)}}.$

14. $\displaystyle\int\frac{dx}{(1-x^2)\sqrt{(1+x^2)}} = \frac{1}{2\sqrt{2}}\log\frac{x\sqrt{2}+\sqrt{(1+x^2)}}{x\sqrt{2}-\sqrt{(1+x^2)}}.$

EXAMPLES. XXVII.

1. A particle moves according to the law

$$\frac{ds}{dt} = u_0 - gt;$$

prove that the space described before it comes to rest is $u_0^2/2g$.

2. If a point start from rest at time $t=0$ and move with a constant acceleration, and if v_1 be the velocity after any interval and $\bar{v}$ the mean velocity in this interval, then

$$\bar{v}=\tfrac{1}{2}v_1.$$

3. If, with the same notation, the acceleration vary as t^n, then

$$\bar{v}=\frac{1}{n+2}v_1.$$

4. A particle moves according to the law

$$\frac{ds}{dt}=v_0\cos nt;$$

prove that the space described from time $t=0$ until it first comes to rest is v_0/n.

5. If the velocity of a particle moving in a resisting medium be given by

$$\frac{ds}{dt}=v_0e^{-kt},$$

prove that the particle never attains a distance v_0/k from its position when $t=0$.

6. A particle moves according to the law

$$\frac{ds}{dt}=v_0e^{-kt}\cos nt;$$

prove that the space described from time $t=0$ until it first comes to rest is

$$\frac{ne^{-k\pi/2n}+k}{n^2+k^2}v_0.$$

7. If the angular velocity of a body rotating about a fixed axis be given by

$$\frac{d\theta}{dt}=2n\,\text{sech}\,nt,$$

prove that $$\theta=4\tan^{-1}e^{nt}-\pi,$$
supposing that θ vanishes for $t=0$.

CHAPTER VI.

DEFINITE INTEGRALS.

86. Definition, and Notation.

Let $y, = \phi(x)$, be a function of x which is regarded as given (and therefore finite) for all values of x ranging from a to b, inclusively. Let the range $b-a$ be subdivided into a number of intervals

$$h_1, h_2, \ldots\ldots h_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

all of the same sign, so that

$$h_1 + h_2 + \ldots + h_n = b - a \ldots\ldots\ldots\ldots\ldots (2).$$

Let y_1 be one of the values which y assumes in the interval h_1, y_2 one of the values which it assumes in the interval h_2, and so on; and let

$$\Sigma = y_1 h_1 + y_2 h_2 + \ldots + y_n h_n \ldots\ldots\ldots\ldots\ldots (3).$$

The value of this sum will in general vary with the mode of subdivision of the range $b-a$, and with the choice of the values $y_1, y_2, \ldots, y_n$ within the respective intervals (1). But if we introduce the condition that none of these intervals is to exceed some assigned magnitude k, then in certain cases, which include all the types of function ordinarily met with in the applications of the Calculus (and more), the value of Σ will tend, as k is diminished, to some definite limiting value S, in the sense that by taking k small enough we can ensure that Σ shall differ from S by less than any assigned magnitude, however small.

If the function $\phi(x)$ admit of graphical representation, Σ will be represented by the sum of a series of rectangles, whose bases

$h_1, h_2, \ldots\ldots h_n$ make up the range $b-a$, and whose altitudes are ordinates of the curve at arbitrarily chosen points in these bases. And the limiting value S, to which Σ tends as the breadths of the rectangles are indefinitely diminished, is known as the 'area' included between the curve, the axis of x, and the extreme ordinates $x=a$, $x=b$. See Fig. 46.

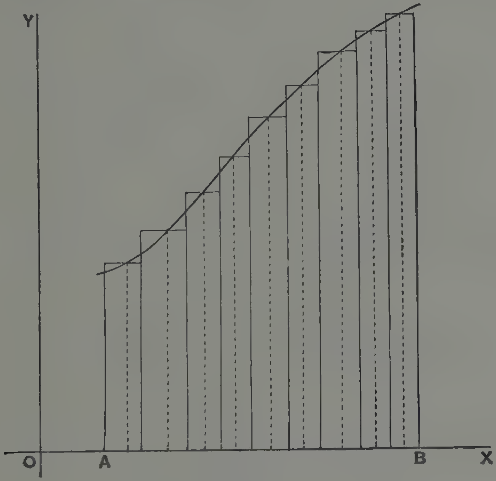

Fig. 46.

The sum which we have denoted by Σ is more fully expressed by

$$\Sigma_a^b y\delta x \text{ or } \Sigma_a^b \phi(x)\,\delta x \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

δx standing for the increments $h_1, h_2, \ldots\ldots h_n$ of x. The limiting value (when it exists) to which this sum converges, as the increments δx are all indefinitely diminished, and their number in consequence indefinitely increased, is called the 'definite integral' of the function $\phi(x)$ between the limits a, and b*, and is denoted by

$$\int_a^b y dx \text{ or } \int_a^b \phi(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots (5),$$

---

* It is a little unfortunate that the word 'limit' has to be used in several different senses. The word 'terminus' might perhaps be substituted in the present case.

the object of this notation being to recall the steps by which the limiting value was approached*.

The connection of the notation (5) with the notation for an indefinite integral used in the preceding chapter has of course yet to be explained. See Art. 92.

Problems in which we require the limiting value of a sum of the type (3) occur in almost every branch of Mathematics. The area of a curve has already been used as an illustration; other simple instances are: the length of a curved arc, regarded as the limit of an inscribed (or circumscribed) polygon; the volume of a solid of revolution, and so on. These will be considered more particularly in Chap. VII.

Again, in Dynamics, the 'impulse' of a variable force, in any interval of time, is defined as the 'time-integral' of the force over that interval; viz. if F be the force, considered as a function of the time t, the impulse in the interval $t_1 - t_0$ is the limiting value of the sum

$$F_1\tau_1 + F_2\tau_2 + \ldots + F_n\tau_n \quad\text{.....................(6)},$$

where $\tau_1, \tau_2, \ldots, \tau_n$ are subdivisions of the interval $t_1 - t_0$, such that

$$\tau_1 + \tau_2 + \ldots + \tau_n = t_1 - t_0 \quad\text{...................(7)},$$

whilst $F_1, F_2, \ldots, F_n$ denote values of the force in these respective intervals. Hence, in our present notation, the impulse is

$$\int_{t_0}^{t_1} F dt \quad\text{..............................(8)}.$$

Newton's Second Law of Motion asserts that the change of momentum of any mass (m) is equal to the impulse which it receives, or

$$mv_1 - mv_0 = \int_{t_0}^{t_1} F dt \quad\text{.......................(9)},$$

where v_0, v_1 are the initial and final velocities.

Again, the work done by a variable force is defined as the space-integral of the force. If F denote the force, regarded now as a function of the position (s) of the body, the work done as s changes from s_0 to s_1 is

$$\int_{s_0}^{s_1} F ds \quad\text{..........................(10)}.$$

---

* The symbol $\int$ is a specialized form of S, the sign of summation employed by the earlier analysts. The mode of indicating the range of integration was introduced by Fourier.

For example, the work done by unit mass of a gas as it expands from volume v_0 to volume v_1 is

$$\int_{v_0}^{v_1} pdv \text{ ....(11)},$$

if p be the pressure when the volume is v. This is seen by supposing the gas to be enclosed, by a piston, in a cylinder of sectional area unity.

The graphical representation of the integral (10) or (11) is frequently employed in practice. Thus, in the case of (10), if a curve be constructed with s as abscissa and F as ordinate, the work is represented by the area included between the curve, the axis of s, and the ordinates corresponding to s_0 and s_1. This is the principle of Watt's indicator-diagram*.

87. Proof of Convergence.

Whenever the sum Σ has a definite limiting value, in the manner above explained, the function $\phi(x)$ is said to be 'integrable.' It may be shewn that every continuous function is integrable *in this sense*†, but as regards the formal proof we shall confine ourselves to the particular case where the range of the independent variable can be divided into a finite number of intervals within each of which the function either steadily increases or steadily decreases. This will be sufficient for all practical purposes.

Before, however, introducing any restriction (beyond that of finiteness) we may note that two fixed limits can be assigned between which Σ must necessarily lie. For if A and B be the lower and upper limits (Art. 24) of the values which the function $\phi(x)$ can assume in the interval $b-a$, it is evident that Σ will lie between

$$A(h_1+h_2+\ldots+h_n), = A(b-a),$$

and

$$B(h_1+h_2+\ldots+h_n), = B(b-a).$$

We will now suppose, for definiteness, that $b > a$, and that $\phi(x)$ steadily increases as x increases from a to b. Consider any particular mode of subdivision

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* See Maxwell, *Theory of Heat*, c. v.; Rankine, *The Steam-Engine*, Art. 43.

† It is not implied that a mathematical formula for the integral can be found

$$h_1,\ h_2, \ldots\ldots h_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

of the range $b-a$, and let

$$\Sigma = y_1h_1 + y_2h_2 + \ldots + y_nh_n \ldots\ldots\ldots\ldots\ldots(2),$$

where, as in Art. 86, y_r denotes some value which the function assumes in the interval h_r.

Now if in (2) we replace $y_1, y_2, \ldots y_n$ by the values which the function has at the *beginnings* of the respective intervals, none of the terms will be increased; and if the resulting sum be denoted by Σ', we shall have

$$\Sigma' \ngtr \Sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

Again, if we replace $y_1, y_2, \ldots y_n$ by the values which the function has at the *ends* of the respective intervals, none of the terms will be decreased; hence if the resulting sum be Σ'', we shall have

$$\Sigma'' \nless \Sigma \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

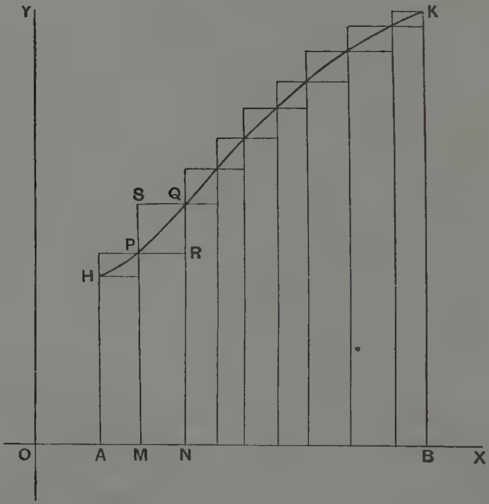

Fig. 47.

In Fig. 47 the quantity Σ' is represented by the sum of a series of rectangles such as PN, and Σ'' by the sum of a series of rectangles such as SN. Hence the difference $\Sigma'' - \Sigma'$ is represented by the sum of a series of rectangles such as SR. The sum of the altitudes of these latter rectangles is $KB - HA$, or $\phi(b) - \phi(a)$, and if k be the greatest of the bases, *i.e.* the greatest of the intervals (1), we shall have

$$\Sigma'' - \Sigma' \not> k\{\phi(b) - \phi(a)\} \text{ ....(5).}$$

Now, considering all possible modes of subdivision of the range $b - a$, the sums Σ', being always less than $B(b-a)$, will have an upper limit, which we will denote by S', and the sums Σ'', being always greater than $A(b-a)$, will have a lower limit, which we will denote by S'', and it is further evident that $S'' \not< S'$. It follows, from (5), that the difference $S'' - S'$ must lie between 0 and $k\{\phi(b) - \phi(a)\}$; and since, in this statement, k may be as small as we please, it appears that S' and S'' cannot but be equal. We will denote their common value by S.

Finally, it is evident that

$$|\Sigma - S| < \Sigma'' - \Sigma' < k\{\phi(b) - \phi(a)\} \text{ ......(6);}$$

hence by taking k small enough we can ensure that $|\Sigma - S|$ shall be less than any assigned quantity, however small*.

A similar proof obviously applies if the function $\phi(x)$ steadily decreases throughout the range $b - a$.

It follows that the final result also holds when the range admits of being broken up into a finite number of smaller intervals within each of which the function either steadily increases or steadily decreases. See Fig. 48.

It has been supposed that $b > a$. If $b < a$, the intervals $h_1, h_2, \ldots, h_n$ will be negative, but the argument is substantially unaltered.

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* The proof is a development of that given by Newton, *Principia*, lib. i., sect. i., lemma iii. (1687). It would be easy to eliminate all geometrical considerations and present the argument in a purely quantitative form.

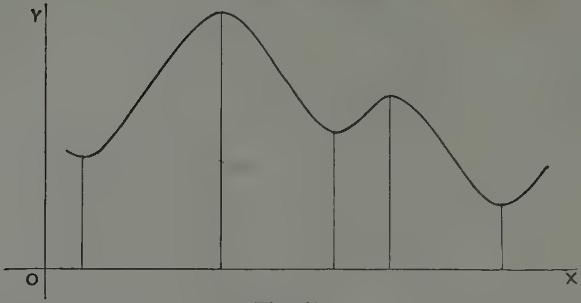

Fig. 48.

88. Examples of Definite Integrals calculated *ab initio.*

The meaning of a definite integral may be further illustrated by the study of a few cases in which the limiting value can be calculated from first principles. The methods to which we are obliged to have recourse for this purpose will at all events enable the student to appreciate the enormous simplification which was introduced into the subject by the invention of the special rule of the Integral Calculus, to which we afterwards proceed (Art. 92).

Ex. 1. To find $$\int_a^b x\,dx \quad\text{..........................}(1).$$

This is equivalent to finding the area of the trapezium $PABQ$ in the figure, where $OA = a$, $OB = b$, and OPQ is the straight line $y = x$.

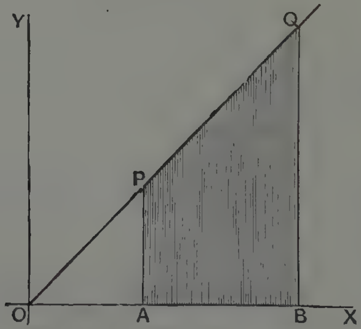

Fig. 49.

Take $$h_1 = h_2 = \ldots\ldots = h_n, = h, \text{ say},$$
so that $$nh = b - a,$$
and let $y_1, y_2, \ldots\ldots y_n$, be the values which the function to be integrated has at the beginnings of the successive intervals, viz.

$$y_1 = a,\ \ y_2 = a + h,\ \ y_3 = a + 2h,\ \ \ldots\ldots\ y_n = a + (n-1)\,h.$$

Then $$\Sigma = ah + (a+h)\,h + (a+2h)\,h + \ldots + \{a + (n-1)\,h\}\,h$$
$$= nah + \{1 + 2 + \ldots + (n-1)\}\,h^2 = nah + \tfrac{1}{2}n\,(n-1)\,h^2$$
$$= a\,(b-a) + \tfrac{1}{2}\left(1 - \frac{1}{n}\right)(b-a)^2.$$

When $h = 0$, we have $n = \infty$, and the limiting value of the above expression is

$$a\,(b-a) + \tfrac{1}{2}\,(b-a)^2 = \tfrac{1}{2}\,(b^2 - a^2).$$

Hence $$\int_a^b x\,dx = \tfrac{1}{2}\,(b^2 - a^2)\ \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Ex. 2. To find $$\int_a^b x^2\,dx\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This is equivalent to finding the area included between the parabola $y = x^2$, the axis of x, and the ordinates $x = a$, $x = b$.

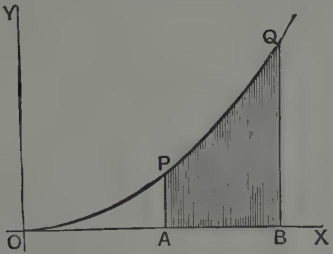

Fig. 50.

Putting $$h_1 = h_2 = \ldots = h_n, = h,$$
$$y_1 = a^2,\ \ y_2 = (a+h)^2,\ \ y_3 = (a+2h)^2,\ \ \ldots\ldots\ y_n = \{a + (n-1)\,h\}^2,$$
we have $$\Sigma = a^2h + (a+h)^2\,h + (a+2h)^2\,h + \ldots + \{a + (n-1)\,h\}^2$$
$$= na^2h + 2\,\{1 + 2 + \ldots + (n-1)\}\,ah^2 + \{1^2 + 2^2 + \ldots + (n-1)^2\}\,h^3$$
$$= na^2h + n\,(n-1)\,ah^2 + \tfrac{1}{6}\,(n-1)\,n\,(2n-1)\,h^3$$
$$= a^2\,(b-a) + \left(1 - \frac{1}{n}\right)a\,(b-a)^2 + \tfrac{1}{3}\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{2n}\right)(b-a)^3.$$

The limiting value of this for $n=\infty$ is

$$a^2(b-a)+a(b-a)^2+\tfrac{1}{3}(b-a)^3, \text{ or } \tfrac{1}{3}(b^3-a^3).$$

Hence $$\int_a^b x^2\,dx=\tfrac{1}{3}(b^3-a^3) \text{ ....} (4).$$

Ex. 3. To find $$\int_a^b e^{kx}\,dx \text{ ....} (5).$$

This is the limit, for $n=\infty$, of

$$\Sigma, =he^{ka}+he^{k(a+h)}+he^{k(a+2h)}+\ldots+he^{k(a+(n-1)h)},$$

where $$h=(b-a)/n.$$

Now $$\Sigma=he^{ka}(1+e^{kh}+e^{2kh}+\ldots+e^{(n-1)kh})$$

$$=he^{ka}\cdot\frac{e^{nkh}-1}{e^{kh}-1}=\frac{kh}{e^{kh}-1}\cdot\frac{1}{k}(e^{kb}-e^{ka}) \text{ ....} (6).$$

And, by Art. 29, the limiting value of the first factor, for $kh=0$, is unity.

Hence $$\int_a^b e^{kx}\,dx=\frac{1}{k}(e^{kb}-e^{ka}) \text{ ....} (7).$$

Ex. 4. To find $$\int_\alpha^\beta \sin x\,dx \text{ ....} (8).$$

This is the limit of

$$\Sigma, =\{\sin(\alpha+\tfrac{1}{2}h)+\sin(\alpha+\tfrac{3}{2}h)+\ldots+\sin(\beta-\tfrac{3}{2}h)+\sin(\beta-\tfrac{1}{2}h)\}\,h,$$

where $$h=(\beta-\alpha)/n,$$

and the values of $\sin x$ at the middles of the respective intervals have been taken. Now

$$\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cdot\Sigma=2\sin\tfrac{1}{2}h\sin(\alpha+\tfrac{1}{2}h)+2\sin\tfrac{1}{2}h\sin(\alpha+\tfrac{3}{2}h)+\ldots$$
$$+\,2\sin\tfrac{1}{2}h\sin(\beta-\tfrac{3}{2}h)+2\sin\tfrac{1}{2}h\sin(\beta-\tfrac{1}{2}h)$$

$$\begin{array}{ll} =\cos\alpha & -\cos(\alpha+h) \\ +\cos(\alpha+h) & -\cos(\alpha+2h) \\ +\ldots\ldots\ldots & \\ +\cos(\beta-2h) & -\cos(\beta-h) \\ +\cos(\beta-h) & -\cos\beta \\ =\cos\alpha & -\cos\beta \text{ ....} (9). \end{array}$$

Hence, proceeding to the limit ($h=0$), we find

$$\int_\alpha^\beta \sin x\,dx = \cos\alpha - \cos\beta \text{ ....(10).}$$

89. Properties of $\int_a^b \phi(x)\,dx$.

1°. If we compare the integrals

$$\int_a^b \phi(x)\,dx \text{ and } \int_b^a \phi(x)\,dx,$$

we see that they may be regarded as limits of the same summation, with this difference, that in one case the increments $h_1, h_2, \dots h_n$ of x, which make up the interval $b-a$ (or $a-b$) have the opposite sign to that which they have in the other. Hence

$$\int_b^a \phi(x) = -\int_a^b \phi(x)\,dx \text{ ....(1).}$$

2°. Again, it follows from the definition that

$$\int_a^c \phi(x)\,dx = \int_a^b \phi(x)\,dx + \int_b^c \phi(x)\,dx \text{.........(2).}$$

This is illustrated graphically in Fig. 51.

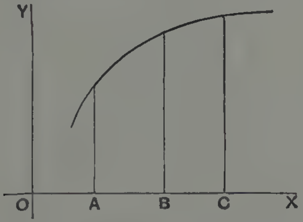

Fig. 51.

3°. If A, B be the least and greatest values which $\phi(x)$ assumes as x ranges from a to b, the integral

$$\int_a^b \phi(x)\,dx,$$

being intermediate in value to $A(b-a)$ and $B(b-a)$, must be equal to $C(b-a)$, where C is some quantity intermediate to A, B. If, as we suppose, $\phi(x)$ is continuous, it assumes within the range $b-a$ *all* values intermediate to A, B. Hence there must be some value (c) of x, between a and b, such that

$$\phi(c) = C.$$

In the graphical representation, Fig. 52, the area $PABQ$ is equal to a rectangle, on the base AB, whose altitude is equal to the ordinate at some point C of the range AB.

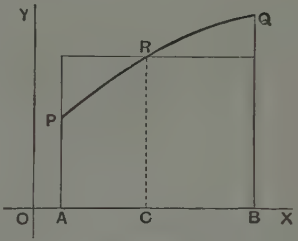

Fig. 52.

We may evidently write

$$c = a + \theta(b-a),$$

where θ is some quantity between 0 and 1. On this understanding,

$$\int_a^b \phi(x)\,dx = (b-a)\,\phi(a+\theta\,\overline{b-a}) \text{ .... } (3).$$

4°. More generally, if u, v, y be three functions such that for values of x ranging from a to b,

$$u > y > v \text{ .... } (4),$$

then the integral

$$\int_a^b y\,dx \text{ .... } (5)$$

will be intermediate in value to

$$\int_a^b u\,dx \text{ and } \int_a^b v\,dx \text{ .... } (6).$$

Suppose, first, that $b > a$. We have

$$\int_a^b u\,dx - \int_a^b y\,dx = \int_a^b (u-y)\,dx.$$

In virtue of (4), every term of the sum, of which the latter integral is the limit, will be positive. Hence

$$\int_a^b y\,dx < \int_a^b u\,dx \text{ .... } (7).$$

Similarly
$$\int_a^b y\,dx > \int_a^b v\,dx \qquad\qquad (8).$$

If $b < a$, the inequalities in (7) and (8) must be reversed.

EXAMPLES. XXVIII.

1. Prove by the method of Art. 88 that

$$\int_a^b x^3\,dx = \tfrac{1}{4}(b^4 - a^4).$$

2. Prove from first principles that

$$\int_\alpha^\beta \cos x\,dx = \sin\beta - \sin\alpha.$$

3. Shew from graphical considerations that

$$\int_a^b k\phi(x)\,dx = k\int_a^b \phi(x)\,dx,$$

$$\int_a^b \phi(x)\,dx = \int_0^{b-a} \phi(x+a)\,dx,$$

$$\int_a^b \phi(kx)\,dx = \frac{1}{k}\int_{ka}^{kb} \phi(x)\,dx.$$

4. By dividing the range $b - a$ into n intervals such that the abscissæ of the points of division are in *geometric* progression, and finally making n infinite, prove that

$$\int_a^b x^m\,dx = \frac{1}{m+1}(b^{m+1} - a^{m+1}). \qquad \text{(Wallis' method.)}$$

90. Differentiation of a Definite Integral with respect to either Limit.

Let
$$I = \int_a^b \phi(x)\,dx \qquad\qquad (1).$$

Evidently, I is a function of the 'limits of integration' a, b, and will in general vary when either of these varies.

Regarding a as fixed, let us form the derived function of I with respect to the upper limit b. We have

$$I+\delta I=\int_a^{b+\delta b}\phi(x)\,dx$$
$$=\int_a^b \phi(x)\,dx+\int_b^{b+\delta b}\phi(x)\,dx \quad \ldots\ldots\ldots(2),$$

by Art 89, 2°. Hence

$$\delta I=\int_b^{b+\delta b}\phi(x)\,dx=\delta b\,.\,\phi(b+\theta\,\delta b)\ldots\ldots\ldots(3),$$

by Art. 89, 3°. This shews that δI vanishes with δb, so that I is a continuous function of b. Also, since

$$\frac{\delta I}{\delta b}=\phi(b+\theta\,\delta b)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4),$$

we have, on proceeding to the limit ($\delta b=0$),

$$\frac{dI}{db}=\phi(b)\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

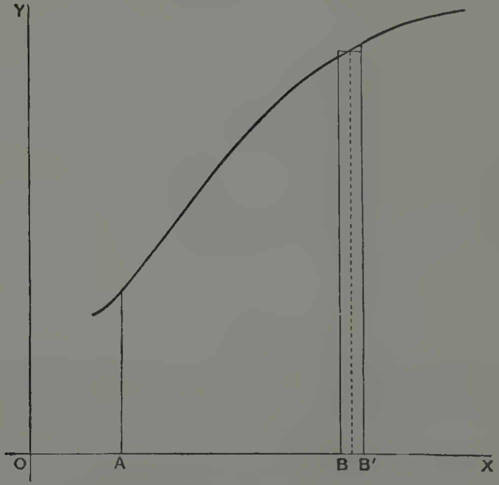

Fig. 53.

In the figure, $OA=a$, $OB=b$, $BB'=\delta b$, and δI is represented by the rectangle having BB' as its base.

In the same way, if we regard the upper limit b as fixed, and the lower limit a as variable, we find that I is a continuous function of a, and that

$$\frac{dI}{da} = -\phi(a) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

91. Existence of an Indefinite Integral.

We can now shew that any function $\phi(x)$, having the character postulated in Art. 87, has an indefinite integral, *i.e.* there exists a definable (but not necessarily calculable) function $\psi(x)$ such that

$$\psi'(x) = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$\psi(x) = D^{-1}\phi(x) \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

For if we write

$$\psi(\xi) = \int_a^\xi \phi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the expression on the right hand is, by Art. 87, a determinate function of ξ, and the investigation just given shews that it satisfies the condition

$$\psi'(\xi) = \phi(\xi) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

The lower limit of integration in (3) is, from the present point of view, arbitrary, and the function $\psi(\xi)$ is therefore indeterminate to the extent of an additive constant. For, by Art. 89, 2°, the substitution of a' for a, as the lower limit in (3), is equivalent to the addition of

$$\int_{a'}^a \phi(x)\,dx$$

to the right-hand side. Cf. Art. 71.

92. Rule for calculating a Definite Integral.

Whenever the analytical form of a function $\psi(x)$, which has a given function $\phi(x)$ as its derivative, has been discovered, the value of the definite integral

$$I = \int_a^b \phi(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

can be written down at once. For, if we regard a as fixed, we have, by Art. 90,

$$\frac{dI}{db} = \phi(b)$$

$$= \psi'(b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

by hypothesis. It follows by Art. 56 that I and $\psi(b)$ can only differ by a 'constant,' *i.e.* a quantity independent of b; thus

$$\int_a^b \phi(x)\,dx = \psi(b) + C \quad \ldots\ldots\ldots\ldots\ldots(3).$$

To find the value of C we may, since it does not vary with b, put $b = a$, whence

$$\psi(a) + C = \int_a^a \phi(x)\,dx = 0 \quad \ldots\ldots\ldots\ldots (4).$$

Hence $C = -\psi(a)$, and

$$\int_a^b \phi(x)\,dx = \psi(b) - \psi(a) \quad \ldots\ldots\ldots\ldots(5).$$

This is the fundamental proposition of the Integral Calculus. It reduces the problem of finding the definite integral of a given function $\phi(x)$ to the discovery of the inverse function $\psi(x)$, or $D^{-1}\phi(x)$. The reason why this inverse function is usually denoted by

$$\int\phi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

is now apparent. The form (6) is simply an abbreviation for

$$\int_a^x \phi(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where a is arbitrary. We have seen that a change in a is equivalent to the addition of a constant.

The notation $$\Big[\psi(b)\Big]_a^b \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

is often used as an abbreviation for $\psi(b) - \psi(a)$.

Ex. 1. To find $$\int_a^b x\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

Here $$\phi(x) = x, \quad \psi(x) = \tfrac{1}{2}x^2,$$

whence $$\int_a^b x\,dx = \psi(b) - \psi(a) = \tfrac{1}{2}(b^2 - a^2) \ldots\ldots\ldots\ldots(10).$$

Ex. 2. To find
$$\int_a^b x^2 dx \quad\text{.........................(11).}$$

Here
$$\phi(x) = x^2, \quad \psi(x) = \tfrac{1}{3}x^3,$$

whence
$$\int_a^b x^2 dx = \tfrac{1}{3}(b^3 - a^3) \quad\text{.................. (12).}$$

Ex. 3. To find
$$\int_a^b e^{kx} dx \quad\text{......................(13).}$$

Here
$$\phi(x) = e^{kx}, \quad \psi(x) = \frac{1}{k} e^{kx},$$

whence
$$\int_a^b e^{kx} dx = \frac{1}{k}(e^{kb} - e^{ka}) \quad\text{..............(14).}$$

The above results agree with those obtained, by much greater labour, in Art. 88.

93. Cases where the function $\phi(x)$, or the limits of integration, become infinite.

Before proceeding to further examples, it will be convenient to extend somewhat the definition of an integral given in Art. 86. It was there assumed that the limits of integration a, b were finite, and also that the function $\phi(x)$ was finite throughout the range $b - a$. We proceed to explain how, under certain conditions, these conditions may be relaxed.

1°. Suppose $\phi(x)$ to be finite and continuous for values of x ranging from a to ∞, and consider the integral
$$\int_a^\omega \phi(x)\, dx \quad\text{.......................(1),}$$
where $\omega > a$. If, as ω is increased indefinitely, the integral tends to a definite limiting value, this value is denoted by
$$\int_a^\infty \phi(x)\, dx \quad\text{.......................(2).}$$

The integral (1) is then said to be 'convergent' for $\omega = \infty$. As might be anticipated from the theory of infinite series (Art. 6) it is not a sufficient condition for convergence that
$$\lim_{x=\infty} \phi(x) = 0 \quad\text{....................... (3);}$$
this condition is moreover not essential, for there may even be convergence when $\phi(x)$ has no definite limiting value for $x = \infty$.

A similar definition of

$$\int_{-\infty}^{b} \phi(x)\,dx \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

can obviously be framed.

Ex. 1. $$\int_0^\omega e^{-ax}\,dx = \left[-\frac{1}{a}e^{-ax}\right]_0^\omega = \frac{1-e^{-a\omega}}{a} \quad\ldots\ldots\ldots(5).$$

As ω increases this tends to the limit $1/a$. Hence we say that

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Ex. 2. $$\int_1^\omega \frac{dx}{x} = \Big[\log x\Big]_1^\omega = \log\omega \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

This increases without limit with ω. Hence there is no limiting value for $\omega = \infty$, although

$$\lim_{x=\infty} \frac{1}{x} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

2°. Let $\phi(x)$ become infinite at or between the limits of integration.

It will be sufficient to consider the case where there is only one value of x for which $\phi(x) = \infty$. The general case can be reduced to this by breaking up the range $b - a$ into smaller intervals*.

If $\phi(x)$ become infinite at the upper limit (only), we consider in the first place the integral

$$\int_a^{b-\epsilon} \phi(x)\,dx \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where ϵ is positive. If, as ϵ is diminished indefinitely, this integral tends to a definite limiting value, this value is adopted as the definition of

$$\int_a^b \phi(x)\,dx.$$

A similar definition applies to the case where $\phi(x)$ becomes infinite at the lower limit a.

---

* It being assumed that $\phi(x)$ becomes infinite only at a finite number of isolated points.

Ex. 3.
$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

The function $1/\sqrt{(1-x)}$ becomes infinite for $x=1$, but

$$\int_0^{1-\epsilon} \frac{dx}{\sqrt{(1-x)}} = \Big[-2\sqrt{(1-x)}\Big]_0^{1-\epsilon} = 2 - 2\sqrt{\epsilon} \ \ldots(11),$$

and as ϵ is indefinitely diminished this tends to the limit 2. Hence

$$\int_0^1 \frac{dx}{\sqrt{(1-x)}} = 2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

If $\phi(x)$ becomes infinite *between* the limits a, b, say for $x=c$, we consider the sum

$$\int_a^{c-\epsilon} \phi(x)\,dx + \int_{c+\epsilon'}^{b} \phi(x)\,dx \ldots\ldots\ldots\ldots(13).$$

If, with diminishing ϵ (and ϵ') each of these integrals tends to a finite limiting value, the sum of these values is adopted as the definition of

$$\int_a^b \phi(x)\,dx^*.$$

The cases where $\phi(x)$ becomes infinite, or is discontinuous, at a finite number of isolated points, are dealt with by dividing the range into shorter intervals bounded by the points of discontinuity.

94. Applications of the Rule of Art. 92.

We give a few more typical examples of the evaluation of definite integrals.

---

* Cases may arise in which each of the integrals

$$\int_a^{c-\epsilon} \phi(x)\,dx \quad\text{and}\quad \int_{c+\epsilon'}^{b} \phi(x)\,dx$$

is ultimately infinite, whilst, *if some special relation be assumed* between the ultimately vanishing quantities ϵ, ϵ', the infinite elements of the two integrals cancel in such a way that the sum remains finite. The value of the sum will then depend on the nature of the assumed relation. The consideration of such cases is, however, beyond the scope of the present treatise. They do not often occur in physical problems.

Ex. 1.
$$\int_0^{\frac{1}{2}\pi} \sin x\,dx = \Big[-\cos x\Big]_0^{\frac{1}{2}\pi} = 1 \text{ ....(1);}$$
$$\int_0^{\frac{1}{2}\pi} \cos x\,dx = \Big[\sin x\Big]_0^{\frac{1}{2}\pi} = 1 \text{ ....(2);}$$
$$\int_0^{\frac{1}{2}\pi} \sin x \cos x\,dx = \Big[\tfrac{1}{2}\sin^2 x\Big]_0^{\frac{1}{2}\pi} = \tfrac{1}{2} \text{ ....(3).}$$

Ex. 2. By Art. 79, we have
$$\int_0^{\omega} e^{-ax}\sin\beta x\,dx = -\left[\frac{a\sin\beta x + \beta\cos\beta x}{a^2+\beta^2}e^{-ax}\right]_0^{\omega}$$
$$= \frac{\beta}{a^2+\beta^2} - \frac{a\sin\beta\omega + \beta\cos\beta\omega}{a^2+\beta^2}e^{-a\omega}.$$

If a be positive the last term tends, as ω is increased indefinitely, to the limiting value 0. Hence
$$\int_0^{\infty} e^{-ax}\sin\beta x\,dx = \frac{\beta}{a^2+\beta^2} \text{ ....(4).}$$

Similarly
$$\int_0^{\infty} e^{-ax}\cos\beta x\,dx = \frac{a}{a^2+\beta^2} \text{ ....(5).}$$

Ex. 3. We have
$$\int_0^{\omega} \frac{dx}{1+x^2} = \Big[\tan^{-1} x\Big]_0^{\omega} = \tan^{-1}\omega - \tan^{-1} 0.$$

The function $\tan^{-1} x$ is many-valued (Art. 21), but it is immaterial which value we take, provided we suppose it to change continuously as x varies through the range of integration. Hence if we take $\tan^{-1} 0 = 0$, we must understand by $\tan^{-1}\omega$ that value which increases continuously from 0 with ω. As ω increases indefinitely, this value tends to the limit $\frac{1}{2}\pi$, so that
$$\int_0^{\infty} \frac{dx}{1+x^2} = \tfrac{1}{2}\pi \text{ ....(6).}$$

Ex. 4. By Art. 77, 4° we have
$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a\sin^2\theta + \beta\cos^2\theta} = \left[\frac{1}{\sqrt{(a\beta)}}\tan^{-1}\left(\sqrt{\frac{a}{\beta}}\,.\tan\theta\right)\right]_0^{\frac{1}{2}\pi}.$$

Now, as θ increases from 0 to $\frac{1}{2}\pi$, $\sqrt{(a/\beta)}\,.\tan\theta$ increases from 0 to ∞, and we may therefore suppose that $\tan^{-1}\{\sqrt{(\alpha/\beta)}\,.\tan\theta\}$ increases from 0 to $\frac{1}{2}\pi$. Hence

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{\alpha\sin^2\theta+\beta\cos^2\theta}=\frac{\pi}{2\sqrt{(\alpha\beta)}} \quad\text{............(7).}$$

The student may have remarked in the course of the preceding Chapter that when an 'indefinite' integration is effected by a change of variable (Arts. 76, 78) the most troublesome part of the process consists often in the translation back to the original variable. This part is, however, unnecessary when the object is merely to find the definite integral between given limits. It is then sufficient to substitute the *altered* limits in the indefinite integral as first obtained.

Ex. 5. To find $\int_0^a \sqrt{(a^2-x^2)}\,dx$ ....(8).

We found (Art. 78), putting $x=a\sin\theta$, that

$$\int\sqrt{(a^2-x^2)}\,dx=a^2\int\cos^2\theta\,d\theta=\tfrac{1}{2}a^2(\theta+\tfrac{1}{2}\sin 2\theta).$$

Now, if θ increase from 0 to $\frac{1}{2}\pi$, x will increase from 0 to a. Hence

$$\int_0^a \sqrt{(a^2-x^2)}\,dx=\tfrac{1}{2}a^2\Big[\theta+\tfrac{1}{2}\sin 2\theta\Big]_0^{\frac{1}{2}\pi}=\tfrac{1}{4}\pi a^2 \quad\text{......(9).}$$

EXAMPLES. XXIX.

1. $\int_0^1 \sqrt{x}\,dx=\frac{2}{3}$, $\int_0^1 \frac{dx}{\sqrt{x}}=2$, $\int_0^1 \frac{dx}{\sqrt{(3-2x)}}=\sqrt{3}-1$.

2. $\int_0^1 \frac{dx}{\sqrt{(1-x^2)}}=\frac{1}{2}\pi$, $\int_0^{1/\sqrt{3}} \frac{dx}{\sqrt{(2-3x^2)}}=\frac{\pi}{4\sqrt{3}}$.

3. $\int_0^\infty \frac{dx}{a^2+b^2x^2}=\frac{\pi}{2ab}$.

4. $\int_0^1 \frac{x\,dx}{\sqrt{(1-x^2)}}=1$, $\int_0^1 \frac{x\,dx}{\sqrt{(1+x^2)}}=\sqrt{2}-1$.

5. $\int_0^1 \frac{dx}{\sqrt{(x^2+1)}}=\log(1+\sqrt{2})$, $\int_1^2 \frac{dx}{\sqrt{(x^2-1)}}=\log(2+\sqrt{3})$.

6. $\int_1^\infty \frac{dx}{x(1+x)}=\log 2$, $\int_1^\infty \frac{dx}{x(1+x^2)}=\frac{1}{2}\log 2$.

7. $\int_0^1 \frac{dx}{x^2 - x + 1} = \frac{2\pi}{3\sqrt{3}}$. 8. $\int_0^1 \frac{1 - x^2}{1 + x^2}\,dx = \tfrac{1}{2}\pi - 1$.

9. $\int_0^\infty \frac{dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2ab(a + b)}$.

10. $\int_0^\infty \frac{x^2\,dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi}{2(a + b)}$.

11. $\int_0^\infty \frac{x\,dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{1}{a^2 - b^2} \log \frac{a}{b}$.

12. $\int_1^\infty \frac{dx}{x^2(1 + x)} = 1 - \log 2$, $\int_1^\infty \frac{dx}{x(1 + x)^2} = \log 2 - \tfrac{1}{2}$.

13. $\int_0^\infty \frac{dx}{(1 + x^2)^2} = \tfrac{1}{4}\pi$.

14. $\int_1^\infty \frac{dx}{x\sqrt{(x^2 - 1)}} = \tfrac{1}{2}\pi$, $\int_1^\infty \frac{dx}{x\sqrt{(x^2 + 1)}} = \log(1 + \sqrt{2})$.

15. $\int_0^1 \frac{dx}{\sqrt{\{x(1 - x)\}}} = \pi$, $\int_0^1 \sqrt{\left(\frac{x}{1 - x}\right)}\,dx = \tfrac{1}{2}\pi$.

16. $\int_{-1}^1 \sqrt{\left(\frac{1 - x}{1 + x}\right)}\,dx = \pi$. 17. $\int_\alpha^\beta \frac{dx}{\sqrt{\{(x - \alpha)(\beta - x)\}}} = \pi$.

18. $\int_0^{\frac{1}{4}\pi} \sec^2\theta\,d\theta = 1$, $\int_0^{\frac{1}{4}\pi} \tan^2\theta\,d\theta = 1 - \tfrac{1}{4}\pi$.

19. $\int_0^{\frac{1}{2}\pi} \sin 2\theta\,d\theta = 1$, $\int_0^{\frac{1}{2}\pi} \cos 2\theta\,d\theta = 0$.

20. $\int_0^{\frac{1}{2}\pi} \frac{\cos\theta}{1 + \sin^2\theta}\,d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin\theta}{1 + \cos^2\theta}\,d\theta = \tfrac{1}{4}\pi$.

21. $\int_0^{\frac{1}{4}\pi} \sec^4\theta\,d\theta = \tfrac{4}{3}$, $\int_0^{\frac{1}{4}\pi} \tan^4\theta\,d\theta = \tfrac{1}{4}\pi - \tfrac{2}{3}$.

22. $\int_0^\pi \frac{d\theta}{1 + e\cos\theta} = \frac{\pi}{\sqrt{(1 - e^2)}}$, $[e < 1]$.

23. $\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 + \cos\theta} = \int_0^{\frac{1}{2}\pi} \frac{d\theta}{1 + \sin\theta} = 1$.

24. $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \tan\theta\,d\theta = 0$, $\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \sec\theta\,d\theta = \log(3 + 2\sqrt{2})$.

25. $\int_0^1 \log x\,dx = -1, \quad \int_0^1 x \log x\,dx = -\tfrac{1}{4}.$

26. $\int_0^1 \sin^{-1} x\,dx = \tfrac{1}{2}\pi - 1, \quad \int_0^1 \tan^{-1} x\,dx = \tfrac{1}{4}\pi - \tfrac{1}{2}\log 2.$

27. $\int_0^{\frac{1}{2}\pi} \theta \sin\theta\,d\theta = 1, \quad \int_0^{\frac{1}{2}\pi} \theta\cos\theta\,d\theta = \tfrac{1}{2}\pi - 1.$

28. $\int_0^{\frac{1}{2}\pi} \theta^2 \sin\theta\,d\theta = \pi - 2, \quad \int_0^{\frac{1}{2}\pi} \theta^2\cos\theta\,d\theta = \tfrac{1}{4}\pi^2 - 2.$

29. $\int_0^{\frac{1}{4}\pi} \theta\sec^2\theta\,d\theta = \tfrac{1}{4}\pi - \tfrac{1}{2}\log 2.$

30. $\int_0^{\infty} e^{-x}\cos(x + \tfrac{1}{4}\pi)\,dx = 0.$

31. $\int_{-\infty}^{\infty} \frac{du}{\cosh u} = \pi.$

95. Formulæ of Reduction.

The methods of Arts. 80, 81, when applied to the reduction of *definite* integrals, sometimes lead to specially simple results, owing to the vanishing of the integrated terms at both limits.

1°. If
$$u_n = \int_0^{\frac{1}{2}\pi} \cos^n\theta\,d\theta \qquad\qquad (1),$$
we have, by Art. 81 (2),
$$u_n = \left[\frac{1}{n}\sin\theta\cos^{n-1}\theta\right]_0^{\frac{1}{2}\pi} + \frac{n-1}{n}u_{n-2} \qquad\qquad (2).$$
If $n > 1$, the first part vanishes, since
$$\sin 0 = 0, \quad \cos\tfrac{1}{2}\pi = 0.$$
Hence
$$\int_0^{\frac{1}{2}\pi} \cos^n\theta\,d\theta = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi}\cos^{n-2}\theta\,d\theta \qquad\qquad (3).$$

Similarly, from Art. 81 (6),
$$\int_0^{\frac{1}{2}\pi} \sin^n\theta\,d\theta = \frac{n-1}{n}\int_0^{\frac{1}{2}\pi}\sin^{n-2}\theta\,d\theta \qquad\qquad (4).$$

If n be a positive integer, we can, by successive applications of (3), express

$$\int_0^{\frac{1}{2}\pi} \cos^n \theta \, d\theta$$

in terms of either

$$\int_0^{\frac{1}{2}\pi} \cos \theta \, d\theta, = 1, \quad \text{or} \quad \int_0^{\frac{1}{2}\pi} d\theta, = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots(5),$$

according as n is odd or even. In the same way

$$\int_0^{\frac{1}{2}\pi} \sin^n \theta \, d\theta$$

can be made to depend either on

$$\int_0^{\frac{1}{2}\pi} \sin \theta \, d\theta, = 1, \quad \text{or on} \quad \int_0^{\frac{1}{2}\pi} d\theta, = \tfrac{1}{2}\pi \quad \ldots\ldots\ldots(6).$$

Ex. 1.

$$\int_0^{\frac{1}{2}\pi} \cos^5 \theta \, d\theta = \tfrac{4}{5} \int_0^{\frac{1}{2}\pi} \cos^3 \theta \, d\theta$$

$$= \tfrac{4}{5} \cdot \tfrac{2}{3} \int_0^{\frac{1}{2}\pi} \cos \theta \, d\theta = \tfrac{8}{15}.$$

After working out one or two examples in this way, the student will be able to supply the successive steps mentally, and write down at once the factors of the result; thus

$$\int_0^{\frac{1}{2}\pi} \cos^6 \theta \, d\theta = \tfrac{5}{6} \cdot \tfrac{3}{4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = \tfrac{5}{32}\pi.$$

The general values of the preceding integrals can be written down without difficulty. Thus, if n be odd, we have

$$\int_0^{\frac{1}{2}\pi} \cos^n \theta \, d\theta = \int_0^{\frac{1}{2}\pi} \sin^n \theta \, d\theta = \frac{(n-1)(n-3)\ldots 2}{n(n-2)\ldots 3} \quad \ldots(7),$$

whilst, if n be even,

$$\int_0^{\frac{1}{2}\pi} \cos^n \theta \, d\theta = \int_0^{\frac{1}{2}\pi} \sin^n \theta \, d\theta = \frac{(n-1)(n-3)\ldots 1}{n(n-2)\ldots 2} \cdot \frac{\pi}{2} \quad \ldots(8).$$

Integrals of this type are of frequent occurrence in the physical applications of the Calculus.

2°. If $$u_{m,n}=\int_0^{\frac{1}{2}\pi}\sin^m\theta\cos^n\theta\,d\theta\;\dots\dots\dots\dots(9),$$

we have, by Art. 81 (10),

$$u_{m,n}=\left[\frac{1}{m+n}\sin^{m+1}\theta\cos^{n-1}\theta\right]_0^{\frac{1}{2}\pi}+\frac{n-1}{m+n}u_{m,n-2}\;\dots(10).$$

If $n>1$, the expression in [ ] vanishes at both limits, and we have

$$\int_0^{\frac{1}{2}\pi}\sin^m\theta\cos^n\theta\,d\theta=\frac{n-1}{m+n}\int_0^{\frac{1}{2}\pi}\sin^m\theta\cos^{n-2}\theta\,d\theta\dots(11).$$

In the same way from Art. 81 (11) we obtain, if $m>1$,

$$\int_0^{\frac{1}{2}\pi}\sin^m\theta\cos^n\theta\,d\theta=\frac{m-1}{m+n}\int_0^{\frac{1}{2}\pi}\sin^{m-2}\theta\cos^n\theta\,d\theta\;\dots(12).$$

By means of these formulæ, either index can be reduced by 2, and by repetitions of this process we can, if m, n be positive integers, make the integral (9) depend on one in which each index is 1 or 0. The result therefore finally involves one or other of the following forms:

$$\left.\begin{array}{l}\displaystyle\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\,d\theta,=\tfrac{1}{2};\quad\int_0^{\frac{1}{2}\pi}d\theta,=\tfrac{1}{2}\pi;\\ \displaystyle\qquad\int_0^{\frac{1}{2}\pi}\sin\theta\,d\theta,=1;\quad\int_0^{\frac{1}{2}\pi}\cos\theta\,d\theta,=1\end{array}\right\}\dots(13).$$

Ex. 2. We have

$$\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^3\theta\,d\theta=\tfrac{4}{8}\int_0^{\frac{1}{2}\pi}\sin^3\theta\cos^3\theta\,d\theta=\tfrac{4}{8}\cdot\tfrac{2}{6}\int_0^{\frac{1}{2}\pi}\sin\theta\cos^3\theta\,d\theta,$$

by (12). Again, by (11),

$$\int_0^{\frac{1}{2}\pi}\sin\theta\cos^3\theta\,d\theta=\tfrac{2}{4}\cdot\int_0^{\frac{1}{2}\pi}\sin\theta\cos\theta\,d\theta=\tfrac{2}{4}\cdot\tfrac{1}{2}.$$

Hence $$\int_0^{\frac{1}{2}\pi}\sin^5\theta\cos^3\theta\,d\theta=\tfrac{4}{8}\cdot\tfrac{2}{6}\cdot\tfrac{2}{4}\cdot\tfrac{1}{2}=\tfrac{1}{24}.$$

After a little practice, the result can be written down immediately. Thus

$$\int_0^{\frac{1}{2}\pi}\sin^6\theta\cos^2\theta\,d\theta=\tfrac{5}{8}\cdot\tfrac{3}{6}\cdot\tfrac{1}{4}\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi=\tfrac{5}{256}\pi.$$

The formulæ (11) and (12), as well as (3) and (4), are often required in practice, and should be remembered.

Again, the algebraic integral

$$\int_0^1 x^m (1-x)^n \, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(14),$$

is reduced by the substitution $x = \sin^2\theta$ to the form

$$2\int_0^{\frac{1}{2}\pi} \sin^{2m+1}\theta \cos^{2n+1}\theta \, d\theta \quad \ldots\ldots\ldots\ldots(15),$$

and can therefore be evaluated by means of the formulæ given above, whenever $2m+1$ and $2n+1$ are positive integers or null.

Similarly, if we put $x = \sin\theta$, the integral

$$\int_0^1 x^m (1-x^2)^n \, dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(16)$$

takes the form

$$\int_0^{\frac{1}{2}\pi} \sin^m\theta \cos^{2n+1}\theta \, d\theta \quad \ldots\ldots\ldots\ldots\ldots(17).$$

Ex. 3.
$$\int_0^1 x^2 (1-x)^{\frac{3}{2}} dx = 2\int_0^{\frac{1}{2}\pi} \sin^5\theta \cos^4\theta \, d\theta$$
$$= 2 \cdot \tfrac{4}{9} \cdot \tfrac{2}{7} \cdot \tfrac{3}{5} \cdot \tfrac{1}{3} \cdot 1 = \tfrac{16}{315}.$$

Ex. 4.
$$\int_0^1 x^2 (1-x^2)^{\frac{3}{2}} = \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^4\theta \, d\theta$$
$$= \tfrac{1}{6} \cdot \tfrac{3}{4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = \tfrac{1}{32}\pi.$$

96. **Related Integrals.**

There are various theorems concerning definite integrals which follow almost intuitively from the definition of Art. 86.

For example,

$$\int_0^a \phi(x)\, dx = \int_0^a \phi(a-x)\, dx \quad \ldots\ldots\ldots\ldots\ldots(1).$$

This is proved by writing

$$x = a - x', \quad dx = -dx',$$

the new limits of integration being $x' = a$, $x' = 0$, corresponding to $x = 0$, $x = a$, respectively. Thus

$$\int_0^a \phi(x)\,dx = -\int_a^0 \phi(a - x')\,dx' = \int_0^a \phi(a - x)\,dx,$$

the accent being dropped in the end, as no longer necessary.

This process is equivalent to transferring the origin to the point $x = a$, and reversing the direction of the axis of x. The areas represented by the integrals in (1) are thus seen to be identical.

An important case of (1) is

$$\int_0^{\frac{1}{2}\pi} f(\sin\theta)\,d\theta = \int_0^{\frac{1}{2}\pi} f(\cos\theta)\,d\theta \qquad\qquad (2).$$

Ex. 1. Thus $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^2\theta\,d\theta = \int_0^{\frac{1}{2}\pi} \cos^2\theta\,d\theta.$

Hence each of these integrals

$$= \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} (\sin^2\theta + \cos^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} d\theta = \tfrac{1}{4}\pi.$$

Again, if $\phi(x)$ be an 'even' function of x, that is

$$\phi(-x) = \phi(x) \qquad\qquad (3),$$

we have

$$\int_{-a}^a \phi(x)\,dx = 2\int_0^a \phi(x)\,dx \qquad\qquad (4),$$

the area represented by the former integral being obviously bisected by the axis of y.

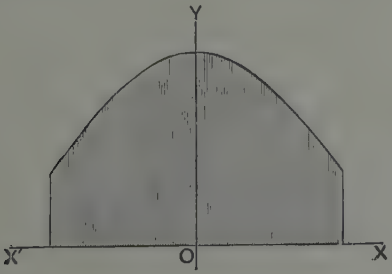

Fig. 54.

On the other hand, if $\phi(x)$ be an 'odd' function of x, so that

$$\phi(-x) = -\phi(x) \qquad (5),$$

we have

$$\int_{-a}^{a} \phi(x)\,dx = 0 \qquad (6),$$

since in the sum, of which the definite integral is the limit (Art. 86), the element $\phi(x)\,\delta x$ is cancelled by the oppositely-signed element $\phi(-x)\,\delta x$

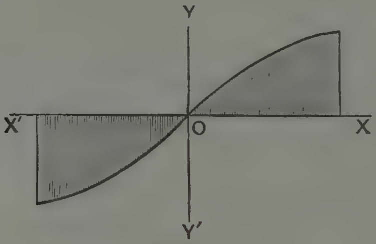

Fig. 55.

Ex. 2. We have

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^2\theta \cos^3\theta\,d\theta = 2\int_0^{\frac{1}{2}\pi} \sin^2\theta\cos^3\theta\,d\theta = 2\,.\,\tfrac{1}{5}\,.\,\tfrac{2}{3}\,.\,1 = \tfrac{4}{15};$$

whilst

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^3\theta\cos^2\theta\,d\theta = 0,$$

since $\sin^3\theta$ changes sign with θ.

For similar reasons, if

$$\phi(a - x) = \phi(x) \qquad (7),$$

we have

$$\int_0^a \phi(x)\,dx = 2\int_0^{\frac{1}{2}a} \phi(x)\,dx \qquad (8);$$

whilst if

$$\phi(a-x) = -\phi(x) \qquad (9),$$

we have

$$\int_0^a \phi(x)\,dx = 0 \qquad (10).$$

As a particular case of (8), we have

$$\int_0^\pi f(\sin\theta)\,d\theta = 2\int_0^{\frac{1}{2}\pi} f(\sin\theta)\,d\theta \qquad (11),$$

since

$$\sin(\pi - \theta) = \sin\theta.$$

Ex. 3.

$$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = 2\int_0^{\frac{1}{2}\pi} \sin^3\theta \cos^2\theta \, d\theta = 2 \,.\, \tfrac{2}{5} \,.\, \tfrac{1}{3} \,.\, 1 = \tfrac{4}{15};$$

whilst
$$\int_0^{\pi} \sin^2\theta \cos^3\theta \, d\theta = 0.$$

EXAMPLES. XXX.

1. Write down the values of the following integrals:

(1) $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^3\theta \, d\theta, \quad \int_0^{\frac{1}{2}\pi} \cos^4\theta \, d\theta, \quad \int_0^{\frac{1}{2}\pi} \sin^7\theta \, d\theta.$

(2) $\displaystyle\int_0^{\frac{1}{2}\pi} \sin^2\theta \cos\theta \, d\theta, \quad \int_0^{\frac{1}{2}\pi} \sin^3\theta \cos^3\theta \, d\theta, \quad \int_0^{\frac{1}{2}\pi} \sin^5\theta \cos^6\theta \, d\theta.$

(3) $\displaystyle\int_0^{\pi} \sin^5\theta \, d\theta, \quad \int_0^{\pi} \cos^5\theta \, d\theta, \quad \int_0^{\pi} \sin^3\theta \cos^4\theta \, d\theta.$

(4) $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^4\theta \, d\theta, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^5\theta \, d\theta, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^5\theta \, d\theta,$

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^8\theta \cos^3\theta \, d\theta.$$

2. Prove from first principles that

$$\int_0^1 x^m (1-x)^n \, dx = \int_0^1 x^n (1-x)^m \, dx.$$

Prove that the common value is

$$\frac{m\,!\,n\,!}{(m+n+1)\,!}.$$

3. Prove that

$$\int_{-a}^{a} \phi(x^2) \, dx = 2\int_0^a \phi(x^2) \, dx, \quad \int_{-a}^{a} \phi(x^2) \, x \, dx = 0.$$

4. Prove that

$$\int_{-a}^{a} \phi(x) \, dx = \int_0^a \{\phi(x) + \phi(-x)\} \, dx,$$

and
$$\int_{-a}^{a} \{\phi(x) - \phi(-x)\} \, dx = 0.$$

5. If
$$u_n = \int_0^{\frac{1}{2}\pi} \tan^n \theta \, d\theta,$$
prove that
$$u_n = \frac{1}{n-1} - u_{n-2}.$$

6. $\displaystyle\int_0^1 x^3 (1-x^2)^{\frac{5}{2}} dx = \tfrac{2}{63}.$ 7. $\displaystyle\int_0^1 x^3 (1-x)^3 dx = \tfrac{1}{140}.$

8. $\displaystyle\int_0^1 x^{\frac{3}{2}} (1-x)^{\frac{3}{2}} dx = \tfrac{3}{128}\pi.$

9. $\displaystyle\int_0^1 \frac{x^5 \, dx}{\sqrt{(1-x^2)}} = \tfrac{8}{15}, \quad \int_0^1 \frac{x^6 \, dx}{\sqrt{(1-x^2)}} = \tfrac{5}{32}\pi.$

10. $\displaystyle\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \theta \sin\theta \cos\theta \, d\theta = \tfrac{1}{4}\pi, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \theta \sin^2\theta \, d\theta = 0.$

11. $\displaystyle\int_0^{\pi} \theta \sin\theta \cos^2\theta \, d\theta = \tfrac{1}{3}\pi, \quad \int_0^{\pi} \theta \sin^2\theta \cos\theta \, d\theta = -\tfrac{4}{9}.$

12. $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin 3\theta}{\sin\theta} d\theta = \tfrac{1}{2}\pi, \quad \int_0^{\frac{1}{2}\pi} \frac{\sin 4\theta}{\sin\theta} d\theta = \tfrac{4}{3}.$

*13. Prove that
$$\int_0^{\infty} \frac{dx}{(1+x^2)^n} = \frac{2n-3}{2n-2} \int_0^{\infty} \frac{dx}{(1+x^2)^{n-1}}.$$
[Put $x = \tan\theta$.]

14. Prove that, if $n > 1$,
$$\int_0^{\infty} \frac{dx}{\{x+\sqrt{(x^2+1)}\}^n} = \frac{n}{n^2-1}.$$
[Put $x = \sinh u$.]

15. Prove that
$$\int_{-1}^{1} (1+x)^m (1-x)^n dx = 2^{m+n+2} \int_0^{\frac{1}{2}\pi} \sin^{2n+1}\theta \cos^{2m+1}\theta \, d\theta.$$

16. Prove that
$$\int_0^{\infty} \frac{du}{\cosh^n u} = \frac{n-2}{n-1} \int_0^{\infty} \frac{du}{\cosh^{n-2} u}.$$
[Put $\cosh u = \sec\theta$.]

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* The Examples which follow are of a more difficult character, and may be passed over on a first reading of the subject.

17. Prove from first principles that

$$\int_0^{\frac{1}{2}\pi} \sin^{n+1}\theta\, d\theta < \int_0^{\frac{1}{2}\pi} \sin^n \theta\, d\theta.$$

Hence show that $\frac{1}{2}\pi$ lies between

$$\frac{2 \,.\, 2 \,.\, 4 \,.\, 4 \,.\, 6 \,.\, 6 \ldots 2n \,.\, 2n}{1 \,.\, 3 \,.\, 3 \,.\, 5 \,.\, 5 \,.\, 7 \ldots (2n-1)(2n+1)}$$

and the fraction obtained by omitting the last factors in the numerator and denominator. (Wallis.)

18. Prove from first principles that

$$\int_0^{\frac{1}{4}\pi} \tan^{n+1}\theta\, d\theta < \int_0^{\frac{1}{4}\pi} \tan^n \theta\, d\theta.$$

Hence, using the result of Ex. 5, shew that

$$\int_0^{\frac{1}{4}\pi} \tan^n \theta\, d\theta$$

lies between

$$\frac{1}{2(n-1)} \text{ and } \frac{1}{2(n+1)}.$$

19. Shew that

$$\int_0^\infty \sin\theta\, d\theta \text{ and } \int_0^\infty \cos\theta\, d\theta$$

are indeterminate.

20. Shew from graphical considerations that

$$\int_0^\infty \frac{\sin\theta}{\theta}\, d\theta$$

is finite and determinate.

21. Prove that if $\phi(x)$ be finite and continuous for values of x ranging from 0 to a, except for $x=0$, when it becomes infinite, the integral

$$\int_0^a \phi(x)\, dx$$

will be finite, provided a positive quantity m can be found, less than unity, and such that

$$\lim_{x=0} x^m \phi(x)$$

is finite. [Put $x=t^n$.]

22. If $\phi(x)$ be finite and continuous for all values of x, the integral

$$\int_0^\infty \phi(x)\, dx$$

will be finite, provided a quantity m can be found, greater than unity, and such that

$$\lim_{x=\infty} x^m \phi(x)$$

is finite. [Put $x = t^{-n}$.]

23. Prove that

$$\int_0^\infty \cos x^2\, dx \text{ and } \int_0^\infty \sin x^2\, dx$$

are finite and determinate.

24. Prove that

$$\int_0^\infty xe^{-x^2}\, dx = \tfrac{1}{2}, \qquad \int_{-\infty}^\infty xe^{-x^2}\, dx = 0.$$

25. Prove that the integral

$$\int_0^\infty x^n e^{-x^2}\, dx$$

(where $n > -1$) is finite and determinate.

26. Prove that, if $n > 1$,

$$\int_0^\infty x^n e^{-x^2}\, dx = \tfrac{1}{2}(n-1) \int_0^\infty x^{n-2} e^{-x^2}\, dx.$$

Hence shew that, if n be a positive integer,

$$\int_0^\infty x^{2n+1} e^{-x^2}\, dx = \tfrac{1}{2} \,.\, n\,!.$$

27. If

$$u_n = \int_0^\infty x^n e^{-ax}\, dx,$$

prove that

$$u_n = \frac{n}{a} u_{n-1},$$

n being positive.

Hence shew that, if n be integral,

$$\int_0^\infty x^n e^{-ax}\, dx = \frac{n\,!}{a^{n+1}}.$$

28. If $f(x)$ and $\phi(x)$ be finite and continuous, and if $\phi(x)$ retain the same sign, throughout the interval from $x=a$ to $x=b$, then

$$\int_a^b f(x)\,\phi(x)\,dx = f[a+\theta(b-a)]\int_a^b \phi(x)\,dx,$$

where $1>\theta>0$.

29. Shew how it follows from the equality

$$\int_1^x \frac{dx}{x} = \log x$$

that the sum of n terms of the harmonic series

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\dots$$

lies between $\log(n+1)$ and $1+\log n$.

Shew that the sum of a million terms of this series lies between 13·8 and 14·8.

30. Shew from graphical considerations that if $f(x)$ steadily diminishes, as x increases from 0 to ∞, the series

$$f(1)+f(2)+f(3)+\dots$$

is convergent, and that its sum lies between I and $I+f(1)$, provided the integral

$$I, = \int_1^\infty f(x)\,dx,$$

be finite.

Apply this to the series

$$\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\frac{1}{(n+3)^2}+\dots$$

EXAMPLES. XXXI.

1. Prove that if the pressure (p) and volume (v) of a gas be connected by the relation

$$pv = \text{const.},$$

the work done in expanding from volume v_0 to volume v_1 is

$$p_0 v_0 \log\frac{v_1}{v_0}.$$

2. Prove also that if the relation be

$$pv^\gamma = \text{const.},$$

the work done is
$$\frac{1}{\gamma-1}(p_0v_0 - p_1v_1).$$

3. If the tension of an elastic string vary as the increase over the natural length, prove that the work done in stretching the string from one length to another is the same as if the tension had been constant and equal to half the sum of the initial and final tensions.

4. Prove that the work done by gravity on a pound of matter, as it is brought from an infinite distance to the surface of the Earth, is n foot-lbs., where n is the number of feet in the Earth's radius. [Assume that the force varies inversely as the square of the distance from the Earth's centre.]

CHAPTER VII.

GEOMETRICAL APPLICATIONS.

97. Definition of an Area.

In Euclid's Elements a system of propositions is developed by means of which we are able to give a precise meaning to the term 'area,' as applied to any figure bounded wholly by straight lines. In particular it is shewn that a rectangle can be constructed equal to the given figure, and having any given base, say the (arbitrarily chosen) unit of length. The 'area' of the figure in question is then measured by the ratio of this rectangle to the square on the unit length.

This process obviously does not apply to a figure bounded, in whole or in part, by *curved* lines, and we require therefore a definition of what is to be understood by the 'area' in such a case. To supply this, we imagine two rectilinear figures to be constructed, one including, and the other included by, the given curved figure. There is an upper limit to the area of the inscribed figure, and a lower limit to that of the circumscribed figure, and these limits can be proved to be identical. The common limiting value is adopted, by definition, as the measure of the 'area' of the given curvilinear figure.

Thus, in the case of a circle, if, in Fig. 4, p. 7, PQ be the side of an inscribed polygon, the area of the polygon will be $\frac{1}{2}\Sigma\,(ON\,.\,PQ)$. Now ON is less than the radius, and $\Sigma\,(PQ)$ is less than the perimeter, of the circle. Hence the upper limit to the area of an inscribed polygon cannot exceed

$\frac{1}{2}a \times 2\pi a$, or πa^2, where a is the radius. Similarly we may shew that the lower limit to the area of a circumscribed polygon cannot be less than πa^2. Moreover, the difference between the area of an inscribed polygon, and that of the corresponding circumscribed polygon, is represented by $\Sigma(PN \, . \, NT)$, and is therefore less than $\Sigma(PN) \, . \, \epsilon$, where ϵ is the greatest value of NT. Since this can be made as small as we please, the upper and lower limits aforesaid must be equal, and each is therefore equal to πa^2.

In the same way we may prove that the area of any sector of a circle of radius a is $\frac{1}{2}a^2\theta$, where θ is the angle of the sector.

98. Formula for an Area, in Cartesian Coordinates.

If the equation of a curve in rectangular coordinates be

$$y = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the area included between the curve, the axis of x, and the ordinates $x = a$, $x = b$, is

$$\int_a^b \phi(x)\,dx \text{ or } \int_a^b y\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

it being assumed that $\phi(x)$ is a function of the type contemplated in Art. 87.

This follows at once from the definition of the preceding Art. and the investigations of Art. 87.

Ex. 1. To find the area included between the curve

$$y = Ax^m \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the axis of x, and the ordinate $x = \alpha$; we have

$$\int_0^\alpha y\,dx = A\left[\frac{x^{m+1}}{m+1}\right]_0^\alpha = \frac{A\alpha^{m+1}}{m+1}$$

$$= \frac{1}{m+1}\alpha\beta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

where β is the length of the extreme ordinate. The area in question is therefore equal to the fraction $1/(m+1)$ of the area of the rectangle contained by the extreme ordinate and its abscissa.

In the case of the parabola

$$y^2 = 4ax \quad\text{.........................}(5),$$

we have $m = \frac{1}{2}$, and the ratio is $\frac{2}{3}$.

Ex. 2. The area of a quadrant of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad\text{.........................}(6)$$

is given by

$$\int_0^a y\,dx = \frac{b}{a}\int_0^a \sqrt{(a^2 - x^2)}\,dx.$$

The value of the definite integral was found in Art. 94 to be $\frac{1}{4}\pi a^2$. Hence the whole area of the ellipse is πab.

Ex. 3. In the rectangular hyperbola

$$x^2 - y^2 = 1 \quad\text{.........................}(7),$$

we may put, for the positive branch,

$$x = \cosh u, \quad y = \sinh u \quad\text{.................}(8),$$

since these satisfy (7), and give the required range of values of x and y.

The area included between the curve, the axis of x, and the ordinate defined by the variable u, is

$$\int_1^x y\,dx = \int_0^u \sinh^2 u\,du = \tfrac{1}{2}\int_0^u (\cosh 2u - 1)\,du$$
$$= \tfrac{1}{4}\sinh 2u - \tfrac{1}{2}u \quad\text{............}(9).$$

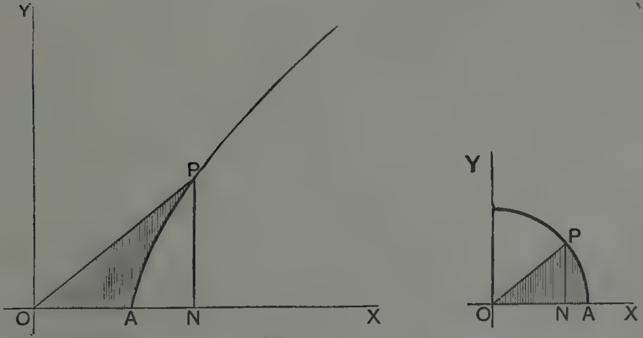

Fig. 56.

This gives the area PAN in the left-hand figure. Hence the area of the hyperbolic sector AOP is

$$\tfrac{1}{2}PN \,.\, ON - \text{area } PAN = \tfrac{1}{2}u \text{ ....(10).}$$

We have here an analogy between the 'amplitude' (u) of the hyperbolic functions $\cosh u$, $\sinh u$, &c., and the amplitude (θ) of the circular functions $\cos\theta$, $\sin\theta$, &c.; viz. the independent variable in each case represents twice the sectorial area AOP corresponding to the point P whose coordinates are ($\cosh u$, $\sinh u$), or ($\cos\theta$, $\sin\theta$), respectively.

In the case of the general hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ ....(11),}$$

the coordinates of any point on the positive branch may be represented by

$$x = a\cosh u, \quad y = b\sinh u \text{ ....(12),}$$

and the sectorial area is $\tfrac{1}{2}ab\,u$.

If the axes of coordinates be oblique, making (say) an angle ω with one another, the elementary rectangles $y\,\delta x$ which occur in the sum, of which the area is the limit, are replaced by elementary parallelograms $y\,\delta x \sin\omega$; the area included between the curve, the axis of x, and two bounding ordinates is therefore given by

$$\sin\omega \int y\,dx \text{ ....(13)}$$

taken between the proper limits.

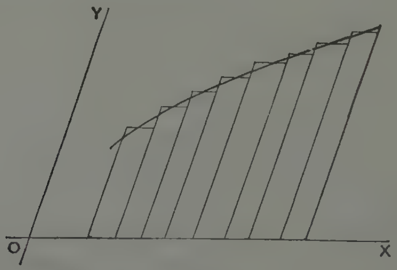

Fig. 57.

Ex. 4. The equation of a parabola, referred to any diameter and the tangent at its extremity, is

$$y^2 = 4a'x \qquad \text{.........................} \quad (14).$$

The area of the segment cut off by the chord $x = \alpha$ is therefore

$$2 \sin \omega \int_0^\alpha y\, dx = 4a'^{\frac{1}{2}} \sin \omega \int_0^\alpha x^{\frac{1}{2}}\, dx = \tfrac{8}{3} a'^{\frac{1}{2}} \alpha^{\frac{3}{2}} \sin \omega$$
$$= \tfrac{4}{3} \alpha\beta \sin \omega \qquad \text{..........................} \quad (15).$$

Hence the area of any segment of a parabola is two-thirds the rectangle contained by the intercept (α) of the chord on its diameter and the projection ($2\beta \sin \omega$) of the chord on the directrix.

99. On the Sign to be attributed to an Area.

It was tacitly implied in Art. 98 that $b > a$, and that the ordinate $\phi(x)$ is positive throughout the range of integration. If we drop these restrictions, it is easily seen that the integral

$$\int_a^b \phi(x)\, dx \qquad \text{......................} \quad (1)$$

is equal to $\pm S$, where S is the area included between the curve, the axis of x, and the extreme ordinates; the sign being + or − according as the area in question lies to the right or left of the curve, supposed described in the direction from P to Q, where PA, QB are the ordinates corresponding to $x = a$, $x = b$, respectively*. If the curve cuts the axis of x between A and B, the integral gives the excess (positive or

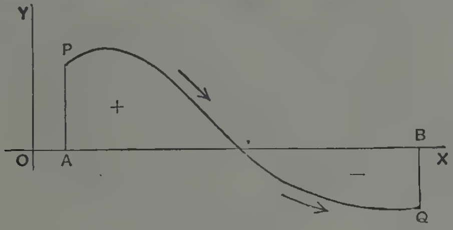

Fig. 58.

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* It is assumed here that the axes of x and y have the relative directions shewn in the figures. In the opposite case, the words 'right' and 'left' must be interchanged.

negative) of the area which lies to the right over that which lies to the left.

Even with these generalizations, the formula

$$\int_a^b \phi(x)\,dx = \pm S \qquad \text{.....................(2)}$$

still applies in strictness only when there is a unique value of y, or $\phi(x)$, for each value of x within the range $b-a$. If however we replace x, as independent variable, by a quantity t such that, as t increases, the corresponding point P moves in a continuous manner along the curve*, the formula

$$\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt \qquad \text{.......................(3)}$$

will give in a generalized sense the area included between the curve, the axis of x, and the ordinates of the points P_0, P_1 for which $t=t_0$, $t=t_1$, respectively, viz. it will give the excess of those portions of the area swept over by the ordinate y as it moves to the right over those swept over as it moves to the left, or *vice versâ*, according as y is positive or negative.

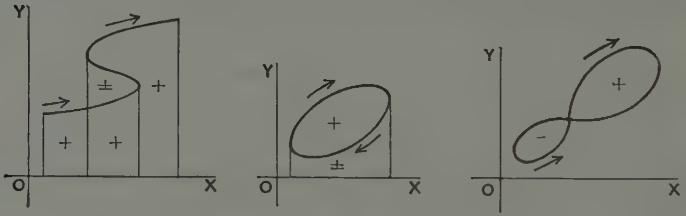

Fig. 59.

If, for a certain value of t, P return to its former position, having described a closed curve, the integral

$$\int y\frac{dx}{dt}\,dt \qquad \text{........................(4),}$$

taken between proper limits of t, will give the area included by the curve, with the sign + or −, according as the area lies to the right or left of P, when this point describes the curve in accordance with the variation of t*. If the curve cut itself, the formula (4) gives the excess of those portions of the included area which lie to the right over those which lie to the left. (See Fig. 59.)

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* For instance, we may take as the new variable the arc s of the curve, measured from some fixed point on it.

It is sometimes convenient, in finding the area of a curve, to use y as independent variable, instead of x. The area included between the curve, the axis of y, and the lines $y=h$, $y=k$, is evidently given, with the same kind of qualification as before, by

$$\int_h^k x\,dy \qquad\qquad (5).$$

The more general formula, analogous to (3), is

$$\int_{t_0}^{t_1} x\frac{dy}{dt}\,dt \qquad\qquad (6),$$

but it will be found on examination that the words 'left' and 'right' must now be interchanged in the rule of signs.

100. Areas referred to Polar Coordinates.

If the equation of a curve in polar coordinates be

$$r=\phi(\theta) \qquad\qquad (1),$$

the area included between the curve and any two radii vectores $\theta=\alpha$, $\theta=\beta$ is given by the formula

$$\tfrac{1}{2}\int_\alpha^\beta r^2 d\theta \text{ or } \tfrac{1}{2}\int_\alpha^\beta \{\phi(\theta)\}^2 d\theta \qquad (2).$$

For we can construct, in the manner indicated by the figure, an including area S, and an included area S', each built up of sectors of circles. The area of any one of these sectors is equal to $\frac{1}{2}r^2\delta\theta$, where r is its radius, and $\delta\theta$ its angle, and the sum of either series of sectors is therefore given by a series of the type

$$\tfrac{1}{2}\Sigma_\alpha^\beta r^2\delta\theta \qquad\qquad (3).$$

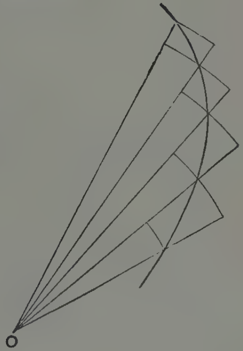

Fig. 60.

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* Thus, in the indicator-diagrams referred to on p. 211, the area enclosed by the curve gives the excess of the work done by the steam on the piston during the forward stroke over the work done by the piston in expelling the steam during the back stroke, and so represents the net energy communicated to the piston in a complete stroke.

Hence either series has the unique limit denoted by (2).

It is here assumed that $\beta > \alpha$ and that each radius vector through the origin intersects the arc considered in one point only. If however we introduce a new independent variable t, such that, as t increases, the corresponding point P moves in a continuous manner along the curve, the expression

$$\tfrac{1}{2}\int_{t_0}^{t_1} r^2 \frac{d\theta}{dt}\,dt \quad\text{......................(4)}$$

will give the net **area** swept over by the radius vector as t varies from t_0 to t_1, *i.e.* the (positive or negative) excess of those parts which are swept over in the direction of θ increasing over those swept over in the contrary direction. Moreover if, as t increases, P at length returns to its original position, having described a closed curve, the expression

$$\tfrac{1}{2}\int r^2 \frac{d\theta}{dt}\,dt \quad\text{........................(5),}$$

taken between suitable limits of t, gives in a generalized sense the area enclosed by the curve; viz., it represents the excess of that part of the area which lies to the left of P (as it describes the curve in accordance with the variation of t) over that part which lies to the right. Cf. Art. 99.

Ex. 1. The area of the circle

$$r = 2a \sin\theta \quad\text{..........................(6)}$$

(see Fig. 39, p. 126) is given by

$$\tfrac{1}{2}\int_0^{\pi} r^2 d\theta = 2a^2 \int_0^{\pi} \sin^2\theta\, d\theta = \pi a^2 \quad\text{..............(7).}$$

Ex. 2. The equation of an ellipse in polar coordinates, the centre being pole, is

$$\frac{1}{r^2} = \frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2} \quad\text{.....................(8).}$$

Hence the area is

$$\tfrac{1}{2}\int_0^{2\pi} \frac{a^2 b^2}{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta = 2a^2b^2 \int_0^{\frac{1}{2}\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta}$$

$$\text{......(9).}$$

The value of the latter integral has been found in Art. 94 to be $\frac{1}{2}\pi/(ab)$. Hence the required area is πab.

101. Area swept over by a Moving Line.

The area swept over by a moving line, of constant or variable length, may be calculated as follows.

Let PQ, $P'Q'$ be two consecutive positions of the line, and let their directions meet in C. Let R, R' be the middle points of PQ, $P'Q'$, and let RS be an arc of a circle with centre C. Then if the angle PCP' be denoted by $\delta\theta$, we have, ultimately,

$$\begin{aligned}\text{area } PQQ'P' &= \triangle QCQ' - \triangle PCP' \\ &= \tfrac{1}{2}CQ^2 . \delta\theta - \tfrac{1}{2}CP^2 . \delta\theta \\ &= PQ . \tfrac{1}{2}(CP + CQ)\,\delta\theta \\ &= PQ . CR . \delta\theta = PQ . RS.\end{aligned}$$

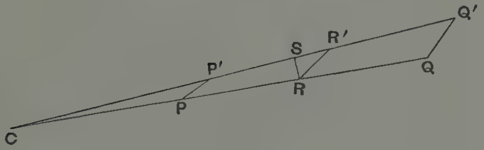

Fig. 61.

Hence, if we denote the length PQ by u, and the elementary displacement of R, estimated in the direction perpendicular to the moving line, by $\delta\sigma$, the area swept over may be represented by

$$\int u\,d\sigma \text{ ....(1).}$$

The student will notice that the formulæ of Arts. 98, 100 are particular cases of this result. Thus in the case of Art. 98 (13) we have $u = y$, $\delta\sigma = \delta x . \sin\omega$.

It is tacitly assumed, in the foregoing proof, that the areas are swept over always in the same direction. It is easy to see, however, that the formula (1) will apply without any such restriction, provided areas be reckoned positive or negative according as they are swept over towards the side of the line PQ on which $\delta\sigma$ is reckoned positive, or the reverse.

For example, the area swept over by a straight line whose middle point is fixed is on this reckoning zero.

We will suppose, for definiteness, that $\delta\sigma$ is positive when the motion of R is to the *left* of PQ as regards a spectator looking along the straight line in the direction from P to Q. If PQ return finally to its original position, its extremities P, Q having described closed curves, the integral (1) will, on the above convention, represent the excess of the area enclosed by the path of Q over that enclosed by the path of P, provided the signs attributed to these areas be in accordance with the rule of Art. 100.

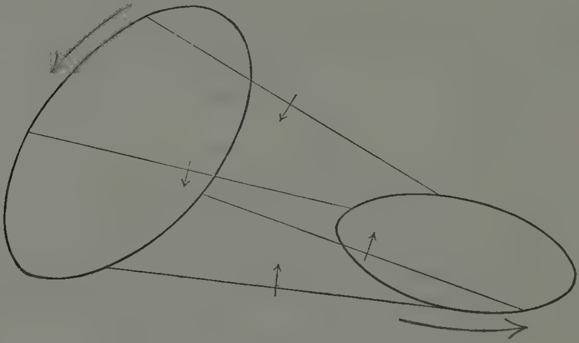

Fig. 62.

102. Theory of Amsler's Planimeter.

A 'planimeter' is an instrument by which the area of any figure drawn on paper is measured mechanically.

Many such instruments have been devised*, but the simplest and most popular is the one invented by Amsler, of Schaffhausen, in 1854. This consists of two bars OP, PQ, freely jointed at P, the former of which can rotate about a fixed point at O. If a tracing point attached to the bar PQ at Q be carried round any closed curve, P will oscillate to and fro along an arc of a circle, describing (as it were) a contour of zero area. Hence by the

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* See Henrici, 'Report on Planimeters,' *Brit. Ass. Rep.*, 1894, p. 496.

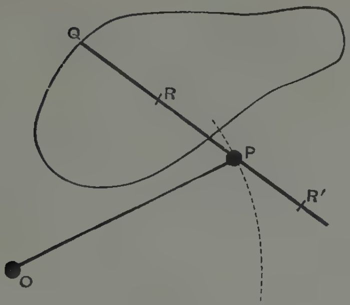

Fig. 63.

theorem stated at the end of Art. 101, the area of the curve described by Q will be equal to

$$\int l\,d\sigma \quad\text{..........}\quad (1),$$

where l is the length PQ, and $\int d\sigma$ represents the integral motion of R, the middle point of PQ, estimated always in the direction perpendicular to PQ*.

Now if, as is generally the case in the actual use of the instrument, PQ return to its original position without making a complete revolution, the *integral* motion of R at right angles to PQ is the same as that of any other point R' in the line PQ. For if $\delta\sigma$, $\delta\sigma'$ be corresponding elements of the paths of R, R', estimated as aforesaid, we have

$$\delta\sigma - \delta\sigma' = R'R\,.\,\delta\theta,$$

where $\delta\theta$ is the angle between the consecutive positions of $R'R$. Hence

$$\int d\sigma = \int d\sigma' + R'R \int d\theta$$
$$= \int d\sigma' \quad\text{..........}\quad (2),$$

since, under the circumstances supposed, we have $\int d\theta = 0$.

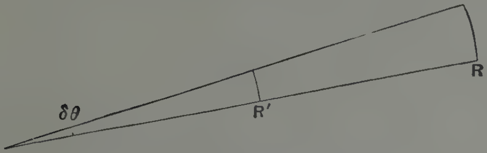

Fig. 64.

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* This is of course *not* in general the same thing as the length of the path of R.

In the instrument, as actually constructed, the integral motion normal to the bar of a point R' in QP produced backwards, is recorded by means of a small wheel having its axis in the direction PQ. As Q describes any curve, the wheel partly rolls and partly slides over the plane of the paper on which the curve is drawn, and the rotation of the wheel is in exact proportion to the displacement of R' perpendicular to its axis. The wheel is graduated, and has a fixed index for the record of partial revolutions; the whole revolutions are recorded by a dial and counter.

There is also an arrangement for varying the length PQ; this merely alters the *scale* of the record.

EXAMPLES*. XXXII.

1. If a curve be such that

$$y^m \propto x^n,$$

the rectangle enclosed between the coordinate axes and lines drawn parallel to them through any point on the curve is divided by the curve into two portions whose areas are as $m : n$.

2. The area included between the hyperbola

$$xy = k^2,$$

the axis of x, and the lines $x = a$, $x = b$, is

$$k^2 \log b/a.$$

3. The area included between the axis of x and one semi-undulation of the curve

$$y = b \sin x/a$$

is $2ab$.

4. The area included between the catenary

$$y = c \cosh x/c,$$

the axis of x, and the lines $x = 0$, $x = x_1$, is

$$c^2 \sinh x_1/c.$$

5. The curve $\quad a^2 y = x^2 (x + a)$

includes, with the axis of x, an area $\frac{1}{12}a^2$.

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[*] Some further Examples for practice will be found in the Chapter (IX) on "Special Curves."

6. The areas included between the axis of x and successive semi-undulations of the curve

$$y = e^{-\alpha x} \sin \beta x$$

form a descending geometric series, the common ratio being $e^{-\pi\alpha/\beta}$.

7. The area included between the positive branch of the curve

$$y = b \tanh x/a,$$

its asymptote, and the axis of y, is $ab \log 2$. (See Fig. 22, p. 43.)

8. The area included between the coordinate axes and the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

is $\frac{1}{6}ab \sin \omega$, where ω is the inclination of the axes.

[Put $x = a \sin^4 \theta$, $y = b \cos^4 \theta$.]

9. The area swept over by the radius vector of the parabola

$$r = \frac{2a}{1 + \cos \theta}$$

is equal to the difference between the initial and final values of

$$a^2 (\tan \tfrac{1}{2}\theta + \tfrac{1}{3} \tan^3 \tfrac{1}{2}\theta).$$

10. A curve AB is traced on a lamina which turns in its own plane about a fixed point O through an angle θ. Prove that the area swept over by the curve is

$$\tfrac{1}{2} (OA^2 \sim OB^2)\, \theta.$$

11. Prove that, with a proper convention as to sign, the area of a closed curve is given by

$$\tfrac{1}{2} \int \left(x \frac{dy}{dt} - y \frac{dx}{dt}\right) dt,$$

provided the total variation of t corresponds to a complete circuit of the curve.

12. The formula (Art. 79 (2)) for integration by parts may be written

$$\int u dv = uv - \int v du;$$

interpret this geometrically in terms of areas.

13. The area common to the two ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

is $4ab \tan^{-1} b/a$.

14. The area common to the two parabolas

$$y^2=4ax, \quad x^2=4ay$$

is $\frac{16}{3}a^2$.

15. Prove by integration that the area of an ellipse is

$$\pi\alpha\beta \sin \omega,$$

where α, β are the lengths of any pair of conjugate semi-diameters, and ω is the angle between these.

16. Prove by transformation to polar coordinates that the area of the ellipse

$$Ax^2+2Hxy+By^2=1$$

is $\pi/\sqrt{(AB-H^2)}$.

17. A weightless string of length l, attached to a fixed point O, passes through a small ring which can slide along a horizontal rod AB in the same vertical plane with O, and the lower portion hangs vertically, carrying a small weight P. Find the locus of P, and prove that the area between this locus and AB is

$$l\sqrt{(l^2-h^2)}-h^2\cosh^{-1} l/h,$$

where h is the depth of AB below O.

18. Prove directly from geometrical considerations that the area included between two focal radii of a parabola and the curve is half that included between the curve, the corresponding perpendiculars on the directrix, and the directrix.

19. What is indicated by the record of the wheel in Amsler's Planimeter when the bar PQ (Fig. 63) makes a complete revolution whilst the point Q traces out the closed curve?

20. If S_A, S_B, S_C be the areas of the closed curves described by three points A, B, C on a bar which moves in one plane, and returns to its original position without performing a complete revolution, prove that

$$BC \,.\, S_A + CA \,.\, S_B + AB \,.\, S_C = 0,$$

where the lines BC, CA, AB have signs attributed to them according to their directions, and the signs of S_A, S_B, S_C are determined by the rule of Art. 99.

21. If P be a point on a bar AB which moves in one plane, and returns to its original position after accomplishing one revolution, prove that

$$S_P = \frac{aS_B + bS_A}{a+b} - \pi ab,$$

where $a = AP$, $b = PB$, and the meanings of S_A, S_B, S_P are as in the preceding question.

Hence shew that if the extremities A, B of the bar move on a closed oval curve

$$S_A - S_P = \pi ab. \qquad \text{(Holditch.)}$$

22. If a straight line AB of constant length move with its extremities on two fixed intersecting straight lines, any point P on it describes an ellipse of area $\pi . AP . PB$.

103. **Volumes of Solids.**

It is impossible to give a general definition of the 'volume,' even of a solid bounded wholly by plane faces, without introducing, in one form or another, the notion of a 'limiting value.'

It may, indeed, be proved by Euclidean methods that two rectangular parallelepipeds are to one another in the ratio compounded of the ratios, each to each, of three concurrent edges of the one to three concurrent edges of the other; and, more generally, that two *prisms* are to one another in the ratio compounded of the ratio of their altitudes and the ratio of their bases. In this way we may define the ratio of any prism to that of the unit cube.

But it is not in general possible to dissect a given polyhedron into a finite number of *prisms*. The simplest general mode of dissection is into *pyramids* having a common vertex at some internal point O, and the faces of the polyhedron as their bases. And the volume of a pyramid cannot be compared with that of a prism without having recourse to the notion of a limiting value. A triangular prism may, indeed, be dissected into three pyramids of equal altitude standing on equal bases (Euc. XII. 7); but we cannot prove these pyramids equal to one another except by a process which involves the consideration of infinitesimal elements (Euc. XII. 5).

A general definition of the volume of a solid bounded by any surfaces, plane or curved, may be framed similar to that of the area of a plane figure (Art. 97). It is always possible to construct two figures, built up of prisms, such that one figure includes, and the other is included by, the given solid, and that the difference between their volumes admits of being made as small as we please. The limiting value to which the volume of either of these figures tends, as the difference between them is indefinitely diminished, is adopted as the definition of the 'volume' of the given solid. We may easily satisfy ourselves, as before, that this limiting value is unique.

The volume of any cylinder (right or oblique), with plane parallel ends, is equal to the product of the area of either end into the perpendicular distance between the two ends. For we may construct an including, and also an included, prismatic figure, whose bases are polygons respectively including, and included by, the base of the cylinder. The above statement is true of each of these figures, and therefore in the limit it is true of the cylinder. Thus the volume of a circular cylinder is $\pi a^2 h$, where a is the radius of the base, and h is the altitude.

Having found the volume of a cylinder with parallel plane ends, we are at liberty, if we find it convenient, to use such cylinders, in place of prisms, to build up the accessory figures employed in the general definition given above. The limit finally obtained in either way must evidently be the same.

104. General expression for the Volume of any Solid.

The axis of x being drawn in any convenient direction, let the area of the section of the solid by a plane perpendicular to this axis, at a distance x from the origin, be $f(x)$.

If we draw a system of planes perpendicular to x, at intervals δx, it is evident that the required volume will be the limit of the sum

$$\Sigma f(x)\,\delta x \quad\text{.......................}\quad (1),$$

since each element of this sum represents the volume of a cylinder of height δx and base $f(x)$. Hence the volume will be given by

$$\int f(x)\,dx \quad\text{.......................}\quad (2),$$

taken between suitable limits of x.

Ex. 1. Thus, in the case of a cone (or a pyramid), right or oblique, on any base, we take the origin O at the vertex, and the axis of x perpendicular to the base. If A be the area of the base, and h the altitude, the area of a section at a distance x from O will be

$$f(x) = \left(\frac{x}{h}\right)^2 . A \quad\text{.......................}\quad (3),$$

since similar areas are proportional to the squares on corresponding lines. Hence the volume, being equal to

$$\frac{1}{h^2} A \int_0^h x^2\,dx, \quad\text{or}\quad \tfrac{1}{3}hA \quad\text{..................}\quad (4),$$

is one-third the altitude into the area of the base.

Ex. 2. The volume of a tetrahedron is

$$\tfrac{1}{6}haa' \sin \alpha \quad\text{.........................}\quad (5),$$

where a, a' are any pair of opposite edges, h their shortest distance, and α the angle between their directions.

Divide the tetrahedron into laminæ by planes parallel to the edges a, a', and therefore perpendicular to the shortest distance h. It is evident, on reference to Fig. 65, that the section made by a plane of the system at a distance x from the edge a is a parallelogram whose sides are

$$\frac{x}{h} . a' \quad\text{and}\quad \frac{h-x}{h} a,$$

and whose area is therefore

$$\frac{aa'}{h^2} x(h-x) \sin \alpha.$$

Hence the volume $= \dfrac{aa}{h^2} \sin \alpha \displaystyle\int_0^h x(h-x)\,dx,$

which reduces to the value given above.

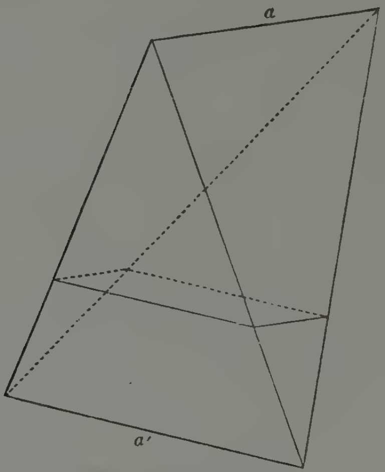

Fig. 65.

105. Solids of Revolution.

Let the equation of the generating curve be

$$y = \phi(x) \text{ .... } (1),$$

the axis of x being that of symmetry, and let the solid be bounded by plane ends perpendicular to x. In this case, the area $f(x)$, being that of a circle of radius y, is πy^2. Hence the required volume is

$$\int \pi y^2 dx \text{ .... } (2),$$

taken between proper limits. Each element of the sum, of which this integral is the limit, represents, in fact, the volume of a circular plate of thickness δx and area πy^2.

Ex. 1. The equation of a circle, referred to a point on its circumference as origin, is

$$y^2 = x(2a - x) \text{ .... } (3).$$

Hence the volume of a segment of a sphere, of height h, is

$$\pi \int_0^h x(2a - x)\, dx = \pi \left[ax^2 - \tfrac{1}{3}x^3 \right]_0^h$$

$$= \pi h^2 (a - \tfrac{1}{3}h) \text{ .... } (4),$$

a being the radius of the sphere. For the complete sphere, we have $h = 2a$, and the volume is $\frac{4}{3}\pi a^3$, or two-thirds the volume $(\pi a^2 \times 2a)$ of the circumscribed circular cylinder.

Ex. 2. The volume of a segment, of height h, of the paraboloid generated by the revolution of the curve

$$y^2 = 4ax \quad \text{..............................(5)}$$

about the axis of x, is

$$\pi \int_0^h y^2\,dx = 4\pi a \left[\tfrac{1}{2}x^2\right]_0^h = 2\pi a h^2 \quad \text{...........(6).}$$

If b be the radius of the base, we have $b^2 = 4ah$. Hence the volume is $\frac{1}{2}\pi b^2 . h$, or one-half that of the cylinder of the same height on the same base.

Ex. 3. To find the volume of the 'anchor-ring,' or 'tore,' generated by the revolution of the circle

$$x^2 + (y - a)^2 = b^2 \quad \text{........................(7)}$$

about the axis of $x\,[a > b]$. See Fig. 70, p. 272.

For each value of $x\,(< b)$ we have two values of y, say y_1, y_2; viz.

$$y_1 = a + \sqrt{(b^2 - x^2)}, \quad y_2 = a - \sqrt{(b^2 - x^2)} \quad \text{.........(8).}$$

The area of a section of the ring by a plane perpendicular to x is therefore

$$\pi {y_1}^2 - \pi {y_2}^2 = 4\pi a \sqrt{(b^2 - x^2)} \quad \text{..................(9)}$$

and the required volume is

$$4\pi a \int_{-b}^{b} \sqrt{(b^2 - x^2)}\,dx = 2\pi^2 a b^2 \quad \text{...........(10);}$$

by Art. 94, Ex. 5.

This is the same as the volume of a cylinder whose section (πb^2) is equal to that of the ring, and whose length $(2\pi a)$ is equal to the circumference of the circle described by the centre of the generating circle.

106. Some other Cases.

We give some further examples of the general formula (2) of Art. 104.

Ex. 1. The section of the elliptic paraboloid

$$2x = \frac{y^2}{p} + \frac{z^2}{q} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

by a plane $x=$ const. is an ellipse of semi-axes $\surd(2px)$ and $\surd(2qx)$, and therefore of area $2\pi\surd(pq)\,x$. Hence the volume of the segment cut off by the plane $x=h$ is

$$2\pi\surd(pq)\int_0^h x\,dx = \pi\surd(pq)\,h^2 \quad \ldots\ldots\ldots\ldots(2).$$

This is one-half the volume of a cylinder of the same height h on the same elliptic base.

Ex. 2. In the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

the section by a plane $x=$ const. is an ellipse of semi-axes

$$b\sqrt{\left(1-\frac{x^2}{a^2}\right)} \text{ and } c\sqrt{\left(1-\frac{x^2}{a^2}\right)} \quad \ldots\ldots\ldots (4),$$

and therefore of area

$$\pi bc\left(1-\frac{x^2}{a^2}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The volume included between any two planes perpendicular to x therefore

$$= \pi bc\int\left(1-\frac{x^2}{a^2}\right)dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

taken between the proper limits of x. For the whole volume the limits of x are $\pm a$, and the result is $\frac{4}{3}\pi abc$.

107. Simpson's Rule.

Most of the preceding results are virtually included in a general formula applicable to all cases where the area of the section by a plane perpendicular to x is a quadratic function of x, say

$$f(x) = A + B\frac{x}{h} + C\frac{x^2}{h^2} \quad \ldots\ldots\ldots\ldots\ldots (1).$$

Integrating this between the limits 0 and h, we find

$$\int_0^h f(x)\,dx = h\left(A + \tfrac{1}{2}B + \tfrac{1}{3}C\right) \quad \ldots\ldots\ldots\ldots (2).$$

Now A is evidently the area of the section $x=0$; and denoting the areas of the sections $x=\frac{1}{2}h$, and $x=h$, respectively, by A' and A'', we have

$$A+\tfrac{1}{2}B+\tfrac{1}{4}C=A', \quad A+B+C=A'' \ldots\ldots (3),$$

whence

$$\int_0^h f(x)\,dx=\tfrac{1}{6}h\,(A+4A'+A'') \ldots\ldots\ldots (4).$$

This gives the volume included between two parallel sections, in terms of the areas of these sections and of the section half-way between them.

The formula (1) is obviously applicable to the case of a cone, pyramid, or sphere, and also to the case of a paraboloid, ellipsoid, or hyperboloid, provided the bounding sections be perpendicular to a principal axis. The student who is familiar with the theory of surfaces of the second degree will easily convince himself, moreover, that the latter condition is not essential.

Another case coming under the present rule is that of a solid bounded by two parallel plane polygonal faces and by plane lateral faces which are triangles or trapeziums. We may even include the case where some or all of the lateral faces are curved surfaces (hyperbolic paraboloids) generated by straight lines moving parallel to the planes of the polygons, and each intersecting two straight lines each of which joins a vertex of one polygon to a vertex of the other (see Fig. 67). And since the number of sides in each polygon may be increased indefinitely, the rule will also apply to a solid bounded by any two plane

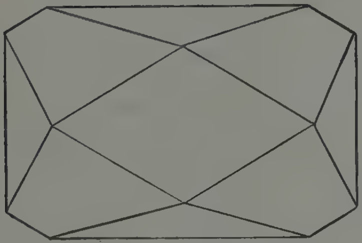

Fig. 66.

parallel faces and by a curved surface generated by a straight line which meets always the perimeters of those faces.

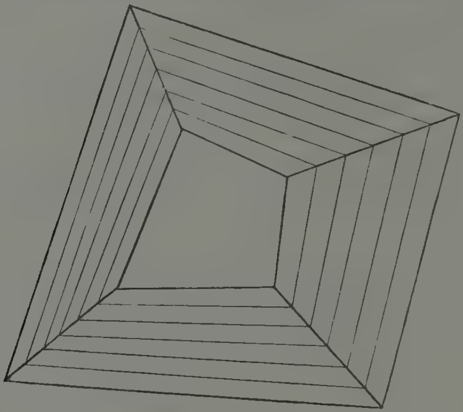

Fig. 67.

EXAMPLES*. XXXIII.

1. The volume generated by the revolution of one semi-undulation of the curve

$$y = b \sin x/a,$$

about the axis of x is one-half that of the circumscribing cylinder.

2. The volume of a frustum of any cone, with parallel ends, is

$$\tfrac{1}{3}h\{A_1 + \sqrt{(A_1A_2)} + A_2\},$$

where A_1, A_2 are the areas of the two ends, and h is the perpendicular distance between them.

3. In the solid generated by the revolution of the rectangular hyperbola

$$x^2 - y^2 = a^2$$

about the axis of x, the volume of a segment of height a, measured from the vertex, is equal to that of a sphere of radius a.

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* See the footnote on p. 252.

4. The volume of a segment of a sphere bounded by two parallel planes at a distance h apart exceeds that of a cylinder of height h and sectional area equal to the arithmetic mean of the areas of the plane ends, by the volume of a sphere of diameter h.

5. If a segment of a parabola revolve about the ordinate, the volume generated is $\frac{8}{15}$ of that of the circumscribing cylinder.

6. The volume of the solid generated by the revolution of a parabola about the tangent at the vertex is $\frac{1}{5}$ that of the circumscribing cylinder.

7. The volume of a frustum of a triangular prism cut off by any two planes is

$$\tfrac{1}{3}(h_1 + h_2 + h_3)A,$$

where h_1, h_2, h_3 are the lengths of the three parallel edges, and A is the area of the section perpendicular to these edges.

8. If b be the radius of the middle section of a cask, and a the radius of either end, prove that the volume of the cask is

$$\tfrac{1}{15}\pi(3a^2 + 4ab + 8b^2)h,$$

where h is the length, it being assumed that the generating curve is an arc of a parabola.

9. An arc of a circle revolves about its chord; prove that the volume of the solid generated is

$$\tfrac{4}{3}\pi a^3 \sin\alpha + \tfrac{2}{3}\pi a^3 \sin\alpha\cos^2\alpha - 2\pi a^3 \alpha\cos\alpha,$$

where a is the radius, and 2α is the angular measure of the arc.

10. The figure bounded by a quadrant of a circle of radius a, and the tangents at its extremities, revolves about one of these tangents; prove that the volume of the solid thus generated is

$$\left(\frac{5}{3} - \frac{\pi}{2}\right)\pi a^3.$$

11. The axis of a right circular cylinder of radius b passes through the centre of a sphere of radius $a\,(> b)$; prove that the volume of that portion of the sphere which is external to the cylinder is

$$\tfrac{4}{3}\pi(a^2 - b^2)^{\frac{3}{2}}.$$

12. The volume enclosed by two right circular cylinders of equal radius a, whose axes intersect at right angles, is $\frac{16}{3}a^3$.

If the axes intersect at an angle α, the volume is $\frac{16}{3}a^3 \operatorname{cosec} \alpha$.

13. If the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

revolve about the axis of x, the volume included between the surface thus generated, the cone generated by the asymptotes, and two planes perpendicular to x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.

14. A right circular cone of semi-angle α has its vertex on the surface of a sphere of radius a, and its axis passes through the centre. Prove that the volume of the portion of the sphere which is exterior to the cone is $\frac{4}{3}\pi a^3 \cos^4 \alpha$.

108. Rectification of Curved Lines.

The perimeter of a *rectilinear* figure is the length obtained by placing end to end in succession, in a straight line, lengths equal to the respective sides of the figure.

But since a *curved* line, however short, cannot be superposed on any portion of a straight line, we require some definition of what is to be understood by the 'length' of a curve. The definition usually adopted is that it is the limit to which the perimeter of an inscribed polygon tends as the lengths of the sides are indefinitely diminished. It will appear that, under proper conditions, this limit is unique; and it can also be shewn that it coincides with the corresponding limit for a circumscribed polygon.

If (x, y) and $(x + \delta x, y + \delta y)$ be the rectangular coordinates of two adjacent points P, Q on a curve, the length of the chord PQ is

$$\sqrt{\{(\delta x)^2 + (\delta y)^2\}}.$$

It has been shewn, in Art. 56, that if y and dy/dx be finite and continuous, the ratio $\delta y/\delta x$ is equal to the value of the derived function dy/dx for *some* point of the curve between P and Q. Hence, with a properly chosen value of dy/dx, we have

$$PQ = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, \delta x.$$

The limiting value of the perimeter of the inscribed polygon is therefore

$$\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx \;\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

taken between proper limits of x. The fact that this limiting value is unique has been established in Art. 87.

If x be regarded as a function of y, the corresponding formula is

$$\int \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy \;\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

If we denote by s the arc of the curve measured from some arbitrary point (x_0), and if as in Art. 53, we put $dy/dx = \tan\psi$, the formula (1) becomes

$$s = \int_{x_0}^{x} \sec\psi\, dx \;\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

There is of course a similar transformation of the formula (2).

Ex. 1. In the catenary

$$y = c\cosh\frac{x}{c} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

we have
$$\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int \sqrt{\left(1 + \sinh^2\frac{x}{c}\right)}\, dx$$
$$= \int \cosh\frac{x}{c}\, dx = c\sinh\frac{x}{c}.$$

Since this vanishes with x, the arc (s) measured from the lowest point is given by

$$s = c\sinh\frac{x}{c} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Ex. 2. In the parabola

$$y^2 = 4ax \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

we have
$$\int \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx = \int \sqrt{\left(\frac{x+a}{x}\right)}\, dx \;\ldots\ldots\ldots (7).$$

This may be integrated by the method of Art. 75, first rationalizing the numerator, or we may put

$$x = a\sinh^2 u,$$

and obtain
$$2a\int \cosh^2 u\,du = a\int (1+\cosh 2u)\,du$$
$$= a\,(u+\tfrac{1}{2}\sinh 2u) \quad\text{............}(8).$$

Since u vanishes with x, this gives the length of the arc measured from the vertex.

For example, at the end of the latus-rectum we have
$$\sinh u = 1, \quad \cosh u = \sqrt{2}, \quad u = \log(1+\sqrt{2}),$$
whence we find that the length of the arc up to this point is
$$a\{\log(1+\sqrt{2})+\sqrt{2}\} = 2{\cdot}296a.$$

109. Generalized Formulæ.

It is a consequence of the definition above given that any infinitely small arc PQ of a curve is ultimately in a ratio of equality to the chord PQ.

This may be verified immediately by differentiating the formula (3) of the preceding Art. with respect to the upper limit (x) of the integral. We thus find
$$\lim \frac{\delta s}{\delta x} = \sec\psi \quad\text{.....................}\ (1).$$

Since, when Q is taken infinitely near to P, $\sec\psi$ is the limiting value of the ratio of the chord PQ to δx, we have, ultimately,
$$\lim \frac{\delta s}{PQ} = 1 \quad\text{.......................}\ (2).$$

This leads to several important formulæ. In the first place, if the coordinates x, y of any point P on the curve be regarded as functions of the arc s, we have, in Fig. 26, p. 66,
$$\cos QPR = \frac{PR}{PQ} = \frac{\delta x}{\delta s}\cdot\frac{\delta s}{PQ}, \quad \sin QPR = \frac{QR}{PQ} = \frac{\delta y}{\delta s}\cdot\frac{\delta s}{PQ},$$
and therefore
$$\cos\psi = \frac{dx}{ds}, \quad \sin\psi = \frac{dy}{ds} \quad\text{............}\ (3).$$

The former of these results might have been written down at once from (1). It follows that
$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = 1 \quad\text{..................}\ (4).$$

Again, if x, y be functions of any other variable t, we have

$$PQ = \sqrt{\left\{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right\}}\,\delta t,$$

and therefore
$$\lim \frac{PQ}{\delta t} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}},$$

or
$$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \quad\ldots\ldots\ldots\ldots\ldots(5).$$

Hence
$$s = \int \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt \quad\ldots\ldots\ldots\ldots(6).$$

This may be regarded as a generalization of Art. 108 (1). The formula referred to was obtained on the supposition that there is only one value of y for each value of x, within the arc considered. The result (6) is free from this restriction; all that is essential is that as t increases, the point P should describe the curve continuously.

In the same way, the formulæ (3) may be taken to apply to any rectifiable curve, provided we understand by ψ the angle which the tangent, drawn in the direction of s increasing, makes with the positive direction of the axis of x.

The formulæ (5) and (6) have an obvious interpretation in Dynamics. If x, y be the rectangular coordinates of a moving point, regarded as functions of the time t, then dx/dt and dy/dt are the component velocities parallel to the coordinate axes, and if v be the actual velocity, we have

$$v = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

The formulæ (5), (6) are thus equivalent to

$$\frac{ds}{dt} = v, \quad s = \int v\,dt \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Ex. In the ellipse

$$x = a\sin\phi, \quad y = b\cos\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

we have
$$\left(\frac{ds}{d\phi}\right)^2 = \left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = a^2\cos^2\phi + b^2\sin^2\phi$$
$$= a^2(1 - e^2\sin^2\phi),$$

where e is the eccentricity. Hence the arc s, measured from the extremity of the minor axis, is

$$s = a\int_0^{\phi} \sqrt{(1 - e^2 \sin^2 \phi)}\, d\phi \quad\text{................(10)}.$$

This cannot be expressed (in a finite form) in terms of the ordinary functions of Mathematics. The integral is called an 'elliptic integral of the second kind,' and is denoted by $E(e, \phi)$. It may be regarded as a known function, having been tabulated by Legendre*. The whole perimeter of the ellipse is expressed by

$$4a\int_0^{\frac{1}{2}\pi} \sqrt{(1 - e^2 \sin^2 \phi)}\, d\phi \quad\text{..................(11)}.$$

The integral in this expression is denoted by $E(e, \frac{1}{2}\pi)$, or more shortly by $E_1(e)$. It is called a 'complete' elliptic integral of the second kind†. The quantity e is called the 'modulus' of the integral.

The calculation of the integral (11) by means of a series will be treated in Chap. XIII.

110. Arcs referred to Polar Coordinates.

Let OP, OP' be two consecutive radii of the curve, and let PN be drawn perpendicular to OP'. If we write

$$OP = r, \quad OP' = r + \delta r, \quad \angle POP' = \delta\theta,$$

then, as in Art. 55, PN will differ from $r\delta\theta$, and NP' from δr, by quantities which are infinitely small in comparison with PN, NP' respectively. Hence PP', or $\sqrt{(PN^2 + NP'^2)}$, is ultimately in a ratio of equality to

$$\sqrt{\{r^2(\delta\theta)^2 + (\delta r)^2\}}.$$

It follows that, if θ be the independent variable,

$$\frac{ds}{d\theta} = \lim \frac{PP'}{\delta\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \quad\text{............(1)},$$

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* *Traité des Fonctions Elliptiques*, t. ii. (1826).

† The elliptic integral of the 'first kind' is

$$\int_0^{\phi} \frac{d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)}}$$

and is denoted by $F(e, \phi)$. The corresponding 'complete' integral (with $\frac{1}{2}\pi$ as the upper limit) is denoted by $F_1(e)$.

and therefore

$$s = \int \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} \, d\theta \quad \ldots\ldots\ldots\ldots (2),$$

provided the integral be taken between the appropriate limits of θ.

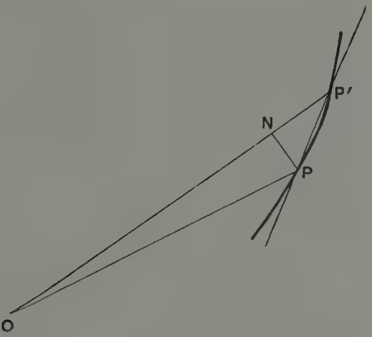

Fig. 68.

If r and θ be given as functions of an independent variable t, we have

$$\frac{ds}{dt} = \lim \frac{PP'}{\delta t} = \sqrt{\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\}} \quad \ldots\ldots\ldots (3),$$

and therefore

$$s = \int \sqrt{\left\{\left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2\right\}} \, dt \quad \ldots\ldots\ldots\ldots (4),$$

which includes (2) as a particular case.

Ex. In the circle $\qquad r = 2a\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$

we have $\qquad r^2 + \left(\dfrac{dr}{d\theta}\right)^2 = 4a^2,$

and therefore $\qquad \dfrac{ds}{d\theta} = 2a, \quad s = 2a\theta + \text{const.} \quad \ldots\ldots\ldots\ldots\ldots (6),$

as is otherwise obvious. See Fig. 39, p. 126.

Again, if r, θ be regarded as functions of the arc s, and if ϕ denote the angle which the positive direction of the radius vector makes with the tangent-line, drawn in the direction of s increasing, we have

$$\cos NP'P = \frac{NP'}{PP'}, \qquad \sin NP'P = \frac{PN}{PP'},$$

and therefore, in the limit,

$$\cos\phi = \frac{dr}{ds}, \qquad \sin\phi = r\frac{d\theta}{ds} \quad \text{............} \quad (7).$$

These results, again, have a dynamical illustration. If v denote the velocity of a moving point, the component velocities along and transverse to the radius are

$$\left.\begin{aligned} v\cos\phi &= \frac{dr}{ds}\frac{ds}{dt} = \frac{dr}{dt}, \\ \text{and} \qquad v\sin\phi &= \frac{rd\theta}{ds}\frac{ds}{dt} = \frac{rd\theta}{dt} \end{aligned}\right\} \quad \text{....................} (8),$$

respectively; and the formula (4) is equivalent to

$$s = \int v\,dt \quad \text{...........................} (9),$$

as before.

111. Areas of Surfaces of Revolution.

To frame a general definition of the area of a curved surface, and to prove that the area so defined has a determinate value, is a matter of some nicety. We shall here only consider the case of a surface of revolution limited (if at all) by planes perpendicular to the axis.

We begin with the case of a circular cylinder. The curved surface may be defined as the limiting value of the sum of the lateral faces of an inscribed prism. These faces have all the same length, and their sum is equal to this common length multiplied by the perimeter of the cross-section of the prism. In the limit this perimeter becomes the perimeter of the cylinder. Hence the curved surface of a right cylinder of radius a and height h is $2\pi ah$.

Take next the surface of a right cone included between two planes perpendicular to the axis. We can inscribe in this a frustum of a pyramid, whose bases are similar and similarly situated regular polygons, inscribed in the two bounding circles. The curved surface in question may be defined as the limit of the lateral area of this frustum. This area is made up of a number of trapeziums having a common altitude, viz. the perpendicular distance between their parallel sides, and is therefore equal to this common altitude multiplied by the arithmetic mean of the perimeters of the two polygons. In the limit, these perimeters become the circumferences of the bounding circles, and the common altitude becomes the length of a generating line of the cone included between these circles. In other words, the curved surface generated by the revolution of a straight line PQ about any axis in the same plane with it is equal to PQ multiplied by the arithmetic mean of the circumferences of the circles described by P and Q. This is the same as the product of PQ into the circumference of the circle described by its middle point.

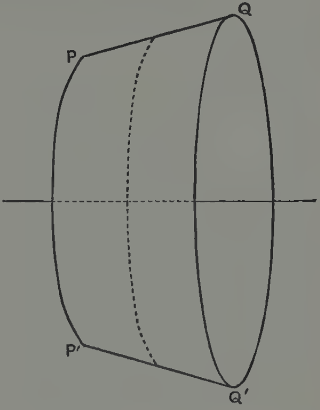

Fig. 69.

Next consider the surface generated by the revolution of any arc of a curve

$$y = \phi(x) \quad \text{.......................(1)},$$

about the axis of x. Taking any number of points in this arc, and joining them by straight lines, we obtain an open polygon; the curved surface is then defined as the limiting value to which the sum of the areas described by the sides of the polygon tends, when the lengths of the sides are indefinitely diminished. Hence if PQ be the chord of any element δs of the generating curve, and y the ordinate of the middle point of PQ, the curved surface is the limiting value of the sum $\Sigma(2\pi y \,.\, PQ)$. Ultimately, PQ is in a ratio of equality to δs, and y may be taken to be the ordinate of the curve; the surface is then equal to the limiting value of $2\pi\Sigma(y\,.\,\delta s)$, that is, to

$$2\pi\int y\,ds \qquad\text{.........................(2),}$$

taken over the proper range of s.

Ex. 1. In the case of the sphere the coordinates of any point of the generating curve may be written

$$x = a\cos\theta,\quad y = a\sin\theta \qquad\text{..................(3),}$$

whence

$$ds/d\theta = a \qquad\text{.........................(4).}$$

Hence the surface of a zone bounded by planes perpendicular to x is

$$2\pi a^2\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta = 2\pi a^2(\cos\theta_1 - \cos\theta_2)$$

$$= 2\pi a(x_1 - x_2) \qquad\text{..................(5).}$$

where the suffixes refer to the bounding circles. Hence the zone is equal in area to the corresponding zone of a circumscribing cylinder having its axis perpendicular to the planes of the bounding circles. In particular, the whole surface of the sphere is $2\pi a\,.\,2a$, or $4\pi a^2$.

Ex. 2. To find the surface of the ring generated by the revolution of a circle of radius b about a line in its plane at a distance a from its centre, we may put

$$x = b\sin\theta,\quad y = a - b\cos\theta,\quad ds/d\theta = b \qquad\text{.........(6),}$$

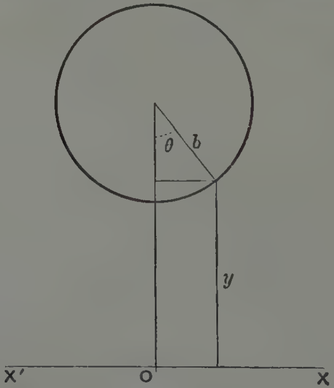

Fig. 70.

$= 25\pi$

and obtain $$2\pi\int y\,ds = 2\pi b\int(a - b\cos\theta)\,d\theta \quad\text{..............(7).}$$

The limits of θ being 0 and 2π, the result is $2\pi b \times 2\pi a$, which is equal to the curved surface of a right cylinder of radius b and length $(2\pi a)$ equal to the circumference described by the centre of the generating circle.

Ex. 3. To find the surface generated by the revolution of the ellipse
$$x = a\sin\phi, \quad y = b\cos\phi \quad\text{....................(8)}$$
about the *major* axis, we have
$$2\pi\int y\,ds = 2\pi\int y\frac{ds}{d\phi}\,d\phi$$
$$= 2\pi ab\int\sqrt{(1 - e^2\sin^2\phi)}\,d(\sin\phi) \quad\text{............(9),}$$
by Art. 109. If we put $e\sin\phi = \sin\theta$ ....(10),

this
$$= \frac{2\pi ab}{e}\int\cos^2\theta\,d\theta = \frac{\pi ab}{e}[\theta + \sin\theta\cos\theta] \quad\text{.........(11).}$$

To find the whole surface we take this between the limits $\phi = \mp\frac{1}{2}\pi$, or $\theta = \mp\sin^{-1}e$. The result is
$$\frac{2\pi ab}{e}\{\sin^{-1}e + e\sqrt{(1 - e^2)}\},$$
or
$$2\pi b^2 + 2\pi ab\frac{\sin^{-1}e}{e} \quad\text{.....................(12).}$$

By a similar process we find, for the surface generated by the revolution of the ellipse about its *minor* axis, the value
$$\frac{2\pi ab}{e'}\{\sinh^{-1}e' + e'\sqrt{(1 + e'^2)}\},$$
where $e' = \sqrt{(a^2 - b^2)}/b$, or
$$2\pi a^2 + 2\pi ab\frac{\sinh^{-1}e'}{e'} \quad\text{..................(13).}$$
This may also be put into the form
$$2\pi a^2 + \pi b^2 . \frac{1}{e}\log\frac{1 + e}{1 - e} \quad\text{..................(14).}$$

EXAMPLES[*]. XXXIV.

1. The length of a complete undulation of the curve of sines

$$y = b \sin x/a$$

is equal to the perimeter of an ellipse whose semi-axes are $\surd(a^2 + b^2)$ and a.

2. Prove the following formula for the length of the perpendicular (p) from the origin on any tangent to a curve.

$$p = x\frac{dy}{ds} - y\frac{dx}{ds}.$$

Also prove that the orthogonal projection of the radius vector on the tangent is

$$x\frac{dx}{ds} + y\frac{dy}{ds} \quad \text{or} \quad r\frac{dr}{ds}.$$

3. The surface generated by the revolution about the directrix of an arc of the catenary

$$y = c \cosh x/c,$$

commencing at the vertex, is

$$\pi(cx + ys),$$

where x, y, s refer to the extremity of the arc.

4. The curved surface cut off from a paraboloid of revolution by a plane perpendicular to the axis is

$$\tfrac{1}{6}\frac{\pi b}{h^2}\{(4h^2 + b^2)^{\frac{3}{2}} - b^3\},$$

where h is the length of the axis, and b the radius of the bounding circle.

5. The curved surface generated by the revolution about the axis of x of the portion of the parabola $y^2 = 4ax$ included between the origin and the ordinate $x = 3a$ is $\frac{56}{3}\pi a^2$.

6. The segment of a parabola included between the vertex and the latus-rectum revolves about the axis; prove that the curved surface of the figure generated is 1·219 times the area of its base.

---

[*] See the footnote on p. 252.

7. A circular arc revolves about its chord; prove that the surface generated is

$$4\pi a^2(\sin \alpha - \alpha \cos \alpha),$$

where a is the radius, and 2α the angular measure of the arc.

8. A quadrant of a circle of radius a revolves about the tangent at one extremity; prove that the area of the curved surface generated is $\pi(\pi - 2)a^2$.

9. A variable sphere of radius r is described with its centre on the surface of a fixed sphere of radius a; prove that the area of its surface intercepted by the fixed sphere is a maximum when $r = \frac{4}{3}a$.

112. Approximate Integration.

Various methods have been devised for finding an approximate value of a definite integral, when the indefinite integral of the function involved cannot be obtained. For brevity of statement, we will consider the problem in its geometrical form; viz. it is required to find an approximate value of the area included between a given curve, the axis of x, and two given ordinates.

The methods referred to all consist in substituting, for the actual curve, another which shall follow the same course more or less closely, whilst it is represented by a function of an easily integrable character.

The simplest, and roughest, mode is to draw n equidistant ordinates of the curve, and to join their extremities by straight lines. The required area is thus replaced by the sum of a series of trapeziums. If h be the distance between consecutive ordinates, and $y_1, y_2, \ldots, y_n$ the lengths of the ordinates, the sum of the trapeziums is

$$\tfrac{1}{2}(y_1+y_2)h+\tfrac{1}{2}(y_2+y_3)h+\ldots+\tfrac{1}{2}(y_{n-2}+y_{n-1})h+\tfrac{1}{2}(y_{n-1}+y_n)h$$
$$=(\tfrac{1}{2}y_1+y_2+y_3+\ldots+y_{n-2}+y_{n-1}+\tfrac{1}{2}y_n)h\ldots(1);$$

that is, we add to the arithmetic mean of the first and last ordinates the sum of the intervening ordinates, and multiply the result by the common interval h.

The value thus obtained will obviously be in excess if the curve is convex to the axis of x, and in defect in the opposite case.

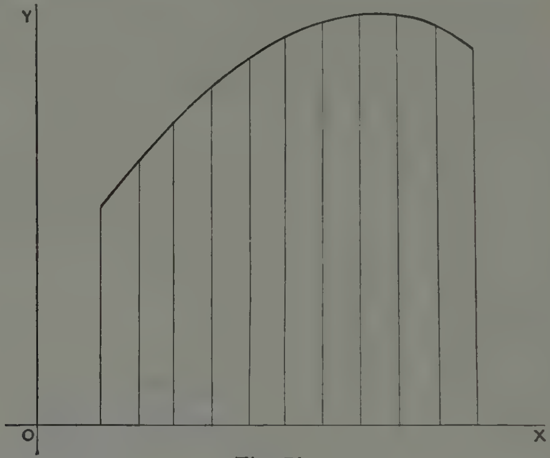

Fig. 71.

Another method, originally given by Newton and Cotes*, is to assume for y a rational integral expression of degree $n-1$, thus

$$y = A_0 + A_1 x + A_2 x^2 + \dots + A_{n-1} x^{n-1} \dots\dots\dots (2),$$

and to determine the coefficients $A_1, A_2, \dots A_{n-1}$ so that, for the n equidistant values of x, y shall have the prescribed values $y_1, y_2, \dots, y_n$. The area is then given by

$$\int y\,dx, = A_0 x + \tfrac{1}{2} A_1 x^2 + \tfrac{1}{3} A_2 x^3 + \dots + \frac{1}{n} A_{n-1} x^n \dots\dots\dots (3),$$

taken between proper limits of x.

Thus, in the case of *three* equidistant ordinates, taking the origin at the foot of the middle ordinate, we assume

$$y = A_0 + A_1 \frac{x}{h} + A_2 \left(\frac{x}{h}\right)^2 \dots\dots\dots\dots (4),$$

---

* See the latter's tract *De Methodo Differentiali*, printed as a supplement to the *Harmonia Mensurarum*, Cambridge, 1722.

with the conditions that

$$\text{for}\qquad \left.\begin{array}{llll} y = & y_1, & y_2, & y_3, \\ x = & -h, & 0, & h, \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$

respectively. These give

$$A_0 - A_1 + A_2 = y_1, \quad A_0 = y_2, \quad A_0 + A_1 + A_2 = y_3 \ldots(6),$$

so that

$$A_0 = y_2, \quad A_1 = \tfrac{1}{2}(y_3 - y_1), \quad A_2 = \tfrac{1}{2}(y_1 + y_3 - 2y_2) \ldots(7).$$

Hence

$$\int_{-h}^{h} y\,dx = 2\left(A_0 + \tfrac{1}{3}A_2\right)h$$
$$= \tfrac{1}{3}(y_1 + 4y_2 + y_3)h \quad \ldots\ldots\ldots\ldots(8).$$

The method here employed is equivalent to replacing the actual curve by an arc of a parabola having its axis vertical; and the result represents the difference between the trapezium

$$\tfrac{1}{2}(y_1 + y_3)\,.\,2h$$

and the parabolic segment

$$\tfrac{2}{3}\,.\,\left\{\tfrac{1}{2}(y_1 + y_3) - y_2\right\}.\,2h;$$

see Art. 98, Ex. 4.

In the case of *four* equidistant ordinates a similar process leads to the formula

$$\tfrac{3}{8}(y_1 + 3y_2 + 3y_3 + y_4)h \quad \ldots\ldots\ldots\ldots\ldots\ldots(9),$$

whilst for *five* ordinates we get

$$\tfrac{2}{45}(7y_1 + 32y_2 + 12y_3 + 32y_4 + 7y_5)h \quad \ldots\ldots\ (10).$$

With an increasing number of ordinates the coefficients in this method become more and more unwieldy*. A simple, but generally less accurate, rule was devised by Simpson†. Taking an odd number of ordinates, the areas between alternate ordinates, beginning with the first, are calculated from the 'parabolic' formula (8), and the results added. We thus obtain

---

* The coefficients for the cases $n=3, 4, 5, \ldots 11$, were calculated by Cotes; see also Bertrand, *Calcul Intégral*, Art. 363.

† *Mathematical Dissertations* (1743).

$$\begin{aligned}
&\tfrac{1}{3}\{y_1 + 4y_2 + y_3 \\
&\qquad\qquad + y_3 + 4y_4 + y_5 \\
&\qquad\qquad\qquad\qquad + y_5 + 4y_6 + y_7 \\
&\qquad\qquad\qquad\qquad\qquad\qquad + \ldots\ldots\ldots\ldots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + y_{2n-1} + 4y_{2n} + y_{2n+1}\}\, h \\
&= \tfrac{1}{3}\{(y_1 + y_{2n+1}) + 2(y_3 + y_5 + \ldots + y_{2n-1}) + 4(y_2 + y_4 + \ldots + y_{2n})\}\, h \qquad \ldots\ldots\ldots(11).
\end{aligned}$$

That is, we take the sum of the first and last ordinates, twice the sum of the intervening odd ordinates, and four times the sum of the even ordinates, and multiply one-third the aggregate thus obtained by the common interval h.

Ex. To calculate the value of π from the formula

$$\tfrac{1}{4}\pi = \int_0^1 \frac{dx}{1+x^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Dividing the range into 10 equal intervals, so that $h = \cdot 1$, we find

$$\begin{array}{lll}
y_1 = 1, & y_2 = \cdot 9900990, & y_3 = \cdot 9615385 \\
 & y_4 = \cdot 9174312, & y_5 = \cdot 8620690, \\
 & y_6 = \cdot 8000000, & y_7 = \cdot 7352941, \\
 & y_8 = \cdot 6711409, & y_9 = \cdot 6097561. \\
y_{11} = \cdot 5, & y_{10} = \cdot 5524862, &
\end{array}$$

Hence

$$\begin{aligned}
y_1 + y_{11} &= 1 \cdot 5, \\
y_3 + y_5 + y_7 + y_9 &= 3 \cdot 1686577, \\
y_2 + y_4 + y_6 + y_8 + y_{10} &= 3 \cdot 9311573.
\end{aligned}$$

The formula (11) then gives

$$\begin{aligned}
\tfrac{1}{4}\pi &= \tfrac{1}{30}(1 \cdot 5 + 6 \cdot 3373154 + 15 \cdot 7246292) \\
&= \cdot 78539815,
\end{aligned}$$

whence, retaining only seven figures,

$$\pi = 3 \cdot 141593.$$

This is correct to the last figure.

The formula (1) would have given

$$\tfrac{1}{4}\pi = \cdot 78498150, \quad \pi = 3\cdot 139926,$$

which is too small by about one part in 2000.

113. Mean Values.

Let $y_1, y_2, \ldots\ldots y_n$ be the values of a function $\phi(x)$ corresponding to n equidistant values of x distributed over the range $b-a$, say to the values of x which mark the middle points of the n equal intervals (h) into which this range may be subdivided. The limiting value to which the arithmetic mean

$$\frac{1}{n}(y_1 + y_2 + \ldots + y_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

tends, as n is indefinitely increased is called the 'mean value' of the function over the range $b-a$.

Since $h = (b-a)/n$, the expression (1) may be written

$$\frac{y_1 h + y_2 h + \ldots + y_n h}{b-a},$$

and the limiting value of this for $n = \infty$, $h = 0$, is

$$\frac{\int_a^b \phi(x)\,dx}{b-a} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

In the geometrical representation the mean value is the altitude of the rectangle on base $b-a$ whose area is equal to that included between the curve $y = \phi(x)$, the extreme ordinates, and the axis of x. See Fig. 52, p. 218.

The theorem of Art. 89, 3° may now be stated as follows: The mean value of a continuous function, over any range of the independent variable, is equal to the value of the function for some value of the independent variable within the range.

In applying the conception of a mean value it is essential to have a clear understanding as to what is the independent variable to which (in the first instance) the equal increments are given. Thus, in the case of a particle descending with a constant acceleration g, the mean value of the velocity in any interval of *time* t_1 from rest is

$$\frac{1}{t_1}\int_0^{t_1} v\,dt = \frac{1}{t_1}\int_0^{t_1} gt\,dt = \tfrac{1}{2}gt_1;$$

i.e. it is *one-half* the final velocity. But if we seek the mean velocity for equal infinitesimal increments of the *space* (s), we have, since $v^2 = 2gs$,

$$\frac{1}{s_1}\int_0^{s_1} v\,ds = \frac{(2g)^{\frac{1}{2}}}{s_1}\int_0^{s_1} s^{\frac{1}{2}}\,ds = \tfrac{2}{3}\,(2gs_1)^{\frac{1}{2}}\,;$$

i.e. it is *two-thirds* the final velocity.

Ex. 1. The mean value of $\sin\theta$ for equidistant intervals of θ ranging from 0 to π is

$$\frac{1}{\pi}\int_0^{\pi} \sin\theta\,d\theta = \frac{2}{\pi} = \cdot 6366.$$

Hence the mean value of the ordinates of a semicircle of radius a, drawn through equidistant points of the *arc*, is $\cdot 6366a$.

If the ordinates had been drawn through equidistant points on the *diameter*, the mean value would have been

$$\frac{1}{2a}\int_{-a}^{a} \sqrt{(a^2 - x^2)}\,dx = \tfrac{1}{2}a\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \cos^2\theta\,d\theta = \tfrac{1}{4}\pi a,$$

or $\cdot 7854a$. It is easily seen *à priori* why this latter mean should be the greater.

Ex. 2. To find the mean latitude of all places north of the equator.

The surface of the hemisphere is here supposed divided into infinitely narrow zones of equal area. If z denote distance from the plane of the equator, the area of the zone included between the planes z and $z + \delta z$ is proportional to δz. Hence, if a be the radius, the mean latitude is

$$\frac{1}{a}\int_0^{a} \sin^{-1}\frac{z}{a}\,dz = \int_0^{\frac{1}{2}\pi} \theta\cos\theta\,d\theta$$

$$= \Big[\theta\sin\theta + \cos\theta\Big]_0^{\frac{1}{2}\pi}$$

$$= \tfrac{1}{2}\pi - 1,$$

or, in degrees, $90° - 57{\cdot}296°, = 32{\cdot}704°$.

The various formulæ of Art. 112 may be interpreted as giving approximate expressions for the mean value of a function, over a given range, in terms of a series of values of the function taken at equidistant intervals covering the range. For example, in terms of three and of four such values, the mean values, as given by Cotes' method, are

$$\tfrac{1}{6}(y_1 + 4y_2 + y_3) \quad \text{and} \quad \tfrac{1}{8}(y_1 + 3y_2 + 3y_3 + y_4),$$

respectively.

EXAMPLES. XXXV.

1. Apply Simpson's rule to calculate $\log_e 2$ from the formula

$$\log_e 2 = \int_0^1 \frac{dx}{1+x}.$$

[The correct value is $\log_e 2 = \cdot 693147\ldots$]

2. Calculate the value of π from the formula

$$\tfrac{1}{6}\pi = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)}}.$$

3. The mean of the squares on the diameters of an ellipse, drawn at equal angular intervals, is equal to the rectangle contained by the major and minor axes.

4. A point is taken at random on a straight line of length a; prove that the mean area of the rectangle contained by the two segments is $\frac{1}{6}a^2$, and that the mean value of the sum of the squares on the two segments is $\frac{2}{3}a^2$.

5. If a point move with constant acceleration, the mean square of the velocities at equal infinitely small intervals of time is equal to

$$\tfrac{1}{3}(v_0{}^2 + v_0 v_1 + v_1{}^2),$$

where v_0, v_1 are the initial and final velocities.

6. Prove that in simple-harmonic motion the mean kinetic energy is one-half the maximum kinetic energy.

7. The mean horizontal range of a particle projected with given velocity, but arbitrary elevation, is $\cdot 6366$ of the maximum range.

8. A particle describes an ellipse with a velocity varying as the conjugate diameter. Prove that the mean kinetic energy is equal to the kinetic energy of the particle when at an extremity of an equiconjugate diameter.

9. A particle describes an ellipse with a velocity varying inversely as the perpendicular from one focus on the tangent line. Prove that the mean kinetic energy is equal to the kinetic energy of the particle when at either extremity of the minor axis.

10. The mean of the focal radii of an ellipse, drawn at equal angular intervals, is equal to the semi-minor axis.

11. The mean distance of points on the curved surface of a hemisphere from the plane of the base is one-half the radius.

12. If the orbits of comets were uniformly distributed through space, their mean inclination to the ecliptic would be equal to the radian (57·296°).

13. The mean distance of the points of a spherical surface of radius a from a point P at a distance c from the centre is

$$c + \tfrac{1}{3}\frac{a^2}{c} \quad \text{or} \quad a + \tfrac{1}{3}\frac{c^2}{a},$$

according as P is external or internal.

14. The mean distance of points on the circumference of a circle of radius a from a fixed point on the circumference is $1{\cdot}273a$.

15. The mean distance of points within a circular area of radius a from a fixed point on the circumference is $1{\cdot}132a$.

16. The mean distance of points within a sphere from a given point on the surface is $\frac{6}{5}a$.

*114. Multiple Integrals.

This book deals mainly with functions of a single variable, and therefore, as regards the Integral Calculus, with problems which depend, or can be made to depend, upon a single integration. Multiple integrals occur however so frequently, as a matter of notation, in the physical applications of the subject, that it may be useful to give here a few explanations concerning them. We shall pass very lightly over theoretical points; what is wanting in this respect may be supplied by a proper adaptation of the method of Art. 87.

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* This Art. may be postponed.

Let z be a continuous and single-valued function of the independent variables x, y; say

$$z = \phi(x, y) \qquad \text{......................(1).}$$

This may be interpreted, geometrically, as the equation of a surface (Art. 45). Take any finite region S in the plane xy, and let a cylindrical surface be generated by a straight line which meets always the perimeter of S, and is parallel to the axis of z. We consider the volume (V) included between this cylinder, the plane xy, and the surface (1). See Fig. 72.

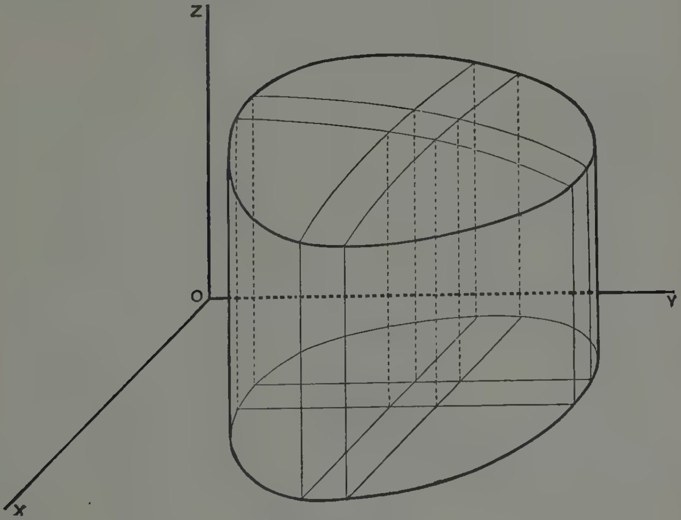

Fig. 72.

If the region S be divided into elements of area δA_1, δA_2, δA_3, ..., and if z_1, z_2, z_3, ... be ordinates of the surface (1) at arbitrarily chosen points within these elements, then, the coordinate axes being supposed rectangular, the sum

$$z_1\delta A_1 + z_2\delta A_2 + z_3\delta A_3 + \ldots \qquad \text{...............(2).}$$

will give us the total volume of a system of cylinders, of altitudes z_1, z_2, z_3, ..., standing on the bases δA_1, δA_2, δA_3, ...

And if the function $\phi(x, y)$ be subject to certain generally satisfied conditions*, the above sum will, when the dimensions of $\delta A_1, \delta A_2, \delta A_3, \ldots$ are taken infinitely small, tend to a unique limiting value, viz. the aforesaid volume V.

If the subdivision of the region S be made by lines drawn parallel to the axes of x and y, the elements $\delta A_1, \delta A_2, \delta A_3, \ldots$ are rectangular areas of the type $\delta x \delta y$, and the sum (2) may be denoted by

$$\Sigma\Sigma z \delta x \delta y \quad \text{.......................(3),}$$

where the sign Σ is duplicated because the summation is in two dimensions. The limiting value of this sum is denoted by

$$\iint z dx dy \quad \text{.........................(4),}$$

and we have the formula

$$V = \iint \phi(x, y)\, dx dy \quad \text{..................(5).}$$

The expression on the right hand is called a 'double integral'; it is of course not determinate unless the range of the variables x, y, as limited by the boundary of S, be specified.

The volume V may, however, be obtained in another way. If $f(x)$ denote the area of a section by a plane parallel to yz, whose abscissa is x, we have, by Art. 104,

$$V = \int_a^b f(x)\, dx \quad \text{....................(6),}$$

where a, b are the extreme values of x belonging to the area S. But, by Art. 98,

$$f(x) = \int_\alpha^\beta z dy \quad \text{......................(7),}$$

where α, β are the extreme values of y in the section $f(x)$, and are therefore in general functions of x. Hence we have

$$V = \int_a^b \left\{ \int_\alpha^\beta \phi(x, y)\, dy \right\} dx \quad \text{..............(8),}$$

---

* The already stipulated condition of continuity is sufficient, but the proof is simplified if we introduce the additional condition that $\phi(x, y)$ shall have only a finite number of maxima and minima within any finite area of the plane xy. Cf. Art. 87.

or, as it is more usually written,

$$V = \int_a^b \int_\alpha^\beta \phi(x, y)\, dx\, dy \quad\text{..............(9)*.}$$

If the limits of *both* integrations be constants, *i.e.* if the region S take the form of a rectangle having its sides parallel to x and y, the volume V is expressed also by

$$\int_\alpha^\beta \left\{\int_a^b \phi(x, y)\, dx\right\} dy \quad\text{...............(10),}$$

and we may assert that

$$\int_a^b \int_\alpha^\beta \phi(x, y)\, dx\, dy = \int_\alpha^\beta \int_a^b \phi(x, y)\, dy\, dx \quad\text{.....(11).}$$

This is illustrated by Fig. 30, p. 96. In other cases the limits of the respective integrations require to be adjusted when we invert the order.

The above explanations have been clothed in a geometrical form, but this is not of the essence of the matter. The same principles are involved, for example, in the calculation of the mass of a plane lamina, having given the density at any point (x, y) of it, and in many other physical problems. See Chapter VIII.

Another mode of decomposition of the area S is often useful. Taking polar coordinates r, θ in the plane xy, we may divide the area into quasi-rectangular elements by means of concentric circles and radii. The area of any one of these elements may be denoted by $r\delta\theta \,.\, \delta r$, if r be the arithmetic mean of the radii of the two curved sides. The formula (8) is then replaced by

$$V = \iint z r\, d\theta\, dr \quad\text{.....................(12),}$$

where z is supposed given as a function of r and θ.

After what precedes, the meaning of a 'triple-integral,'

$$\iiint \phi(x, y, z)\, dx\, dy\, dz \quad\text{...............(13),}$$

will not require much development. If a finite region R be divided into rectangular elements $\delta x\, \delta y\, \delta z$ by planes parallel to the coordinate axes, and if we multiply the volume of each of these elements by the value of the function $\phi(x, y, z)$ at some arbitrarily chosen point of it, the expression (13) is used to denote the limiting value to which (under certain conditions) the sum of the products tends when the dimensions of the elements are infinitely small. The same limiting value may be obtained by a succession of three simple integrations, thus

---

* Viz. the first $\int$ refers to the dx, and the second to the dy. There is not absolute uniformity of usage, however, on this point.

$$\int_a^b \left\{ \int_\alpha^\beta \left(\int_{\mathrm{a}}^{\mathrm{b}} \phi(x, y, z)\, dz \right) dy \right\} dx \quad \text{.........(14)},$$

where the integration is with respect to z between the limits a and b, which are in general functions of x and y, then with respect to y between the limits α and β, which are in general functions of x, and finally with respect to x between the limits a and b. If the limits of integration be all constants, they are unchanged when the order of integration is varied.

As an example, consider the determination of the mass of a solid, whose density is a function of x, y, z.

Ex. 1. To find the volume of the wedge included between the plane $z = 0$, the cylinder

$$x^2 + y^2 = a^2 \quad \text{........................(15)},$$

and the part of the plane $z = x \tan \alpha$ for which z is positive.

We have $$\iint z\,dx\,dy = \tan \alpha \int_0^a \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} x\,dx\,dy \quad \text{..........(16)}.$$

The integration with respect to y gives

$$\Big[xy \Big]_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} = 2x\sqrt{(a^2 - x^2)}.$$

We then have

$$\int_0^a 2x\sqrt{(a^2 - x^2)}\, dx = \Big[-\tfrac{2}{3}(a^2 - x^2)^{\frac{3}{2}} \Big]_0^a = \tfrac{2}{3}a^3.$$

The required volume is therefore

$$\tfrac{2}{3}a^3 \tan \alpha \quad \text{..........................(17)}.$$

Ex. 2. To find the volume included within the sphere

$$x^2 + y^2 + z^2 = a^2 \quad \text{.......................(18)},$$

by the cylinder $$x^2 + y^2 = ax \quad \text{..........................(19)}.$$

(The cylinder has a radius half that of the sphere, and its axis bisects at right angles a radius of the sphere.)

If we introduce polar coordinates in the plane xy, the equation (19) takes the form

$$r = a\cos\theta \quad\text{......................}(20),$$

and (18) gives $$z = \pm\sqrt{(a^2 - r^2)} \quad\text{......................}(21).$$

The required volume is therefore given by

$$2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{a\cos\theta} zr\,d\theta\,dr = 4\int_0^{\frac{1}{2}\pi}\int_0^{a\cos\theta}\sqrt{(a^2-r^2)}\,r\,d\theta\,dr \ldots(22).$$

Now

$$\int_0^{a\cos\theta}\sqrt{(a^2-r^2)}\,r\,dr = \Big[-\tfrac{1}{3}(a^2-r^2)^{\frac{3}{2}}\Big]_0^{a\cos\theta} = \tfrac{1}{3}a^3(1-\sin^3\theta),$$

and $$\int_0^{\frac{1}{2}\pi}(1-\sin^3\theta)\,d\theta = \tfrac{1}{2}\pi - \tfrac{2}{3}.$$

The final result is $$\tfrac{2}{3}\pi a^3 - \tfrac{8}{9}a^3 \quad\text{......................}(23).$$

CHAPTER VIII.

PHYSICAL APPLICATIONS.

115. Mean Density.

Many interesting applications of the process of integration present themselves in physical problems. They consist, often, in the calculation of the mean values (Art. 113) of various physical quantities.

The 'mean density' of any continuous distribution of matter is defined as the ratio of the total mass to the total volume which it occupies. Hence, if we have masses $m_1, m_2, \ldots, m_n$, distributed over volumes $v_1, v_2, \ldots v_n$, respectively, their mean densities $\rho_1, \rho_2, \ldots \rho_n$ are given by

$$\rho_1 = m_1/v_1, \ \rho_2 = m_2/v_2, \ldots \rho_n = m_n/v_n,$$

and the mean density of the whole is

$$\bar{\rho} = \frac{m_1 + m_2 + \ldots + m_n}{v_1 + v_2 + \ldots + v_n} = \frac{\rho_1 v_1 + \rho_2 v_2 + \ldots + \rho_n v_n}{v_1 + v_2 + \ldots + v_n} \quad \ldots(1).$$

If the volumes $v_1, v_2, \ldots v_n$ be all equal, this becomes

$$\bar{\rho} = \frac{1}{n}(\rho_1 + \rho_2 + \ldots + \rho_n) \quad \ldots\ldots\ldots\ldots\ldots(2),$$

or, the mean density of the whole is the arithmetic mean of the mean densities of the several parts.

A continuous distribution of matter is specified completely when the 'density at a point' is given as a function of the coordinates (x, y, z). Denoting this density by ρ, and an element of volume containing the point (x, y, z) by δV, the mass of the body is the limit of the sum

$$\Sigma(\rho\delta V),$$

and the mean density is given by

$$\bar{\rho} = \lim \frac{\Sigma(\rho\delta V)}{\Sigma(\delta V)} = \frac{\iiint \rho\, dx\, dy\, dz}{\iiint dx\, dy\, dz} \quad\text{..........}(3).$$

The above definitions relate to the case of 'volume-density.' If a finite mass be supposed concentrated in a surface, or in a line, we are led to the conceptions of 'surface-density' and 'line-density,' which may be defined in a similar manner.

Thus if σ be the surface-density of a plane film of matter, and δA an element of area, the mean surface-density is

$$\bar{\sigma}' = \lim \frac{\Sigma(\sigma\delta A)}{\Sigma(\delta A)} = \frac{\iint \sigma\, dx\, dy}{\iint dx\, dy} \quad\text{..............}(4).$$

And if μ be the line-density of a linear distribution of mass, and δs an element of the line, the mean line-density is

$$\bar{\mu} = \lim \frac{\Sigma(\mu\delta s)}{\Sigma(\delta s)} = \frac{\int \mu\, ds}{\int ds} \quad\text{.................}(5).$$

Ex. 1. To find the mean density of a semicircular lamina, whose density varies as the distance from the bounding diameter.

Taking the centre as origin, and the medial line as axis of x, we have

$$\bar{\sigma} = \frac{\int_0^a \sigma y\, dx}{\int_0^a y\, dx} \quad\text{.......................}(6),$$

where a is the radius, and $y = \sqrt{(a^2 - x^2)}$. Putting $\sigma = kx$, the numerator becomes

$$k\int_0^a x\sqrt{(a^2 - x^2)}\, dx = ka^3 \int_0^{\frac{1}{2}\pi} \sin\theta \cos^2\theta\, d\theta = \tfrac{1}{3}ka^3,$$

and the denominator is of course $= \frac{1}{4}\pi a^2$. Hence

$$\bar{\sigma} = \frac{4}{3\pi} \times ka = \cdot 4244 \times \text{maximum density} \ldots\ldots\ldots (7).$$

Ex. 2. If the density of a sphere be a function of the distance r from the centre, taking as the element of volume

$$\delta V = \delta\left(\tfrac{4}{3}\pi r^3\right) = 4\pi r^2 \delta r,$$

we have, if a be the external radius,

$$\bar{\rho} = \frac{4\pi \int_0^a \rho r^2 dr}{\frac{4}{3}\pi a^3} = \frac{3 \int_0^a \rho r^2 dr}{a^3} \ldots\ldots\ldots\ldots\ldots (8).$$

Thus, if $\rho \propto r^n$, we find that the mean density is $3/(n+3)$ of the density at the surface.

Again, assuming that in the Earth

$$\rho = \rho_0 \left(1 - k\frac{r^2}{a^2}\right) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9),$$

we find $$\bar{\rho} = \rho_0 \left(1 - \tfrac{3}{5}k\right) = \tfrac{1}{5}\left(2\rho_0 + 3\rho_1\right) \ldots\ldots\ldots\ldots (10),$$

where ρ_1 is the density at the surface ($r = a$). If the above law of density be really applicable (as it appears to be, approximately,) to the case of the Earth, then since $\bar{\rho} = 2\rho_1$, roughly, we infer that $\rho_0 = \frac{7}{2}\rho_1$, or the density at the centre is $3\frac{1}{2}$ times the density at the surface.

EXAMPLES. XXXVI.

1. If the line-density of a rod vary as the square of the distance from one end, the mean density is equal to the actual density at a distance of $1/\sqrt{3}$ of the length from that end.

2. The mean thickness of a circular plate, whose thickness varies as the nth power of the distance from the centre, is equal to $2/(n+2)$ of the thickness at the edge.

3. If, in a spherical mass whose density ρ is a function of the distance (r) from the centre, D denote the mean density of the matter included within a concentric sphere of radius r, then

$$\rho = D + \tfrac{1}{3} r \frac{dD}{dr}.$$

$M = DV$ [V : volume] $dM = V dD + D dV.$

$\rho\, 4\pi r^2 dr = \frac{4}{3}\pi r^3 dD + D\, 4\pi r^2 dr$

4. If the density of a solid hemisphere of radius a vary as the distance from the base, the mean density is equal to the density at a distance $\frac{3}{8}a$ from the base.

5. A disk has the form of a very flat oblate ellipsoid of revolution, prove that its mean thickness is two-thirds of the thickness at the centre.

6. A rod has the form of a very elongate prolate ellipsoid of revolution, prove that its mean sectional area is two-thirds that at the centre.

7. If the density at a distance r from the centre of the Earth be given by the formula

$$\rho = \rho_0 \frac{\sin kr}{kr},$$

where k is a constant, prove that the mean density is

$$3\rho_0 \frac{\sin ka - ka \cos ka}{k^3 a^3},$$

a denoting the Earth's radius.

116. Centre of Mass.

The 'centre of inertia,' or 'centre of mass,' of any system of particles is a geometrical point whose position may be defined in various ways*. For our present purpose it is sufficient to recall that the coordinates of the centre of mass of a system of particles $m_1, m_2, \ldots\, m_n$, situate at the points $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots\ldots (x_n, y_n, z_n)$, are given by

$$\bar{x} = \frac{\Sigma(mx)}{\Sigma(m)}, \quad \bar{y} = \frac{\Sigma(my)}{\Sigma(m)}, \quad \bar{z} = \frac{\Sigma(mz)}{\Sigma(m)} \quad \ldots\ldots(1),$$

whether the axes be rectangular or oblique. The summations Σ are supposed to include every particle of the system; for example $\Sigma(mx)$ stands for $m_1x_1 + m_2x_2 + \ldots + m_nx_n$.

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* The simplest and best being the vector definition proposed by Grassmann, *Ausdehnungslehre* (1844).

If the origin be at the centre of mass we have

$$\Sigma(mx) = 0,\ \Sigma(my) = 0,\ \Sigma(mz) = 0 \quad \ldots\ldots\ldots (2).$$

If the masses $m_1, m_2, \ldots m_n$ be all equal, the formulæ (1) reduce to

$$\bar{x} = \frac{1}{n}\Sigma(x),\quad \bar{y} = \frac{1}{n}\Sigma(y),\quad \bar{z} = \frac{1}{n}\Sigma(z) \quad \ldots\ldots (3).$$

If the masses $m_1, m_2, \ldots m_n$, though unequal, be commensurable, then each may be regarded as made up by superposition of a finite number of particles, all of the same mass, and the formulæ (3) will then still apply. And since incommensurable magnitudes may be regarded as the limits of commensurable magnitudes, the formulæ (1) may in all cases be interpreted as expressing that the distance of the centre of mass from any plane is in a sense equal to the mean distance of the whole mass from that plane.

An important principle, constantly made use of in the determination of centres of mass, is that any group of particles in the system may be supposed replaced by a particle equal in mass to the whole group, and situate at the mass-centre of the group. This is an easy deduction from (1).

In the case of a continuous distribution of matter, if we denote by δM an element of mass situate about the point (x, y, z), the formulæ become

$$\bar{x} = \lim \frac{\Sigma(x\delta M)}{\Sigma(\delta M)},\quad \bar{y} = \lim \frac{\Sigma(y\delta M)}{\Sigma(\delta M)},\quad \bar{z} = \lim \frac{\Sigma(z\delta M)}{\Sigma(\delta M)} \quad \ldots\ldots (4).$$

or

$$\bar{x} = \frac{\iiint x\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},\quad \bar{y} = \frac{\iiint y\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz},\quad \bar{z} = \frac{\iiint z\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz} \quad \ldots\ldots (5).$$

In the application to particular problems or classes of problems the integrations can be greatly simplified. This is illustrated in the following paragraphs.

117. Line-Distributions.

If μ be the line-density, and δs an element of the arc, we have

$$\bar{x} = \frac{\int x\mu ds}{\int \mu ds}, \quad \bar{y} = \frac{\int y\mu ds}{\int \mu ds} \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

We consider only plane curves, and take the axes of x, y always in the plane, so that $\bar{z} = 0$.

Ex. In the case of a uniform circular arc, if the origin be taken at the centre, and the axis of x along the medial line, we have $\bar{y} = 0$, by symmetry. Also, putting $\mu = 1$, as we may do, without loss of generality, we have, writing $x = a\cos\theta$, $\delta s = a\delta\theta$,

$$\bar{x} = \frac{\int_{-\alpha}^{\alpha} a\cos\theta \,.\, ad\theta}{\int_{-\alpha}^{\alpha} ad\theta} = \frac{\int_0^{\alpha} \cos\theta d\theta}{\alpha} \,.\, a = \frac{\sin\alpha}{\alpha} \,.\, a \ldots\ldots(2),$$

if 2α denote the angle which the whole arc subtends at the centre.

As α increases from an infinitely small value to π, $\bar{x}$ decreases from a to 0. For the semicircle, we have $\alpha = \frac{1}{2}\pi$, and

$$\bar{x} = \frac{2}{\pi} a = \cdot 637a.$$

118. Plane Areas.

If the surface-density be uniform, and if the area in question possess a line of symmetry, then taking this line as axis of x, we have, if y be the ordinate of the bounding line,

$$\bar{x} = \frac{\int x \,.\, ydx}{\int ydx}, \quad \bar{y} = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

the integrals being taken between the proper limits.

Ex. 1. For an isosceles triangle, taking the origin at the vertex, we may write $y = mx$, and therefore

$$\bar{x} = \frac{\int_0^h x^2 dx}{\int_0^h xdx} = \tfrac{2}{3}h \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

h being the altitude.

Ex. 2. For a semicircular area of radius a, we find

$$\bar{x} = \frac{\int_0^a x\sqrt{(a^2 - x^2)}\, dx}{\int_0^a \sqrt{(a^2 - x^2)}\, dx} \text{ ....(3).}$$

The integrations are exactly the same as in Art. 115, Ex. 1; and the result is

$$\bar{x} = \frac{4}{3\pi} a = \cdot 4244a \text{ ....(4).}$$

Ex. 3. For a segment of the parabola

$$y^2 = 4ax \text{ ....(5)}$$

bounded by the double ordinate $x = h$, we have

$$\bar{x} = \frac{\int_0^h xy\, dx}{\int_0^h y\, dx} = \frac{\int_0^h x^{\frac{3}{2}}\, dx}{\int_0^h x^{\frac{1}{2}}\, dx} = \tfrac{3}{5}h \text{ ....(6).}$$

The formulæ (1) will apply also to the case of oblique axes, provided the axis of x bisect all chords drawn within the area parallel to the axis of y. For instead of an elementary rectangle $y\delta x$ we have now an elementary parallelogram $y\delta x \sin\omega$, where ω is the inclination of the axes. The constant factor $\sin\omega$, occurring both in the numerator and in the denominator of the expression for $\bar{x}$, will cancel.

Thus, with the same integrations as in Exs. 1, 2, 3, above, we ascertain that the mass-centre of a triangle is in any median line, at a distance of $\frac{2}{3}$ of the length, from the vertex; that the mass-centre of a semi-ellipse bounded by any diameter lies in the conjugate semi-diameter, at a distance $4/3\pi$ of its length from the centre; and that the mass-centre of any segment of a parabola is in the diameter bisecting the bounding chord, and divides the breadth in the ratio 3 : 2.

In the case of any area included between a curve $y = \phi(x)$, the axis of x, and two bounding ordinates, the mass-centre of an elementary parallelogram $y\delta x \sin\omega$ will be at the point $(x, \frac{1}{2}y)$. Hence

$$\bar{x} = \frac{\int xy\,dx}{\int y\,dx}, \quad \bar{y} = \frac{\int \frac{1}{2}y^2dx}{\int y\,dx} \quad \text{...............(7)},$$

the factor $\sin\omega$ cancelling as before.

Ex. 4. The mass-centre of the area included between the parabola

$$y^2 = 4ax \quad \text{...........................(8)},$$

the axis of x, and any ordinate $x = h$, is given by

$$\bar{x} = \tfrac{3}{5}h, \quad \bar{y} = \frac{a^{\frac{1}{2}}\int_0^h x\,dx}{\int_0^h x^{\frac{1}{2}}dx} = \tfrac{3}{4}(ah)^{\frac{1}{2}} = \tfrac{3}{8}k \quad \text{......... (9)},$$

if k be the ordinate corresponding to $x = h$.

In *polar* coordinates, we may resolve any sectorial area into elementary triangles $\frac{1}{2}r^2\delta\theta$. The mass-centre of any one of these is ultimately at the point $(\frac{2}{3}r, \theta)$. Hence referring to rectangular axes, of which the axis of x coincides with the origin of θ, we have

$$\left.\begin{aligned} \bar{x} &= \frac{\int \frac{2}{3}r\cos\theta\,.\,\frac{1}{2}r^2d\theta}{\int \frac{1}{2}r^2d\theta} = \frac{\frac{1}{3}\int r^3\cos\theta\,d\theta}{\frac{1}{2}\int r^2d\theta}, \\ \bar{y} &= \frac{\int \frac{2}{3}r\sin\theta\,.\,\frac{1}{2}r^2d\theta}{\int \frac{1}{2}r^2d\theta} = \frac{\frac{1}{3}\int r^3\sin\theta\,d\theta}{\frac{1}{2}\int r^2d\theta} \end{aligned}\right\} \quad \text{......(10)},$$

the integrals being taken between proper limits of θ.

Ex. 5. In the case of a circular sector of angle 2α, taking the origin at the centre, and the initial line along the bisector of the angle, we have $\bar{y} = 0$, and

$$\bar{x} = \frac{\frac{1}{3}a^3\int_{-\alpha}^{\alpha}\cos\theta\,d\theta}{a^2\alpha} = \frac{2}{3}\frac{\sin\alpha}{\alpha}a \quad \text{..............(11)}.$$

If the surface-density be not uniform, it is sometimes convenient to take as the element of area the area included between two consecutive lines of equal density ($\sigma = \text{const.}$).

Ex. 6. For example, take the case of a semicircular lamina whose density varies as the distance from the bounding diameter (or, say, an infinitely thin wedge cut from a uniform solid sphere by two planes meeting in a diameter). With the same axes and the same notation as in Art. 115, Ex. 1, and putting $\sigma = kx$, as before, we have $\bar{y} = 0$, and

$$\bar{x} = \frac{\int_0^a x \,.\, kx \,.\, 2y dx}{\int_0^a kx \,.\, 2y dx} = \frac{\int_0^a x^2 \sqrt{(a^2 - x^2)}\, dx}{\int_0^a x \sqrt{(a^2 - x^2)}\, dx}$$

$$= \frac{a \int_0^{\frac{1}{2}\pi} \sin^2\theta \cos^2\theta \, d\theta}{\int_0^{\frac{1}{2}\pi} \sin\theta \cos^2\theta \, d\theta} = \frac{3\pi}{16} a = \cdot 589a \quad\ldots\ldots\ldots\ldots(12).$$

119. Mean Pressure. Centre of Pressure.

A system of parallel forces distributed continuously over a plane area will in general have a single resultant equal to their sum. The quotient of this resultant by the area is called the 'mean intensity' of the force over the area. The 'intensity at a point' is defined as the mean intensity over an infinitely small element of the area, including the point in question.

Thus, in Hydrostatics, we speak of the 'mean pressure-intensity' over an area, and of the 'pressure-intensity at a point' of the area.

If the pressure-intensity (p) at every point of an area be given, the total pressure is the limiting value of the sum

$$\Sigma (p \,.\, \delta A) \quad\ldots\ldots\ldots\ldots(1),$$

where δA is an element of the area. The mean pressure-intensity ($\bar{p}$, say) is then given by

$$\bar{p} = \lim \frac{\Sigma (p \,.\, \delta A)}{\Sigma (\delta A)} \quad\ldots\ldots\ldots\ldots(2).$$

In the limit the summations are to be replaced by integrals, thus

$$\bar{p} = \frac{\iint p \, dx \, dy}{\iint dx \, dy} \quad\ldots\ldots\ldots\ldots(3).$$

In a homogeneous liquid under gravity, the pressure-intensity varies as the depth below the free surface. It is therefore uniform over any horizontal area. In the case of an inclined area we may say that the pressure-intensity varies as the distance (x) from the straight line in which the plane of the area meets the plane of the free surface; or

$$p = kx \text{ .... (4).}$$

Hence, in this case,

$$\bar{p} = \lim \frac{\Sigma(kx \,.\, \delta A)}{\Sigma(\delta A)} = k\bar{x} \text{ .... (5),}$$

where $\bar{x}$ refers to the mass-centre of the area. That is, the mean intensity is equal to the intensity at the mass-centre of the area.

The 'centre of pressure' of a system of parallel forces distributed over a plane area is the point in which the line of action of the resultant pressure on the area meets the plane. Its position may be found, on statical principles, by taking moments about axes x, y in the plane of the area. Thus, denoting the coordinates of the centre of pressure by (ξ, η), we have

$$\left.\begin{aligned} \xi &= \lim \frac{\Sigma(x \,.\, p\,\delta A)}{\Sigma(p \,.\, \delta A)} = \lim \frac{\Sigma(x \,.\, p\,\delta A)}{\bar{p} \,.\, \Sigma(\delta A)} = \frac{\iint xp\,dx\,dy}{\bar{p}\iint dx\,dy} \\ \eta &= \lim \frac{\Sigma(y \,.\, p\,\delta A)}{\Sigma(p \,.\, \delta A)} = \lim \frac{\Sigma(y \,.\, p\,\delta A)}{\bar{p} \,.\, \Sigma(\delta A)} = \frac{\iint yp\,dx\,dy}{\bar{p}\iint dx\,dy} \end{aligned}\right\} \text{...(6).}$$

The axes here may be rectangular or oblique.

In the particular case of a liquid under gravity, if the axis of y be the intersection of the plane of the area with the free surface, we put $p = kx$, and therefore

$$\xi = \lim \frac{\Sigma(x^2 \,.\, \delta A)}{\bar{x}\Sigma(\delta A)}, \quad \eta = \lim \frac{\Sigma(xy \,.\, \delta A)}{\bar{x}\Sigma(\delta A)} \text{ ......(7);}$$

or, if we write $y\,\delta x \sin\omega$ for δA, where ω is the angle between the axes of x, y,

$$\xi = \frac{\int x^2 y\,dx}{\bar{x}\int y\,dx}, \quad \eta = \frac{\int xy^2\,dx}{\bar{x}\int y\,dx} \text{ .... (8).}$$

Ex. 1. To find the centre of pressure of a rectangle, immersed in a liquid, with one pair of sides horizontal.

Taking the axis of x along the line which bisects this pair of sides, we have obviously $\eta = 0$; and if we denote the distances (measured in the plane of the rectangle) of these sides from the free surface by $h \pm a$, we have

$$\xi = \frac{\int_{h-a}^{h+a} x^2 dx}{h \,.\, 2a} = h + \tfrac{1}{3}\frac{a^2}{h} \quad\text{..................(9).}$$

Hence the centre of pressure is beneath the centre of figure, at the distance $\frac{1}{3}a^2/h$. As h increases, this interval diminishes, as we should expect, since the pressure-intensity over the area becomes more and more nearly uniform.

Ex. 2. In the case of a triangle having a vertex in the free surface, and the opposite side horizontal, the origin is conveniently taken at this vertex, and the axis of x along the medial line. We have, then, $\eta = 0$, and, since the breadth of the triangle at any point varies as x,

$$\xi = \frac{\int_0^h x^3\, dx}{\int_0^h x^2 dx} = \tfrac{3}{4}h \quad\text{....................(10),}$$

where h is the length of the medial line.

Ex. 3. In the case of a triangle with one side in the surface, taking the origin at the middle point of this side, and the axis of x along the medial line, the breadth of the triangle will vary as $h - x$; so that

$$\xi = \frac{\int_0^h x^2 (h - x)\, dx}{\int_0^h x\,(h - x)\, dx} = \tfrac{1}{2}h \quad\text{................(11).}$$

Ex. 4. To find the centre of pressure of a semicircular area having its base in the free surface.

This is analytically the same problem as Art. 118, Ex. 6, and the result is

$$\xi = \tfrac{3}{16}\pi a = \cdot 589a \quad\text{....................(12).}$$

It may happen that the sum of a system of parallel forces distributed over an area is zero. In this case there is no single resultant, but (unless the forces balance) a 'couple.'

An example of this occurs in the theory of the flexure of beams. In a pure flexure the algebraic sum of the tensions at the various parts of a cross-section is zero, and the action across the section reduces to a couple. See Art. 130.

EXAMPLES. XXXVII.

✓ 1. Prove by integration that the mass-centre of a trapezium divides the line joining the middle points of the parallel sides in the ratio $2a + b : a + 2b$, where a, b are the lengths of the parallel sides.

✓ 2. The mass-centre of the area included between one semi-undulation of the curve

$$y = b \sin x/a$$

and the axis of x is at a distance $\frac{1}{8}\pi b$ from this axis.

3. The mass-centre of the area included between the curve

$$y = \frac{a^3}{a^2 + x^2},$$

first x = constant

y = constant

and the axis of x is at the point $(0, \frac{1}{4}a)$.

4. The coordinates of the mass-centre of the 'parabolic spandril' bounded by the curve

$$y^2 = 4ax,$$

the tangent at the vertex, and the line $y = k$, are

$$\tfrac{3}{10}h, \ \tfrac{3}{4}k,$$

where h is the abscissa of the ordinate k.

5. Prove that the mass-centre of the area of the circular spandril formed by a quadrant of a circle and the tangents at its extremities is at a distance $\cdot 2234a$ from either tangent, a being the radius.

6. The mass-centre of a segment of a circle of radius a is at a distance

$$\frac{2}{3} \frac{\sin^3 \alpha}{\alpha - \sin \alpha \cos \alpha} a$$

from the centre of the circle, 2α being the angular measure of the arc.

7. The mass-centre of the area included between the coordinate axes and the parabola

$$\left(\frac{x}{a}\right)^{\frac{1}{2}} + \left(\frac{y}{b}\right)^{\frac{1}{2}} = 1$$

is at the point $(\frac{1}{5}a, \frac{1}{5}b)$. [Put $x = a\sin^4\theta$, $y = b\cos^4\theta$.]

8. If the density (σ) of a semi-circular lamina be a function of the distance (r) from the centre, the distance of the centre of mass from the base is given by

$$\bar{x} = \frac{2}{\pi}\int_0^a \sigma r^2 dr \div \int_0^a \sigma r\, dr,$$

where a is the radius.

Hence shew that if $\sigma \propto r$, $\bar{x} = \cdot 4775a$, and that if $\sigma \propto r^2$, $\bar{x} = \cdot 5093a$.

9. If the density of a parabolic arc whose axis is vertical vary as the cosine of the inclination to the horizon, the coordinates of the mass-centre, referred to horizontal and vertical axes will be

$$\bar{x} = \tfrac{1}{2}(x_1 + x_2), \qquad \bar{y} = \tfrac{1}{6}(y_1 + 4y' + y_2),$$

where (x_1, y_1), (x_2, y_2) are the coordinates of the extremities of the arc, and y' is the ordinate half-way between y_1 and y_2.

10. A uniform rod of length l is bent into the form of a circular arc of radius (R) large compared with l. Prove that the displacement of the mass-centre is $\frac{1}{24}l^2/R$, approximately.

11. A triangular area is immersed in liquid, with one side in the free surface. Prove that if it be divided into two portions by a horizontal line through the centre of pressure, the resultant pressures on these two portions are equal.

12. The centre of pressure of a trapezium having one of the two parallel sides (a, b) in the surface divides the medial line in the ratio $a + 3b : a + b$, where a is the side which is in the surface.

13. The centre of pressure of a rhombus immersed in a liquid with a diagonal vertical and one angular point in the surface is at a depth equal to $\frac{7}{12}$ of the diagonal.

120. Mass-Centre of a Surface of Revolution.

If the axis of x coincide with the axis of symmetry, and if δs denote an element of arc of the generating curve, and y its ordinate, the annular element of surface will be represented as in Art. 111 by

$$2\pi y\,\delta s.$$

Hence if the surface-density be uniform, the position of the mass-centre of a zone of the surface included between planes perpendicular to x is given by

$$\bar{x} = \frac{\int x\,.\,2\pi y\,ds}{\int 2\pi y\,ds} = \frac{\int xy\,ds}{\int y\,ds} \;\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If the surface-density (σ) be a function of x, the formula must be replaced by

$$\bar{x} = \frac{\int x\,.\,\sigma\,.\,2\pi y\,ds}{\int \sigma\,.\,2\pi y\,ds} = \frac{\int \sigma xy\,ds}{\int \sigma y\,ds} \;\ldots\ldots\ldots\ldots(2).$$

Ex. 1. For a zone of a spherical surface, putting

$$x = a\cos\theta, \quad y = a\sin\theta, \quad \delta s = a\,\delta\theta \;\ldots\ldots\ldots\ldots(3),$$

we have

$$\bar{x} = a\,\frac{\int_\alpha^\beta \cos\theta\sin\theta\,d\theta}{\int_\alpha^\beta \sin\theta\,d\theta} = \tfrac{1}{2}a\,\frac{\cos^2\alpha - \cos^2\beta}{\cos\alpha - \cos\beta}$$

$$= \tfrac{1}{2}a\,(\cos\alpha + \cos\beta) = \tfrac{1}{2}\,(x_1 + x_2) \;\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

if α, β be the limits of θ, and x_1, x_2 the abscissæ of the bounding circles. Hence the mass-centre of the zone is on the axis, half-way between the planes of the bounding circles.

For example, the mass-centre of a uniform hemispherical surface bisects the axial radius.

These results might also have been inferred immediately from the equality of area of corresponding zones on the sphere and on an enveloping cylinder (Art. 111, Ex. 1).

Ex. 2. To find the mass-centre of a spherical 'lune,' *i.e.* of the area on the surface of a sphere included between two meridians.

The 'angle of the lune' is the angular distance between the two bounding meridians. We shall denote it by 2α, and the radius of the sphere by a.

Divide the lune into elementary lunes of equal infinitely small angle. The mass-centre of any one of these will be at a certain distance x from the centre of the sphere. Supposing the mass of each elementary lune to be transferred to its centre of mass, we obtain a uniform circular arc of radius x, and angle 2α. The mass-centre of this is in the bisecting radius, at a distance

$$x\frac{\sin\alpha}{\alpha}$$

from the centre, by Art. 117. Now in the case of the hemisphere $(\alpha=\frac{1}{2}\pi)$ this distance must be equal to $\frac{1}{2}a$, by Ex. 1, above. Hence $x=\frac{1}{4}\pi a$; and the centre of mass of the given lune is at a distance

$$\tfrac{1}{4}\pi a\frac{\sin\alpha}{\alpha} \quad\text{..........................(5)}$$

from the centre of the sphere.

Ex. 3. In the case of a thin hemispherical shell, whose thickness varies as the distance (x) from the plane of the rim, we have

$$\bar{x}=\frac{\int x^2y\,ds}{\int xy\,ds}=a\frac{\int_0^{\frac{1}{2}\pi}\cos^2\theta\sin\theta\,d\theta}{\int_0^{\frac{1}{2}\pi}\cos\theta\sin\theta\,d\theta}=\tfrac{2}{3}a \quad\text{..........(6)},$$

where a is the radius.

Ex. 4. A simple rule for finding the mass-centre of any portion of a uniform *spherical* surface may be obtained as follows.

Let δS denote any element of the superficial area, x its distance from a fixed plane through the centre. The distance of the centre of mass from this plane will be

$$\bar{x}=\lim\frac{\Sigma(x\,.\,\delta S)}{\Sigma(\delta S)} \quad\text{........................(7)}.$$

Now if θ be the angle which the normal to δS makes with the normal to the plane of reference, we have $x=a\cos\theta$, where a is the radius of the sphere, and therefore

$$x\delta S=a\cos\theta\,\delta S=a\delta\Sigma \quad\text{....................(8)},$$

where $\delta\Sigma$ is the orthogonal projection of the area δS on the plane of reference. Hence

$$\bar{x}=\frac{\Sigma}{S}a \quad\text{...........................(9)},$$

if S be the total area of the portion of the spherical surface considered, and Σ the area of its orthogonal projection on the plane of reference.

Thus, for a hemispherical surface, projecting on the plane of the bounding circle, we have $S = 2\pi a^2$, $\Sigma = \pi a^2$, and therefore

$$\bar{x} = \tfrac{1}{2}a \quad\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

as before.

For a spherical lune, of angle 2α, we project on the plane perpendicular to the central radius. The two bounding meridians project into the two halves of an ellipse whose semi-axes are a and $a \sin\alpha$. Hence $S = 4\alpha a^2$, $\Sigma = \pi a^2 \sin\alpha$, and therefore

$$\bar{x} = \tfrac{1}{4}\pi \cdot \frac{\sin\alpha}{\alpha} \cdot a \quad\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

as above.

121. Mass-Centre of a Solid.

In the case of a *homogeneous* solid, if the area of a section by a plane perpendicular to x be denoted as in Art. 104 by $f(x)$, the x-coordinate of the mass-centre of the volume included between two such sections is obviously given by the formula

$$\bar{x} = \frac{\int x f(x)\,dx}{\int f(x)\,dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

taken between the proper limits of x.

It will sometimes happen that the mass-centres of a system of parallel sections lie in a straight line; in this case, taking the straight line in question as axis of x, we have $\bar{y} = 0$, and $\bar{z} = 0$.

In the case of a solid of revolution, taking the axis of x coincident with the axis of symmetry, we have

$$f(x) = \pi y^2,$$

if y be the ordinate of the generating curve. Hence

$$\bar{x} = \frac{\int x y^2\,dx}{\int y^2\,dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

Ex. 1. In the case of a right circular cone, the origin being at the vertex, $f(x) \propto x^2$, so that

$$\bar{x} = \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx} = \tfrac{3}{4}h \text{ .... (3),}$$

if h be the altitude.

Ex. 2. For the segment of an elliptic paraboloid

$$2x = \frac{y^2}{p} + \frac{z^2}{q} \text{ .... (4)}$$

cut off by a plane $x = h$, since $f(x) \propto x$, as in Art. 106, Ex. 1, we have

$$\bar{x} = \frac{\int_0^h x^2\,dx}{\int_0^h x\,dx} = \tfrac{2}{3}h \text{ ....(5).}$$

Ex. 3. For a hemisphere of radius a, putting $y^2 = a^2 - x^2$, we have

$$\bar{x} = \frac{\int_0^a x(a^2 - x^2)\,dx}{\int_0^a (a^2 - x^2)\,dx} = \tfrac{3}{8}a \text{ ....(6).}$$

The same formula gives the position of the mass-centre of the half of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ ....(7),}$$

which lies on the positive side of the plane yz, since $f(x)$ in this case also varies as $a^2 - x^2$. See Art. 106, Ex. 2.

Ex. 4. In the case of the more general formula

$$f(x) = A + B\frac{x}{h} + C\frac{x^2}{h^2} \text{ ....(8),}$$

the x-coordinate of the mass-centre of the volume included between the planes $x = 0$ and $x = h$ is, by (1),

$$\bar{x} = \frac{\tfrac{1}{2}A + \tfrac{1}{3}B + \tfrac{1}{4}C}{A + \tfrac{1}{2}B + \tfrac{1}{3}C}h = \frac{2A' + A''}{A + 4A' + A''}h \text{ ....(9),}$$

where, as in Art. 107, A, A', A'' denote the areas of the sections $x=0$, $x=\frac{1}{2}h$, $x=h$, respectively. The distance of the centre of mass from the middle section is therefore

$$\bar{x}-\tfrac{1}{2}h=\frac{A''-A}{2(A+4A'+A'')}h\ \ldots\ldots\ldots\ldots\ldots\ldots(10).$$

This result has the same degree of generality as that of Art. 107.

The application of the formula (1) is easily extended to the case of oblique axes. Denoting by $f(x)$ the area of a section parallel to the plane yz, the appropriate element of volume is

$$f(x)\,\delta x\sin\lambda,$$

where λ is the inclination of the axis of x to the plane yz. The constant factor $\sin\lambda$, occurring both in the numerator and in the denominator of the expression for $\bar{x}$, will cancel, and we are left with the same form as before.

Ex. 5. In the case of a cone, or a pyramid, on a plane base, taking the origin O at the vertex, and the axis of x along the line joining O to the mass-centre G of the area of the base, the area of any section parallel to the base will vary as the square of its intercept x on OG. We are thus led as before to the result

$$\bar{x}=\tfrac{3}{4}h,$$

where h now $=OG$. Hence the mass-centre of the pyramid, or cone, is at a point H in OG, such that $OH=\frac{3}{4}OG$.

In a similar manner the investigations Exs. 2, 3, above, can be modified so as to apply to any segment of a paraboloid, and to a semi-ellipsoid cut off by any diametral plane.

For special forms of solid other methods of decomposition into elements will suggest themselves.

Ex. 6. Thus in the case of a 'spherical sector,' *i.e.* the portion cut out of a solid sphere by a right circular cone having its vertex at the centre, the volume of a thin spherical stratum of radius r is proportional to $r^2\delta r$. Also the distance of the mass-centre of this stratum from the vertex is, by Art. 120, Ex. 1,

$$\tfrac{1}{2}(r+r\cos a),=r\cos^2\tfrac{1}{2}a,$$

where α is the semi-angle of the cone. Hence the distance of the mass-centre of the sector from the vertex is

$$\bar{x} = \frac{\int_0^a r^3\,dr}{\int_0^a r^2\,dr} \cos^2 \tfrac{1}{2}\alpha = \tfrac{3}{4} a \cos^2 \tfrac{1}{2}\alpha \quad \ldots\ldots\ldots\ldots(11),$$

where a is the external radius.

For the hemisphere we have $\alpha = \frac{1}{2}\pi$, and $\bar{x} = \frac{3}{8}a$, as in Ex. 3, above.

Ex. 7. To find the mass-centre of a wedge cut from a solid sphere by two planes meeting in a diameter.

Let a be the radius of the sphere, and 2α the angle between the planes. Divide the wedge, by planes through the aforesaid diameter, into elementary wedges of infinitely small angle, and let x be the distance from the diameter of the mass-centre of any one of these. Transferring the mass of each elementary wedge to its centre of mass, we obtain, as in Art. 120, Ex. 2, a uniform circular arc, whose centre of mass will be at a distance from the centre equal to $(x \sin \alpha)/\alpha$. Since, for $\alpha = \frac{1}{2}\pi$, this must equal $\frac{3}{8}a$, we infer that

$$x = \frac{3\pi}{16} a,$$

in agreement with Art. 118, Ex. 6. Hence the distance, from the edge, of the mass-centre of the given wedge of angle 2α, is

$$\bar{x} = \frac{3\pi}{16} \cdot \frac{\sin \alpha}{\alpha} \cdot a \quad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

122. Solid of Variable Density.

In a solid of variable density, if the surfaces of equal density be parallel planes, and if $f(x)$ be the area of the section made by one of these planes, supposed expressed in terms of the intercept on the axis of x, we have, in place of Art. 121, (1),

$$\bar{x} = \frac{\int x\rho f(x)\,dx}{\int \rho f(x)\,dx} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For example, in the case of a solid of revolution, in which the surfaces of equal density are planes perpendicular to the axis (x), we have $f(x) = \pi y^2$, where y refers to the generating curve, and therefore

$$\bar{x} = \frac{\int \rho x y^2 dx}{\int \rho y^2 dx} \text{ .... } (2).$$

Ex. 1. Thus in the case of a hemisphere whose density ρ varies as the distance (x) from the bounding plane, writing $y^2 = a^2 - x^2$, we have

$$\bar{x} = \frac{\int_0^a x^2 (a^2 - x^2)\, dx}{\int_0^a x (a^2 - x^2)\, dx} = \tfrac{8}{15} a \text{ .... } (3).$$

For other laws of density, other methods of decomposition may suggest themselves. For example, when the density is a function of the distance from a fixed point, a decomposition into concentric spherical shells is indicated.

123. Theorems of Pappus.

1°. If an arc of a plane curve revolve about an axis in its plane, not intersecting it, the *surface* generated is equal to the length of the *arc* multiplied by the length of the path of its centre of mass.

Let the axis of x coincide with the axis of rotation, and let y be the ordinate of the generating curve. The surface generated in a complete revolution is, by Art. 111, equal to

$$2\pi \int y\, ds,$$

the integration extending over the arc. But if $\bar{y}$ refer to the mass-centre of the arc, we have

$$\bar{y} = \frac{\int y\, ds}{\int ds},$$

by Art. 117. Hence

$$2\pi \int y\, ds = 2\pi \bar{y} \times \int ds \text{ .... } (1),$$

which is the theorem.

2°. If a plane area revolve about an axis in its plane, not intersecting it, the *volume* generated is equal to the *area* multiplied by the length of the path of its centre of mass.

If δA be an element of the area, the volume generated in a complete revolution is

$$\lim \Sigma (2\pi y \,.\, \delta A).$$

But if $\overline{y}$ refer to the centre of mass of the area, we have

$$\overline{y} = \lim \frac{\Sigma (y\,\delta A)}{\Sigma (\delta A)},$$

by Art. 116. Hence

$$\lim \Sigma (2\pi y \,.\, \delta A) = 2\pi \overline{y} \times \lim \Sigma (\delta A) \text{.........} (2),$$

which is the theorem*.

The revolutions have been taken to be complete, but the restriction is obviously unessential.

Ex. 1. The ring generated by the revolution of a circle of radius b about a line in its own plane at a distance a from its centre.

The surface is $2\pi b \times 2\pi a, = 4\pi^2 ab$;

and the volume is $\pi b^2 \times 2\pi a, = 2\pi^2 ab^2$.

Cf. Art. 105, Ex. 3, and Art. 111, Ex. 2.

Ex. 2. A segment of the parabola $y^2 = 4ax$, bounded by the double ordinate $x = h$, revolves about this ordinate.

If $2k$ be the length of the double ordinate, the area of the segment is $\frac{4}{3}hk$, by Art. 98; and the distance of the centre of gravity from the ordinate is $\frac{2}{5}h$, by Art. 118. Hence the volume generated is

$$\tfrac{4}{3}hk \times \tfrac{4}{5}\pi h = \tfrac{16}{15}\pi h^2 k.$$

The theorems may be used, conversely, to find the mass-centre of a plane arc, or of a plane area, when the surface, or the volume, generated by its revolution is known independently.

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* These theorems are contained in a treatise on Mechanics by Pappus, who flourished at Alexandria about A.D. 300. They were given as new by Guldinus, *de centro gravitatis* (1635—1642). (Ball, *History of Mathematics*.)

Ex. 3. Thus, for a semicircular *arc* revolving about the diameter joining its extremities, we have

$$\pi a \times 2\pi\bar{y} = 4\pi a^2,$$

whence

$$\bar{y} = \frac{2}{\pi} a.$$

Again, for a semicircular *area* revolving about its bounding diameter,

$$\tfrac{1}{2}\pi a^2 \times 2\pi\bar{y} = \tfrac{4}{3}\pi a^3,$$

whence

$$\bar{y} = \frac{4}{3\pi} a.$$

Cf. Arts. 117, 118.

124. Extensions of the Theorems.

A similar calculation leads to a simple formula for the volume of a prism or a cylinder (of any form of cross-section) bounded by plane ends.

In the first place we will suppose that one of the ends, which we will call the base, is perpendicular to the length. Let P be any point of the base, and let z be the length of the ordinate PP' drawn parallel to the length, to meet the opposite end in P', and let $\bar{z}$ be the ordinate of the mass-centre of the oblique end. If δA, $\delta A'$ be corresponding elements of area at P and P', we have

$$\bar{z} = \lim \frac{\Sigma(z\,.\,\delta A')}{\Sigma(\delta A')} = \lim \frac{\Sigma(z\,.\,\delta A)}{\Sigma(\delta A)},$$

since δA, being the orthogonal projection of $\delta A'$, is in a constant ratio to it. Hence the volume of the solid

$$= \Sigma(z\,.\,\delta A) = \bar{z} \times \Sigma(\delta A) \ldots\ldots\ldots\ldots\ldots(1);$$

that is, it is equal to the area of the base multiplied by the ordinate of the mass-centre of the opposite face. It is easily seen (Art. 131) that this is the same as the ordinate drawn through the mass-centre of the base.

A prism or a cylinder with both ends oblique may be regarded as the sum or as the difference of two prisms or cylinders each having one end perpendicular to the length.

We infer that in all cases the volume is equal to the area of the cross-section multiplied by the distance between the mass-centres of the two ends.

Ex. The volume of the wedge-shaped solid cut off from a right circular cylinder by a plane through the centre of the base, making an angle α with the plane of the base, is

$$\tfrac{1}{2}\pi a^2 \times \frac{4}{3\pi} a \tan\alpha = \tfrac{2}{3} a^3 \tan\alpha\,;$$

cf. Art. 114, Ex. 1.

The theorems of Pappus may be generalized in various ways; but it may be sufficient here to state the following extension of the second theorem.

If a plane area, constant or continuously variable, move about in any manner in space, but so that consecutive positions of the plane do not intersect within the area, the volume generated is equal to

$$\int S\,d\sigma \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where S is the area, and $d\sigma$ is the projection of an element of the locus of the mass-centre of the area on the normal to the plane. If ds denote an element of this locus, and θ the angle between ds and the normal to the plane, the formula may also be written

$$\int S \cos\theta\, ds \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

This theorem is the three-dimensional analogue of the proposition of Art. 101, relating to the area swept over by a moving line. It is a simple corollary from the theorem above proved.

EXAMPLES. XXXVIII.

1. A quadrant of a circle revolves about the tangent at one extremity; prove that the distance of the mass-centre of the curved surface generated, from the vertex, is $\cdot 876a$.

2. The mass-centre of either half of the surface of an anchor-ring cut off by the equatorial plane is at a distance $2b/\pi$ from this plane, where b is the radius of the generating circle.

3. Two equal circular holes of angular radius α are made in a uniform thin spherical shell, and the angular distance of their centres is 2β. Prove that the distance of the mass-centre of the remainder from the centre of the sphere is

$$\tfrac{1}{2}a \sin^2 \alpha \sec \alpha \cos \beta,$$

where a is the radius.

4. A portion of a paraboloid of revolution is bounded by two planes perpendicular to the axis. Prove that the distance of the centre of mass of the solid thus defined from the middle point of its axis is

$$\frac{1}{6}\frac{a^2-b^2}{a^2+b^2}h,$$

where h is the length of the axis, and a, b are the radii of the two circular ends.

5. The distances from the centre of a sphere of radius a of the centres of mass of the two segments into which it is divided by a plane at a distance c from the centre of figure are

$$\frac{3}{4}\frac{(a \pm c)^2}{2a \pm c}.$$

6. By dividing a tetrahedron into plane laminæ parallel to a pair of opposite edges, as in Art. 104, Ex. 2, prove that the mass-centre bisects the line joining the middle points of these edges.

7. The figure formed by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of these tangents; prove that the distance of the mass-centre of the solid thus generated from the vertex is $\cdot 869a$.

8. A solid ogival shot has the form produced by rotating a portion APN of a parabolic area, where A is the vertex, and PN an ordinate, about PN; prove that the mass-centre divides the axis in the ratio $5 : 11$.

9. AP is an arc of a parabola beginning at the vertex, and PN is a perpendicular on the tangent at the vertex; prove that the mass-centre of the solid generated by the revolution of the figure APN about AN is at a distance from A equal to $\frac{5}{6}AN$.

10. A right circular cone is divided into two halves by a plane through the axis; prove that the distance from the axis of the mass-centre of either half is a/π, where a is the radius of the base.

11. The mass-centre of the volume included between two equal circular cylinders, whose axes meet at right angles, and the plane of these axes, is at a distance from this plane equal to $\frac{3}{8}$ of the common radius.

12. The mass-centre of a hemispherical shell whose inner and outer radii are a and b is at a distance

$$\frac{3}{8}\frac{(a+b)(a^2+b^2)}{a^2+ab+b^2}$$

from the centre.

13. The mass-centre of a hemisphere of radius a whose density varies as the nth power of the distance from the base is at a distance

$$\frac{(n+1)(n+3)}{(n+2)(n+4)}a$$

from the centre.

14. If the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
revolve about the axis of x, the mass-centre of the curved surface generated by either of the two halves into which the curve is divided by the axis of y is at a distance

$$\frac{2}{3}\frac{a^2+ab+b^2}{a+b}\cdot\frac{a}{b+a(\sin^{-1}e)/e}$$

from the centre, where e is the eccentricity, it being supposed that $b<a$.

Obtain the corresponding result when $b>a$.

15. Apply the theorems of Pappus to find the volume and the curved surface of a right circular cone, and of a frustum of such a cone.

16. A groove of semicircular section, of radius b, is cut round a cylinder of radius a; prove that the volume removed is

$$\pi^2ab^2-\tfrac{4}{3}\pi b^3.$$

Also that the surface of the groove is

$$2\pi^2ab-4\pi b^2.$$

17. A screw-thread of rectangular section is cut on a cylinder of radius R. Prove that the volume of one turn of the thread is $2\pi abR + \pi ab^2$, where a, b are the sides of the rectangle, b being that side which is at right angles to the surface of the cylinder.

18. The mass-centre of either half of the volume of an anchor-ring cut off by the equatorial plane is at a distance $4b/3\pi$ from the plane, where b is the radius of the generating circle.

125. Moment of Inertia. Radius of Gyration.

If in any system of particles the mass of each particle be multiplied by the square of its distance from a given line, the sum of the products thus obtained is called the 'moment of inertia' of the system with respect to that line. In symbols, if m_1, m_2, m_3, ... be the masses of the several particles, p_1, p_2, p_3, ... their distances from the line, and if I denote the moment of inertia, we have

$$I = m_1p_1^2 + m_2p_2^2 + m_3p_3^2 + \ldots = \Sigma(mp^2) \quad\ldots\ldots(1).$$

In the dynamical theory of the rotation of a solid body about a fixed axis it is shewn that the moment of inertia as above defined is the proper measure of the inertia of the body as regards rotation, just as the mass of the body measures its inertia in respect of translation. Thus if M be the mass of a body moving in a straight line with velocity u, its momentum is Mu; and if F be the extraneous force, we have

$$\frac{d}{dt}(Mu) = F \quad\ldots\ldots(2).$$

In like manner, if I be the moment of inertia of a body rotating about a fixed axis with angular velocity ω, its angular momentum is $I\omega$; and if G be the extraneous couple, we have

$$\frac{d}{dt}(I\omega) = G \quad\ldots\ldots(3).$$

The moment of inertia of a body about a given line is often most conveniently specified by means of a linear magnitude called the 'radius of gyration.' This is a quantity k such that

$$I = Mk^2 \quad\ldots\ldots(4),$$

where M is the total mass of the system.

It is evident from the dynamical principles above referred to that, as regards rotation about the fixed line, the body will behave exactly as if its mass were concentrated into a ring of radius k, having its axis coincident with that line*.

The formula (4) is equivalent to

$$k^2 = \frac{m_1 p_1^2 + m_2 p_2^2 + m_3 p_3^2 + \dots}{m_1 + m_2 + m_3 + \dots} = \frac{\Sigma (mp^2)}{\Sigma (m)} \quad \dots\dots(5).$$

Hence k^2 may be regarded as the mean square of the distances of all the particles of the system from the given line. It is easily seen that in calculating k^2 we may replace any group of the particles by a single mass equal to their sum and situate at a distance from the line equal to the radius of gyration of the group about that line.

126. Two-Dimensional Examples.

In the case of bodies whose mass is distributed over lines, surfaces, or volumes, and not condensed into isolated points, the summations in the formulæ (1) and (5) of Art. 125 must of course be replaced by integrations. We begin with a few simple examples in two dimensions.

Ex. 1. To find the radius of gyration of a uniform straight bar about a line through its centre perpendicular to its length.

If $2a$ be the length, we have†

$$k^2 = \frac{\int_{-a}^{a} x^2 dx}{\int_{-a}^{a} dx} = \tfrac{1}{3}a^2 \quad \dots\dots\dots\dots\dots\dots\dots(1).$$

The same result evidently holds for the radius of gyration of a rectangle about a line of symmetry, if $2a$ be the length perpendicular to this line.

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* Hence the name 'swing-radius' was proposed by Clifford, as the equivalent of 'radius of gyration.'

† The line-density, when constant, may be put equal to unity since, whatever its value, it appears in both numerator and denominator of the expression for k^2, and so cancels. A similar remark applies to many subsequent examples.

Ex. 2. The radius of gyration of a uniform circular wire of radius a about its axis is evidently a.

To find the radius of gyration about a diameter, we have

$$k^2 = \frac{4\int_0^{\frac{1}{2}\pi} a^2 \sin^2\theta \, . \, a d\theta}{2\pi a} = \tfrac{1}{2}a^2 \quad \text{.............. (2).}$$

Ex. 3. To find the radius of gyration of a circular plate about an axis through its centre, perpendicular to its plane.

If we divide the disk into concentric annuli, the area of one of these may be represented by $2\pi r\delta r$, and its radius of gyration by r. Hence

$$k^2 = \frac{\int_0^a r^2 \, . \, 2\pi r dr}{\pi a^2} = \tfrac{1}{2}a^2 \quad \text{..................(3).}$$

Ex. 4. To find the radius of gyration of a circular disk about a diameter, we make use of the result of Ex. 2; viz. that the square of the radius of gyration of the annulus $2\pi r\delta r$ is $\frac{1}{2}r^2$. Hence the final result will be $\frac{1}{2}$ of that in Ex. 3; or

$$k^2 = \tfrac{1}{4}a^2 \quad \text{..........................(4).}$$

To find the radius of gyration, about the axis of x, of the area included between a curve

$$y = \phi(x) \quad \text{........................(5),}$$

the axis of x, and two bounding ordinates, we may divide the area into elementary strips $y\delta x$. The square of the radius of gyration of a strip is $\frac{1}{3}y^2$, by Ex. 1, above. Hence

$$k^2 = \frac{\int \frac{1}{3}y^2 \, . \, y dx}{\int y dx} = \frac{\frac{1}{3}\int y^3 dx}{\int y dx} \quad \text{............... (6),}$$

the integrals being taken between the proper limits of x.

Ex. 5. To find the radius of gyration of an isosceles triangular area, about its line of symmetry.

Taking the origin at the vertex, and the axis of x along the line of symmetry, the equations of the two sides will be

$$y = \pm \frac{a}{h} x,$$

where a is half the base, and h the altitude. The radius of gyration for the whole triangle is evidently the same as for either half, whence

$$k^2 = \frac{\frac{1}{3}\int_0^h \frac{a^3}{h^3} x^3\, dx}{\frac{1}{2}ah} = \tfrac{1}{6}a^2 \text{ ....(7).}$$

The same result obviously holds for the radius of gyration of a rhombus about a diagonal.

Ex. 6. To find the radius of gyration of the area bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ....(8),}$$

about the axis of x, we have

$$k^2 = \frac{\frac{1}{3}\int_{-a}^{a} y^3\, dx}{\frac{1}{2}\pi ab} \text{ ....(9).}$$

If we put $x = a\cos\phi$, $y = b\sin\phi$, this becomes

$$k^2 = \frac{4}{3\pi} b^2 \int_0^{\frac{1}{2}\pi} \sin^4\phi\, d\phi = \tfrac{1}{4}b^2 \text{..............(10).}$$

Similarly, for the radius of gyration about the minor axis, we should find

$$k^2 = \tfrac{1}{4}a^2 \text{ ....(11).}$$

127. Three-Dimensional Problems.

The following problems in three dimensions are important.

Ex. 1. To find the radius of gyration of a uniform thin spherical shell of radius a about a diameter.

Take the origin at the centre, and the axis of x along the diameter in question. If the shell be divided into narrow zones by planes perpendicular to x, the area of any one of these may be denoted by $2\pi a\delta x$, by Art. 111, Ex. 1, and its radius of gyration by y. Hence

$$k^2 = \frac{\int_{-a}^{a} y^2 . 2\pi a\, dx}{4\pi a^2} = \frac{1}{a}\int_0^a (a^2 - x^2)\, dx = \tfrac{2}{3}a^2 \text{ ...... (1).}$$

Ex. 2. To find the radius of gyration of a uniform solid sphere about a diameter.

Dividing the volume into thin concentric shells, and using the result of Ex. 1, we have

$$k^2 = \frac{\int_0^a \frac{2}{3}r^2 \, . \, 4\pi r^2 dr}{\frac{4}{3}\pi a^3} = \frac{2}{5}a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

A general formula for the radius of gyration, about its axis, of a uniform solid of revolution, is easily obtained. Dividing the solid into circular laminæ by planes perpendicular to the axis, which is taken as axis of x, the volume of any one of these laminæ may be represented by $\pi y^2 \delta x$, where y is the ordinate of the generating curve, and the square of its radius of gyration by $\frac{1}{2}y^2$ (see Art. 126, Ex. 3). Hence

$$k^2 = \frac{\int \frac{1}{2}y^2 \, . \, \pi y^2 dx}{\int \pi y^2 dx} = \frac{\frac{1}{2}\int y^4 dx}{\int y^2 dx} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 3. To find the radius of gyration of a right circular cone about its axis, we put

$$y = \frac{a}{h}x,$$

where a is the radius of the base, and h the altitude. Thus

$$k^2 = \frac{1}{2}\frac{a^2}{h^2}\frac{\int_0^h x^4 dx}{\int_0^h x^2 dx} = \frac{3}{10}a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Ex. 4. For a solid sphere of radius a, we have

$$k^2 = \frac{\frac{1}{2}\pi \int_{-a}^a y^4 dx}{\frac{4}{3}\pi a^3} = \frac{3}{4a^3}\int_0^a (a^2 - x^2)^2 dx = \frac{2}{5}a^2 \quad \ldots\ldots(5),$$

as in (2), above. A similar result can be obtained for an ellipsoid of revolution.

Ex. 5. A plane area having a line of symmetry, revolves about a parallel axis in the same plane. To find the radius of gyration, about the axis of rotation, of the solid generated.

Let y be the distance of any point of the area from the axis of rotation, and let us write

$$y = a + \eta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6),$$

where a is the distance of the line of symmetry of the area from the axis. If δA be an element of the area, and k the required radius of gyration, we have

$$k^2 = \frac{\Sigma(y^2 \,.\, 2\pi y\delta A)}{\Sigma(2\pi y\delta A)} = \frac{\Sigma(y^3\delta A)}{\Sigma(y\delta A)}$$

$$= \frac{a^3\Sigma(\delta A) + 3a^2\Sigma(\eta\delta A) + 3a\Sigma(\eta^2\delta A) + \Sigma(\eta^3\delta A)}{a\Sigma(\delta A) + \Sigma(\eta\delta A)}.$$

Now, in consequence of the assumed symmetry of the area the sums $\Sigma(\eta\delta A)$ and $\Sigma(\eta^3\delta A)$ must be unaltered when the sign of η is reversed, and must therefore vanish. Also if κ be the radius of gyration of the given area about the line of symmetry, we have

$$\kappa^2 = \frac{\Sigma(\eta^2\delta A)}{\Sigma(\delta A)} \quad\text{........................ (7).}$$

Hence the above formula reduces to

$$k^2 = a^2 + 3\kappa^2 \quad\text{........................(8).}$$

The same result holds whenever the generating curve has a *centre*, *i.e.* a point such that any chord through it is bisected there*. The proof may be left to the reader.

For example, in the case of the solid ring whose sectional area is a circle of radius b, we have $\kappa^2 = \frac{1}{4}b^2$, and therefore

$$k^2 = a^2 + \tfrac{3}{4}b^2 \quad\text{........................(9),}$$

a result easily verified by a direct calculation.

A similar investigation applies to the case of the *surface* generated by the revolution of a plane arc, having a line of symmetry, about a parallel axis in its own plane. The only difference is that the element of area (δA) is now to be replaced by the element of arc (δs). The result has the same form (8) as before.

Thus the radius of gyration of the *surface* of the anchor-ring, about the axis, is given by

$$k^2 = a^2 + \tfrac{3}{2}b^2 \quad\text{........................(10).}$$

This, again, is easily verified independently.

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* Townsend, *Quart. Jour. Math.*, t. x. (1870), and t. xvi. (1879).

128. Mean-Square of the Distances of a System of Particles from a Plane.

With a view to further calculations we may introduce the conception of the mean square of the distance of the mass from a given plane, or from a given point. Thus if $x_1, x_2, x_3, \ldots$ be the distances of the particles $m_1, m_2, m_3, \ldots$, respectively, from a given plane we write

$$\overline{x^2} = \frac{m_1 x_1^2 + m_2 x_2^2 + m_3 x_3^2 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\Sigma(mx^2)}{\Sigma(m)} \quad \ldots\ldots(1),^*$$

and, if $r_1, r_2, r_3, \ldots$ be the distances from a given point,

$$\overline{r^2} = \frac{m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\Sigma(mr^2)}{\Sigma(m)} \quad \ldots\ldots (2).$$

Thus, in two dimensions, if k denote the radius of gyration about a line through the origin perpendicular to the plane of the system,

$$k^2 = \frac{\Sigma m(x^2 + y^2)}{\Sigma(m)} = \overline{x^2} + \overline{y^2} \quad \ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 1. In the case of a rectangle, taking the origin at the centre, and the axes of x, y parallel to the edges $(2a, 2b)$, we have

$$\overline{x^2} = \tfrac{1}{3}a^2, \quad \overline{y^2} = \tfrac{1}{3}b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

by Art. 126, Ex. 1, and therefore

$$k^2 = \tfrac{1}{3}(a^2 + b^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Ex. 2. In the case of a circular hoop, taking the origin at the centre, we have

$$\overline{x^2} = \overline{y^2} = \tfrac{1}{2}k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the equality of the first two members being due to the symmetry about the origin. Since k is evidently equal to the radius (a), we infer that

$$\overline{x^2} = \overline{y^2} = \tfrac{1}{2}a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Cf. Art. 126, Ex. 2.

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[*] The symbol $\overline{x^2}$ must not be confounded with $\bar{x}^2$. For example, in the case of two equal particles we have

$$\overline{x^2} = \tfrac{1}{2}(x_1^2 + x_2^2), \qquad \bar{x}^2 = \{\tfrac{1}{2}(x_1 + x_2)\}^2.$$

Ex. 3. Similarly, in the case of a uniform circular disk, we infer that

$$\overline{x^2} = \overline{y^2} = \tfrac{1}{4}a^2 \quad\text{..........................}(8),$$

since $k^2 = \frac{1}{2}a^2$, by Art. 126, Ex. 3.

In three dimensions, if k_x, k_y, k_z be the radii of gyration with respect to rectangular axes Ox, Oy, Oz, respectively, we have

$$k_x{}^2 = \overline{y^2} + \overline{z^2}, \quad k_y{}^2 = \overline{z^2} + \overline{x^2}, \quad k_z{}^2 = \overline{x^2} + \overline{y^2} \quad\text{......}(9).$$

Also, if r denote the distance of any particle from the origin, we have

$$\overline{r^2} = \frac{\Sigma(mr^2)}{\Sigma(m)} = \frac{\Sigma m(x^2+y^2+z^2)}{\Sigma(m)} = \overline{x^2} + \overline{y^2} + \overline{z^2} \text{...}(10).$$

Hence, and from (9),

$$k_x{}^2 + k_y{}^2 + k_z{}^2 = 2\overline{r^2} \quad\text{...............}(11).$$

Ex. 4. For a rectangular parallelepiped, taking the origin at the centre, and the axes of x, y, z parallel to the edges ($2a$, $2b$, $2c$), we have by a calculation similar to that of Art. 126, Ex. 1,

$$\overline{x^2} = \tfrac{1}{3}a^2, \quad \overline{y^2} = \tfrac{1}{3}b^2, \quad \overline{z^2} = \tfrac{1}{3}c^2 \quad\text{..............}(12),$$

and therefore

$$k_x{}^2 = \tfrac{1}{3}(b^2+c^2), \quad k_y{}^2 = \tfrac{1}{3}(c^2+a^2), \quad k_z{}^2 = \tfrac{1}{3}(a^2+b^2) \text{...}(13).$$

Ex. 5. In the case of a uniform thin spherical shell of radius a, taking the origin at the centre, we have

$$k_x = k_y = k_z,$$

by symmetry, and $\overline{r^2} = a^2$, whence

$$k_x{}^2 = k_y{}^2 = k_z{}^2 = \tfrac{2}{3}a^2 \quad\text{....................}(14),$$

as in Art. 127, Ex. 1.

Ex. 6. For a uniform solid sphere, we have

$$\overline{r^2} = \frac{\int_0^a r^2 \,.\, 4\pi r^2 dr}{\frac{4}{3}\pi a^3} = \tfrac{3}{5}a^2 \quad\text{.................}(15),$$

and therefore

$$k_x{}^2 = k_y{}^2 = k_z{}^2 = \tfrac{2}{5}a^2 \quad\text{...................}(16);$$

cf. Art. 127, Exs. 2, 4.

Ex. 7. In a uniform solid ellipsoid bounded by the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ ....}(17),$$

since the section by a plane perpendicular to x is an ellipse of area

$$\pi bc\left(1-\frac{x^2}{a^2}\right),$$

we have

$$\overline{x^2}=\frac{\pi bc\int_{-a}^{a} x^2\left(1-\frac{x^2}{a^2}\right)dx}{\frac{4}{3}\pi abc}=\tfrac{1}{5}a^2 \text{ ....}(18),$$

and similarly $\overline{y^2}=\frac{1}{5}b^2,\quad \overline{z^2}=\frac{1}{5}c^2.$

Hence $k_x{}^2=\frac{1}{5}(b^2+c^2),\quad k_y{}^2=\frac{1}{5}(c^2+a^2),\quad k_z{}^2=\frac{1}{5}(a^2+b^2)$...(19).

129. Comparison of Moments of Inertia about Parallel Axes.

The following theorems give a simple means of comparing moments of inertia about different parallel axes.

1°. The mean square of the distances of the particles of a system from any given plane exceeds the mean square of the distances from a parallel plane through the centre of mass by the square of the distance between these planes.

Let x_1, x_2, x_3, ... be the distances of the particles m_1, m_2, m_3, ..., respectively, from the first-mentioned plane; and let $\bar{x}$ denote the distance of the centre of mass. If we write

$$x_1=\bar{x}+\xi_1,\quad x_2=\bar{x}+\xi_2,\quad x_3=\bar{x}+\xi_3,\ \ldots,\ \ldots(1),$$

we have

$$\overline{x^2}=\frac{\Sigma(mx^2)}{\Sigma(m)}=\frac{\Sigma\{m(\bar{x}+\xi)^2\}}{\Sigma(m)}$$

$$=\frac{\Sigma(m).\bar{x}^2+2\bar{x}\,.\,\Sigma(m\xi)+\Sigma(m\xi^2)}{\Sigma(m)}.$$

Now $\Sigma(m\xi)=0$, by Art. 116; and the fraction $\Sigma(m\xi^2)/\Sigma(m)$ is denoted, in accordance with our previous notation, by $\overline{\xi^2}$. Hence

$$\overline{x^2}=\overline{\xi^2}+\bar{x}^2 \text{ ....}(2).$$

2°. The mean square of the distances of the particles of a system from a given line exceeds the mean square of the distances from a parallel line through the centre of mass by the square of the distance between these lines.

If the first-mentioned line be taken as axis of z, and if the coordinates be rectangular, we have, by the preceding case,

$$\overline{x^2+y^2} = \overline{x^2} + \overline{y^2} = \overline{\xi^2+\eta^2} + (\bar{x}^2 + \bar{y}^2) \quad \text{.........(3).}$$

Hence if k denote the radius of gyration of the system about an axis through the centre of mass, and k' the radius of gyration about any parallel axis, we have

$$k'^2 = k^2 + h^2 \quad \text{.........................(4),}$$

where h is the perpendicular distance between the two axes.

This is a very important result in the dynamical application of the subject.

Ex. 1. The radius of gyration of a rectangle about a side is given by

$$k'^2 = a^2 + \tfrac{1}{3}a^2 = \tfrac{4}{3}a^2 \quad \text{...................... (5),}$$

if $2a$ be the length perpendicular to that side. This may be easily verified by direct integration.

Ex. 2. With the same notation as in Art. 128, Ex. 4, the radius of gyration of a rectangular parallelepiped about an edge $(2c)$ is given by

$$k'^2 = (a^2+b^2) + \tfrac{1}{3}(a^2+b^2) = \tfrac{4}{3}(a^2+b^2) \quad \text{...........(6).}$$

Ex. 3. The radius of gyration of a uniform circular disk of radius a about an axis through a point on the circumference, normal to the plane of the disk, is given by

$$k'^2 = a^2 + \tfrac{1}{2}a^2 = \tfrac{3}{2}a^2 \quad \text{........................(7).}$$

Similarly, the radius of gyration about a tangent line is given by

$$k'^2 = a^2 + \tfrac{1}{4}a^2 = \tfrac{5}{4}a^2 \quad \text{.......................(8).}$$

130. Application to Distributed Stresses.

The calculations of radii of gyration of plane areas have an application in the theory of stresses distributed over plane areas.

Thus, in determining the centre of pressure of an area in contact with a homogeneous liquid, if the axis of y be the line in which the plane of the area cuts the free surface, the x-coordinate of the centre of pressure is, by Art. 119,

$$\xi = \lim \frac{\Sigma(x^2\delta A)}{\Sigma(x\delta A)} \quad\text{.................. (1).}$$

In our present notation, we have

$$\Sigma(x^2\delta A) = \overline{x^2} \times \Sigma(\delta A), \quad \Sigma(x\delta A) = \bar{x} \times \Sigma(\delta A),$$

ultimately, and therefore

$$\xi = \frac{\overline{x^2}}{\bar{x}} \quad\text{.......................... (2).}$$

Ex. In the case of a circular area, having its centre at a distance h from the line in which its plane meets the surface, we have

$$\overline{x^2} = h^2 + \tfrac{1}{4}a^2, \quad \bar{x} = h,$$

and therefore

$$\xi = h + \tfrac{1}{4}\frac{a^2}{h} \quad\text{..........................(3).}$$

Again, in the theory of flexure, referred to in Art. 119, the intensity of the force at any point of the cross-section of a beam is equal to

$$\frac{Ey}{R} \quad\text{..............................(4),}$$

where y is the distance from a certain line in the plane of the section, called the 'neutral line,' R is the radius of the curve into which the beam is bent, and E is a certain coefficient depending on the material. If δA be an element of area, the total force across the section is the limit of

$$\frac{E}{R}\Sigma(y\delta A) \quad\text{..........................(5).}$$

In a pure flexure, this force is, by hypothesis, zero; hence, by Art. 116, the neutral line will pass through the mass-centre of the section.

The stresses on the cross-section now reduce to a couple. The moment of this couple about the neutral line (or about any line parallel to it) is got by multiplying the force on each element

δA by its distance from the neutral line. In this way we get, as the value of the 'flexural couple,'

$$\frac{E}{R} \times \lim \Sigma (y^2 \delta A) \quad \text{..................} \quad (6),$$

or

$$EAk^2/R \quad \text{..........................} \quad (7),$$

where A is the area of the cross-section, and k is its radius of gyration about the neutral line.

The ratio of the flexural couple to the curvature $(1/R)$ is called the 'flexural rigidity' of the bar. For bars of the same material it varies as Ak^2.

131. Homogeneous Strain in Two Dimensions.*

Taking first the case of two dimensions, let us suppose that, in any plane figure, the rectangular coordinates of a point (x, y) are changed to (x', y'), where

$$x' = \alpha x, \quad y' = \beta y \quad \text{..................} \quad (1),$$

α and β being given constants. The resulting deformation is of the kind called 'homogeneous strain'; the coordinate axes are called the 'principal directions' of the strain; and the constants α, β are called the 'principal ratios.'

A particular case is the method of 'orthogonal projection.' If the axis of x be the common section of the two planes, we have $\alpha = 1$, $\beta = \cos\theta$, where θ is the inclination of the plane of the original figure to the plane of projection.

Since the substitution (1) is of the first degree, it follows that straight lines will transform into straight lines. Also, since infinitely distant points transform into infinitely distant points, parallel straight lines will transform into parallel lines, and therefore parallelograms into parallelograms. Hence, further, equal and parallel straight lines will transform into equal and parallel straight lines; so that lines having originally any given direction are altered in a constant ratio, the ratio varying however (in general) with the direction. The new direction of a straight line is of course in general different from the original direction.

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* This is the same as Rankine's 'Method of Parallel Projection,' *Applied Mechanics*, Arts. 61, 82, 580.

Again, any algebraic curve whatever transforms into a curve of the same degree. In particular, a circle

$$x^2 + y^2 = a^2 \text{.......................} (2),$$

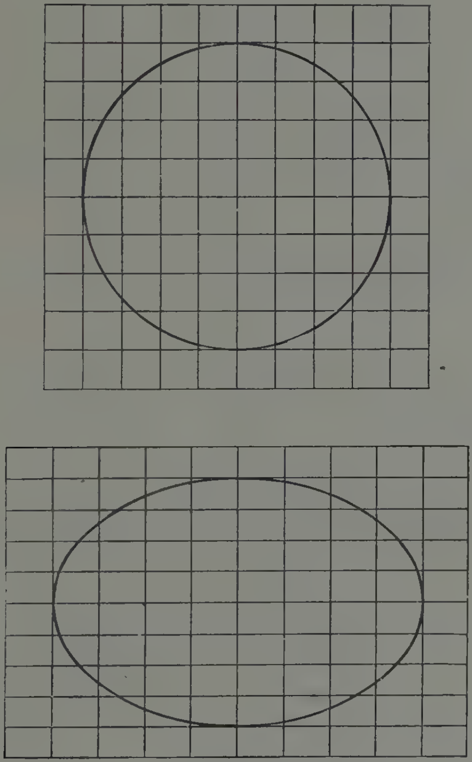

Fig. 73.

transforms into an ellipse

$$\frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} = 1, \text{.....................} (3),$$

where

$$a' = \alpha a, \quad b' = \beta a \text{; ....} (4),$$

and it is evident that by a proper choice of the ratios α, β a circle can be transformed into an ellipse of any given dimensions, and *vice versâ*. Also since a system of parallel chords, and the diameter bisecting them, transform into a system of parallel chords, and the diameter bisecting them, it is evident that perpendicular diameters of the circle transform into conjugate diameters of the ellipse.

Further, areas are altered by transformation in the constant ratio $\alpha\beta$. For this is evidently true of any rectangle having its sides parallel to the principal directions of the strain; and any area whatever can be approximated to as closely as we please by the sum of a system of rectangles of this type.

Ex. 1. Thus, the area of the ellipse (3) is $\alpha\beta$ times that of the circle (2); and so

$$= \alpha\beta \,.\, \pi a^2 = \pi \,.\, \alpha a \,.\, \beta a = \pi a' b'.$$

Again, a chord cutting off a segment of constant area from a circle touches a fixed concentric circle. Hence, a chord cutting off a segment of constant area from an ellipse touches a similar, similarly situated, and concentric ellipse.

Again, centres of mass of *areas*, considered as sheets of matter of uniform surface-density, transform into centres of mass. For, if δA, $\delta A'$ be corresponding elements of area, we have

$$\overline{x'} = \lim \frac{\Sigma\,(x'\delta A')}{\Sigma\,(\delta A')} = \alpha \times \lim \frac{\Sigma\,(x\delta A)}{\Sigma\,(\delta A)},$$

since $$\delta A' = \alpha\beta\,\delta A.$$

Hence, and by similar reasoning,

$$\overline{x'} = \alpha\overline{x}, \quad \overline{y'} = \beta\overline{y} \quad \text{..................} \quad (5).$$

Ex. 2. The centre of mass of a semicircular area is on the radius perpendicular to the bounding diameter, at a distance $4/3\pi$ of its length from the centre. Hence, the mass-centre of a semi-ellipse, bounded by any diameter, lies on the conjugate semi-diameter, at a distance of $4/3\pi$ of its length from the centre.

Finally, mean squares of distances from the axes of x, y transform into mean squares of distances. Thus

$$\overline{x'^2} = \lim \frac{\Sigma(x'^2\delta A')}{\Sigma(\delta A')} = \alpha^2 \times \lim \frac{\Sigma(x^2\delta A)}{\Sigma(\delta A)}.$$

Hence, and by similar reasoning,

$$\overline{x'^2} = \alpha^2\overline{x^2}, \quad \overline{y'^2} = \beta^2\overline{y^2} \;\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Ex. 3. The mean squares of the distances of points within the circle (2) from the coordinate axes are

$$\overline{x^2} = \tfrac{1}{4}a^2, \quad \overline{y^2} = \tfrac{1}{4}a^2.$$

Hence, for the ellipse (3),

$$\overline{x^2} = \tfrac{1}{4}\alpha^2 a^2 = \tfrac{1}{4}a'^2, \quad \overline{y^2} = \tfrac{1}{4}\beta^2 a^2 = \tfrac{1}{4}b'^2 \;\ldots\ldots\ldots\ldots(7).$$

The radius of gyration of an elliptic area about a line through the centre normal to the plane of the area is therefore given by

$$k^2 = \tfrac{1}{4}(a'^2 + b'^2) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),$$

where a', b' are the principal semi-axes.

132. Homogeneous Strain in Three Dimensions.

There is a similar method of transformation in three dimensions, the formulæ of transformation being now

$$x' = \alpha x, \quad y' = \beta y, \quad z' = \gamma z \;\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where the axes are supposed rectangular.

It is easily seen that parallel planes transform into parallel planes; and equal and parallel straight lines into equal and parallel straight lines.

Also, the sphere

$$x^2 + y^2 + z^2 = a^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

transforms into the ellipsoid

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} + \frac{z^2}{c'^2} = 1 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

where $$a' = \alpha a, \quad b' = \beta a, \quad c' = \gamma a \;\ldots\ldots\ldots\ldots(4);$$

and a set of three mutually perpendicular diameters of the sphere transform into a set of conjugate diameters of the ellipsoid.

Again, volumes are altered by the transformation in the constant ratio $\alpha\beta\gamma$. For this is obviously true of any rectangular parallelepiped having its edges parallel to the coordinate axes; and any volume whatever can be approximated to as closely as we please by the sum of a system of such parallelepipeds.

Ex. 1. The volume of the ellipsoid (3) is

$$\alpha\beta\gamma\,.\,\tfrac{4}{3}\pi a^3 = \tfrac{4}{3}\pi a'b'c'.$$

Again, a plane cutting off a segment of constant volume from an ellipsoid touches a similar, similarly situated, and concentric ellipsoid.

By reasoning similar to that employed in the preceding Art., we learn that centres of mass of *volumes*, considered as occupied by matter of uniform density, transform into centres of mass.

Also that mean squares of distances from the coordinate planes transform into mean squares of distances.

Ex. 2. The mass-centre of a uniform solid hemisphere is on the radius perpendicular to the bounding plane, at a distance of $\frac{3}{8}$ of its length from the centre. Hence the mass-centre of a semi-ellipsoid cut off by any diametral plane is on the radius conjugate to that plane, at a distance of $\frac{3}{8}$ of its length from the centre.

Ex. 3. The mean squares of the distances of points within the sphere (2) from the coordinate planes being assumed to be

$$\overline{x^2} = \tfrac{1}{5}a^2, \quad \overline{y^2} = \tfrac{1}{5}a^2, \quad \overline{z^2} = \tfrac{1}{5}a^2 \quad \ldots\ldots\ldots\ldots\ldots(5),$$

it follows that, for the ellipsoid (3),

$$\overline{x^2} = \tfrac{1}{5}a'^2, \quad \overline{y^2} = \tfrac{1}{5}b'^2, \quad \overline{z^2} = \tfrac{1}{5}c'^2 \quad \ldots\ldots\ldots\ldots\ldots(6).$$

The radii of gyration about the principal axes of the ellipsoid are therefore given by

$$k_x^2 = \tfrac{1}{5}(b'^2 + c'^2), \quad k_y^2 = \tfrac{1}{5}(c'^2 + a'^2), \quad k_z^2 = \tfrac{1}{5}(a'^2 + b'^2) \ldots(7).$$

EXAMPLES. XXXIX.

1. The squares of the radii of gyration of a rhombus about its diagonals, and about an axis through its centre normal to its plane are

$$\tfrac{1}{6}b^2, \quad \tfrac{1}{6}a^2, \quad \tfrac{1}{6}(a^2+b^2),$$

respectively, where $2a$, $2b$ are the lengths of the diagonals.

2. The radius of gyration, about the axis, of the area of a parabolic segment cut off by a double ordinate $2b$, is given by

$$k^2 = \tfrac{1}{5}b^2.$$

3. The radius of gyration of the same segment about the tangent at the vertex is given by

$$k^2 = \tfrac{3}{7}h^2,$$

where h is the length of the axis of the segment.

4. The square of the radius of gyration of a semicircular area of radius a, about an axis through its centre of mass perpendicular to its plane, is

$$\left(\tfrac{1}{2} - \frac{16}{9\pi^2}\right)a^2.$$

5. The radius of gyration, about the axis, of a segment of a paraboloid of revolution, cut off by a plane perpendicular to the axis, is given by

$$k^2 = \tfrac{1}{3}b^2,$$

where b is the radius of the base.

6. Find by direct calculation the radii of gyration of the volume and surface of an anchor-ring about its axis.

7. The square of the radius of gyration of a uniform circular arc of radius a and angle 2α, about the middle radius, is

$$\tfrac{1}{2}a^2\left(1 - \frac{\sin 2\alpha}{2\alpha}\right).$$

8. The radius of gyration of a uniform circular arc of radius a and angle 2α about an axis through the centre of mass, perpendicular to the plane of the arc, is given by

$$k^2 = a^2\left(1 - \frac{\sin^2\alpha}{\alpha^2}\right);$$

and the radius of gyration about a parallel axis through the middle point of the arc is given by

$$k^2 = 2a^2\left(1 - \frac{\sin\alpha}{\alpha}\right).$$

9. The square of the radius of gyration, about the axis, of a solid ring whose section is a rectangle with the sides parallel and perpendicular to the axis, is

$$\tfrac{1}{2}(a^2 + b^2),$$

where a, b are the inner and outer radii.

10. The mean square of the distance, from the centre, of points within an ellipse of semi-axes a, b, is

$$\tfrac{1}{4}(a^2 + b^2).$$

11. The mean square of the distance, from the centre, of points within an ellipsoid of semi-axes a, b, c, is

$$\tfrac{1}{5}(a^2 + b^2 + c^2).$$

12. The mean square of the distance, from an equatorial plane, of the surface of an anchor-ring is $\frac{1}{2}b^2$, where b is the radius of the generating circle.

13. The mean square of the distance, from the same plane, of the volume of the ring, is $\frac{1}{4}b^2$.

14. Explain how the method of homogeneous strain can be applied to simplify the determination of centres of pressure in certain cases; and employ it to find the centre of pressure of a semi-ellipse, bounded by a principal axis, when this axis is in the surface of a liquid.

15. The centre of pressure of an elliptic area is in the diameter $P'CP$ which bisects the horizontal chords, and is at a distance

$$\tfrac{1}{4}CP^2/CH$$

from the centre C, where H is the point in which PP' produced meets the surface of the liquid.

16. The flexural rigidity of a beam of rectangular section varies as the breadth and as the cube of the depth.

17. The flexural rigidity of a beam of circular section is to that of a beam of square section as $3 : \pi$, if the areas of the sections be equal.

18. If the thickness of a semi-circular lamina of radius a vary as the distance from the bounding diameter, the square of the radius of gyration with respect to this diameter is $\frac{2}{5}a^2$.

19. If ds be an element of arc of an ellipse, and β the parallel semi-diameter, the value of the integral

$$\int \frac{ds}{\beta},$$

taken round the curve, is 2π.

CHAPTER IX.

SPECIAL CURVES.

133. Algebraic Curves with an Axis of Symmetry.

The method of tracing algebraic curves of the type

$$y = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

where $f(x)$ is a rational function, including the determination of asymptotes, maximum and minimum ordinates, and points of inflexion, has been illustrated in various parts of this book; see Arts. 14, 15, 50, 68.

The study of algebraic curves in general is beyond our limits, but a little space may be devoted to the discussion of curves of the type

$$y^2 = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Two points of novelty here present themselves. Since the equation gives two equal, but oppositely-signed, values of y for every value of x, the curve will be symmetrical with respect to the axis of x; also since y^2 must be positive, there can be no real part of the curve within those ranges of x (if any) for which $f(x)$ is negative.

Thus if $f(x)$ contain a *simple* factor $x - x_1$, so that the equation is of the form

$$y^2 = (x - x_1)\,\phi(x) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the right-hand member will change sign as x passes through the value x_1. Hence on one side of the point $(x_1, 0)$ the ordinate is imaginary.

Also, we have, at this point,

$$\left(\frac{\delta y}{\delta x}\right)^2 = \frac{y^2}{(x-x_1)^2} = \frac{\phi(x)}{x-x_1},$$

and therefore, $dy/dx = \infty$. The tangent is therefore perpendicular to Ox.

If, on the other hand, $f(x)$ contain a *double* factor, say

$$y^2 = (x-x_1)^2\,\phi(x) \quad\dots\dots\dots\dots\dots\dots\quad (4),$$

the right-hand side does not change sign as x passes through the value x_1. Hence the ordinate is real on both sides of the point $(x_1, 0)$, or imaginary on both sides. In the former case we have two branches of the curve intersecting at an angle and forming what is called a 'node'; in the latter case $(x_1, 0)$ is an isolated or 'conjugate' point on the locus. The directions of the tangent-lines at the node are given by

$$\left(\frac{dy}{dx}\right)^2 = \lim \frac{y^2}{(x-x_1)^2} = \phi(x_1).$$

If $f(x)$ contain a *triple* factor, say

$$y^2 = (x-x_1)^3\,\phi(x) \quad\dots\dots\dots\dots\dots\dots\quad (5),$$

the right-hand side changes sign at the point $(x_1, 0)$; the curve is therefore imaginary on one side of this point. Also since dy/dx here $= 0$, the curve touches the axis of x.

We proceed to some examples; beginning with cases where $f(x)$ is integral as well as rational.

Ex. 1. In the cases where $f(x)$ is of the first or second degree, say

$$y^2 = Ax + B, \quad y^2 = Ax^2 + Bx + C \dots\dots\dots\dots\dots (6),$$

the curve is a conic having the axis of x as a principal axis.

Ex. 2. The cubical curves

$$y^2 = Ax^3 + Bx^2 + Cx + D \dots\dots\dots\dots\dots\dots (7),$$

include some interesting varieties.

(*a*) If the linear factors of the right-hand side be real and distinct, we may write

$$ay^2 = (x-\alpha)(x-\beta)(x-\gamma) \dots\dots\dots\dots\dots\dots (8),$$

and there is no loss of generality in supposing that a is positive and $\alpha < \beta < \gamma$. The ordinates are then imaginary for $x < \alpha$, and for $\beta < x < \gamma$. Between $(\alpha, 0)$ and $(\beta, 0)$ there is a maximum value of y^2. The curve consists therefore of a closed oval, and of an infinite branch. For large values of x we have

$$\frac{y^2}{x^2} = \frac{x}{a}\left(1-\frac{\alpha}{x}\right)\left(1-\frac{\beta}{x}\right)\left(1-\frac{\gamma}{x}\right),$$

so that the curve tends to become more and more nearly perpendicular to the axis of x.

(*b*) If the expression on the right-hand of (7) has only one real factor, we may write

$$ay^2 = (x-\alpha)(x^2+px+q) \dots\dots\dots\dots\dots\dots (9),$$

where $p^2 < 4q$. The curve then only meets the axis of x once.

(*c*) The transition from the form (8) to the form (9) may be imagined to take place in two ways. In the first way, the intermediate critical case is marked by the coalescence of the two greater of the quantities α, β, γ, so that

$$ay^2 = (x-\alpha)(x-\beta)^2 \dots\dots\dots\dots\dots\dots (10).$$

Here y is imaginary for $x < \alpha$, and real for $x > \alpha$, but vanishes for $x = \beta$. The point $(\beta, 0)$ is here a node; it may be regarded as due to the union of the oval in the former case with the infinite branch.

(*d*) If, however, the two smaller of the quantities α, β, γ coalesce, so that

$$ay^2 = (x-\alpha)^2(x-\gamma) \dots\dots\dots\dots\dots\dots (11),$$

y will be imaginary for $x < \gamma$, except for $x = \alpha$, when it vanishes. The point $(\alpha, 0)$ is therefore an isolated point; it may be regarded as due to the evanescence of the oval in the first case.

All these cases are illustrated in Fig. 74. Beginning on the right we have a curve of the type (9), consisting of a single infinite branch. Next to it comes the case of an infinite branch associated with an isolated point (at O), the equation being of the type (11). Next in order comes an infinite branch, and with it an oval surrounding the point O; the equation is now of the type (8). In the next stage, the oval and the infinite branch have united to form a node with a loop, the corresponding type of equation being (10). Finally, we have a single branch passing outside the loop in the last case; the equation is again of the type (9)*.

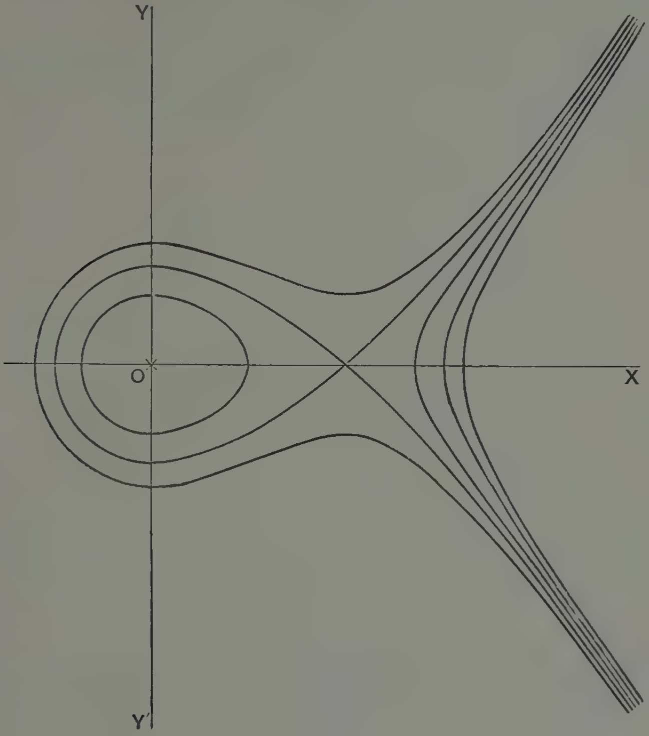

Fig. 74.

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* The curves in the figure have been traced from the equation

$$y^2 = \tfrac{1}{4}(x^3 - 3x^2 + C),$$

where $C = -2, 0, 2, 4, 6$. The relation between them is most easily conceived by regarding them as successive contour-lines of a surface (Art. 45), as in the neighbourhood of a pinnacle on a mountain side.

(e) In the very special case where all three quantities α, β, γ, in (8), coincide, so that

$$ay^2 = (x-\alpha)^3 \quad\text{.......................}(12),$$

the curve is known as the 'semi-cubical parabola.' It has a 'cusp' at $(\alpha, 0)$; this may be regarded as an extreme form of a node, due to the evanescence of the loop. See Fig. 75, where $\alpha = 0$.

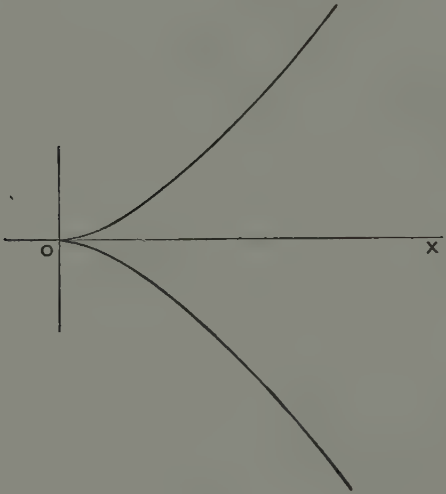

Fig. 75.

If, in the equation (2), $f(x)$ be rational but not integral, the real roots (if any) of the denominator will give asymptotes parallel to y, provided that, for values of x differing infinitely little from these roots, y^2 be positive.

Ex. 3.
$$\frac{y^2}{a^2} = \frac{x-a}{x} \quad\text{..........................}\ (13).$$

The axis of y is an asymptote. Also, for large values of x we have $y = \pm a$, nearly. There is no real part of the curve between $x = 0$ and $x = a$. See Fig. 76.

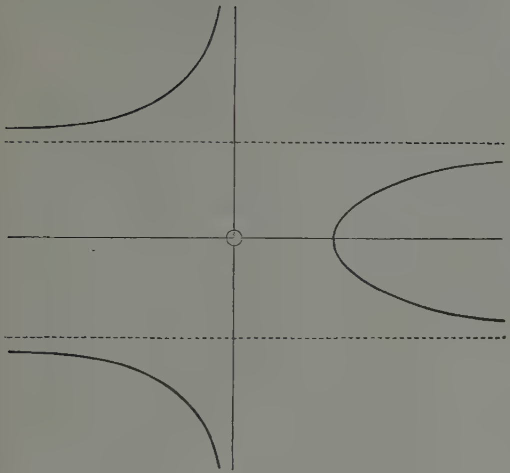

Fig. 76.

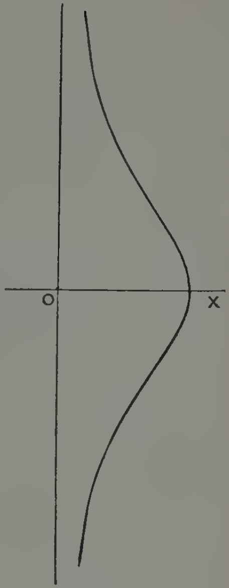

Fig. 77.

Ex. 4.
$$\frac{y^2}{a^2} = \frac{a-x}{x} \quad \text{..........................(14)}.$$

Here y is imaginary for x negative, and for $x > a$. See Fig. 77. The curve is known as the 'witch' of Agnesi.

Ex. 5.
$$y^2 = x^2 \frac{a+x}{b-x} \quad \text{..........................(15)}.$$

There is a node at the origin, and the curve cuts the axis of x again at $(-a, 0)$. For $x > b$, and $x < -a$, y is imaginary. The line $x = b$ is an asymptote. See Fig. 78.

Ex. 6.
$$y^2 = \frac{x^3}{b-x} \quad \text{..........................(16)}.$$

This is obtained by putting $a = 0$ in (15). The loop now shrinks into a cusp; see Fig. 79. The curve is known as the 'cissoid.

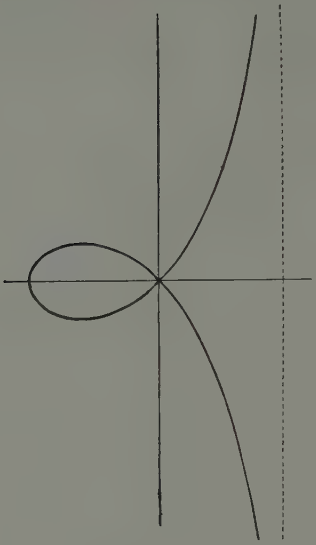

Fig. 78.

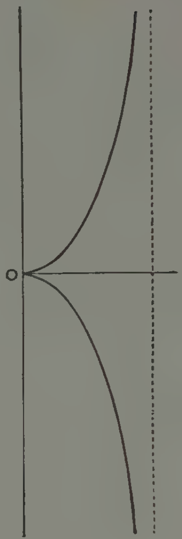

Fig. 79.

Ex. 7.
$$y^2 = x^2\frac{x+a}{x-a} \quad\text{..........................(17).}$$

Since y is imaginary for $a > x > -a$, except for $x = 0$, the origin is an isolated point. To find the oblique asymptotes we have

$$\frac{y}{x} = \pm\left(\frac{1+\dfrac{a}{x}}{1-\dfrac{a}{x}}\right)^{\frac{1}{2}} = \pm\left(1+\frac{a}{x}\right)\left(1-\frac{a^2}{x^2}\right)^{-\frac{1}{2}}$$

$$= \pm\left(1+\frac{a}{x}+\tfrac{1}{2}\frac{a^2}{x^2}+\dots\right) \quad\text{..........................(18).}$$

Hence the lines

$$y = \pm(x+a) \quad\text{..........................(19)}$$

are asymptotes. See Fig. 80.

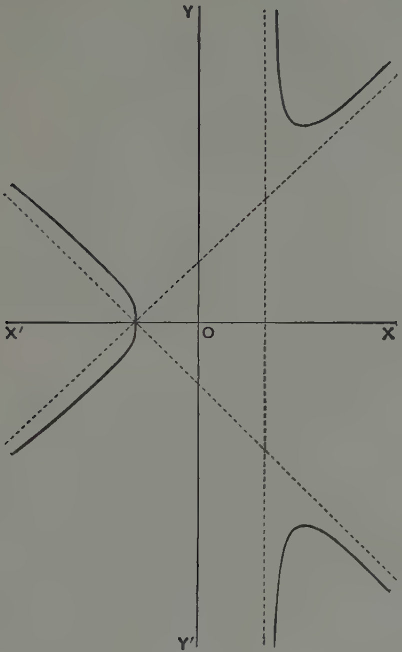

Fig. 80.

EXAMPLES. XL.

1. Trace the curves

$$y^2 = 4x(1-x), \quad y^2 = x^2 + x + 1.$$

2. Trace the curve

$$ay^2 = x^2(a-x),$$

and shew that it forms a loop of area $\frac{8}{15}a^2$.

Find where the breadth of the loop is greatest. $[x = \frac{2}{3}a]$

3. Trace the curve

$$a^2y^2 = x^2(a^2-x^2),$$

and shew that it forms two loops, each of area $\frac{2}{3}a^2$.

4. Trace the curves

$$y^2 = x(x^2 - 1), \quad y^2 = x^3(1 - x).$$

5. Trace the curve

$$a^4y^2 = x^4(a^2 - x^2),$$

and shew that it encloses an area $\frac{1}{4}\pi a^2$.

6. Trace the curve

$$a^2y^4 = x^4(a^2 - x^2),$$

and shew that it encloses an area $\frac{8}{5}a^2$.

7. The length of an arc of the curve

$$ay^2 = x^3$$

(Fig. 75), from the vertex to the point whose abscissa is x, is

$$\frac{1}{27\sqrt{a}}(9x + 4a)^{\frac{3}{2}} - \tfrac{8}{27}a.$$

8. The mass centre of the area included between the curve $ay^2 = x^3$ and the line $x = h$ is at the point $(\frac{5}{7}h, 0)$.

9. If the curve $ay^2 = x^3$ revolve about the axis of x, the volume included between the surface generated, and any plane perpendicular to the axis, is one-fourth that of a cylinder of the same length on the same circular base.

10. Trace the curves

$$y^2 = \frac{1}{x}, \quad y^2 = \frac{1}{x(1 - x)}.$$

11. The area included between the curve

$$\frac{y^2}{a^2} = \frac{a - x}{x}$$

(Fig. 77) and its asymptote is πa^2.

If the same curve revolve about its asymptote, the volume of the solid generated is $\frac{1}{2}\pi^2 a^3$.

12. Trace the curves

$$y^2 = \frac{x^2 + 1}{x}, \quad y^2 = \frac{x^2 - 1}{x}.$$

13. Trace the curves

$$y^2 = \frac{x^2+1}{x^2}, \quad y^2 = \frac{x^2-1}{x^2}.$$

14. Trace the curves

$$y^2 = \frac{x}{1+x^2}, \quad y^2 = \frac{x}{1-x^2}.$$

Determine the maximum and minimum ordinates (if any), and the points of inflexion.

15. The area included between the curve

$$y^2 = \frac{x^3}{a-x},$$

(Fig. 79) and its asymptote is $\frac{3}{4}\pi a^2$.

If the same curve revolve about its asymptote the volume of the solid generated is $\frac{1}{4}\pi^2 a^3$.

16. Trace the curve

$$y^2 = \frac{a^2x^2}{a^2+x^2},$$

and shew that the area included between its two branches and either asymptote is $2a^2$.

17. Shew that the area included between the curve

$$y^2 = x^2\frac{a+x}{a-x},$$

(Fig. 78) and its asymptote is $\frac{1}{2}(\pi+4)a^2$.

18. Trace the curve

$$y^2 = \frac{x^4}{a^2-x^2},$$

and shew that the area included between the curve and either asymptote is $\frac{1}{2}\pi a^2$.

19. Trace the curve

$$y^2 = x^2\frac{a^2-x^2}{a^2+x^2},$$

and shew that it forms a loop of area $\frac{1}{2}(\pi-2)a^2$.

20. Trace the curve

$$y^2 = \frac{x^2}{a^2}(2a - x)(x - a),$$

and shew that it encloses an area $\frac{3}{8}\pi a^2$.

134. Transcendental Curves; Catenary, Tractrix.

We proceed to the discussion of some important curves, mainly transcendental, which are most conveniently defined by equations of the type already referred to in Art. 54, viz.

$$x = \phi(t), \quad y = \chi(t) \text{ .... } (1),$$

where t is a variable parameter.

The 'catenary' is the curve in which a uniform chain hangs freely under gravity. It appears from elementary statical principles that if s be the arc of the curve measured from the lowest point (A) up to any point P, and ψ the inclination to the horizontal of the tangent at P, then

$$s = a \tan \psi \text{ .... } (2),$$

where a is a constant. Hence if x, y be horizontal and vertical coordinates, we have

$$\left.\begin{aligned} \frac{dx}{d\psi} &= \frac{dx}{ds}\frac{ds}{d\psi} = \cos\psi \,.\, a \sec^2\psi = a \sec\psi, \\ \frac{dy}{d\psi} &= \frac{dy}{ds}\frac{ds}{d\psi} = \sin\psi \,.\, a \sec^2\psi = a \tan\psi \sec\psi, \end{aligned}\right\} \text{...} (3).$$

Integrating, we find

$$x = a \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi), \quad y = a \sec\psi \text{ ...... } (4).$$

The omission of the additive constants merely amounts to a special choice of the origin, which was so far undetermined. Since the formulæ (4) make, $x = 0$, $y = a$ for $\psi = 0$, it appears that the origin is at a distance a vertically beneath A.

From (4) the Cartesian equation can be deduced without difficulty. We have

$$\frac{x}{a} = \log \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\psi) = \log(\sec\psi + \tan\psi),$$

whence and therefore
$$\left.\begin{aligned}\sec\psi+\tan\psi&=e^{x/a},\\ \sec\psi-\tan\psi&=e^{-x/a}.\end{aligned}\right\}\text{..........}\quad\text{...(5)}.$$

Hence, by addition and subtraction,

$$\left.\begin{aligned}y&=a\sec\psi=a\cosh\frac{x}{a},\\ s&=a\tan\psi=a\sinh\frac{x}{a}\end{aligned}\right\}\text{..............(6)}.$$

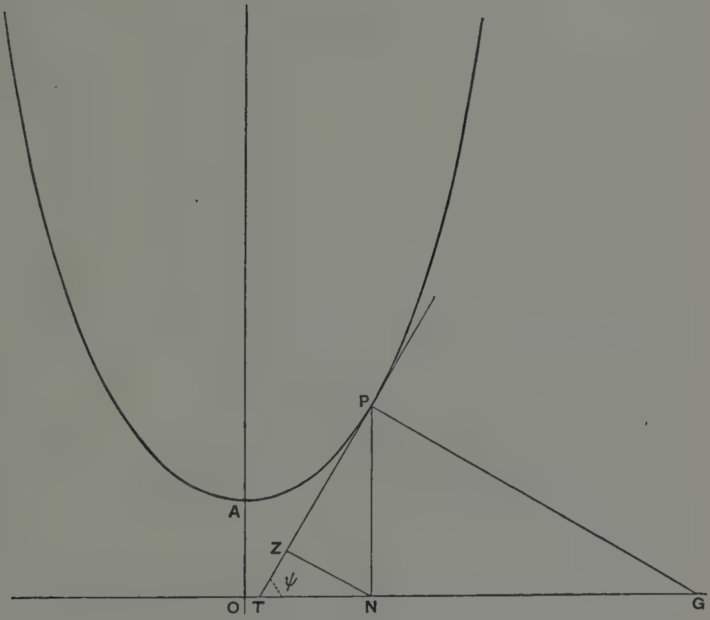

Fig. 81.

Some further properties follow easily from a figure. If PN be the ordinate, PT the tangent, PG the normal, NZ the perpendicular from the foot of the ordinate on the tangent, we have

$$NZ=y\cos\psi=a,\quad PZ=a\tan\psi=s.$$

Since PZ is equal to the arc of the catenary, it is easily seen that the consecutive position of Z is in ZN; in other words, ZN is a tangent to the locus of Z. Hence this locus possesses the property that its tangent ZN is of constant length. The curve thus characterized is called the 'tractrix,' from the fact that it is the path of a heavy particle dragged along a rough horizontal plane by a string, the other end (N) of which is made to describe a straight line (OX).

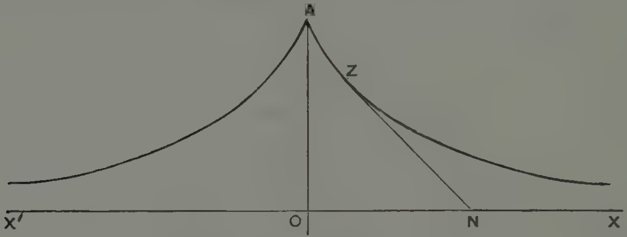

Fig. 82.

The curve has a cusp at A, and the axis of x is an asymptote.

Many properties of the tractrix follow immediately from the constancy (in length) of the tangent. For example, since two consecutive tangents make an angle $\delta\psi$ with one another, the area swept over by the tangent is given by

$$\tfrac{1}{2}\int a^2 d\psi,$$

taken between the proper limits. The whole area between the curve and its asymptote is thus found to be $\frac{1}{2}\pi a^2$.

135. Lissajous' Curves.

These curves, which are of importance in Acoustics, result from the composition of two simple-harmonic motions in perpendicular directions. They may therefore be represented by

$$x = a\cos(nt + \epsilon), \quad y = b\cos(n't + \epsilon') \ldots\ldots\ldots(1),$$

and it is further obvious that we may give any convenient value to *one* of the quantities ϵ, ϵ', since this amounts merely to a special choice of the origin of t.

When the periods $2\pi/n$, $2\pi/n'$ are commensurable, we can by elimination of t obtain the relation between x and y in an algebraic form.

Ex. 1. In the case $n' = n$, we may write

$$x = a\cos(nt + \epsilon), \quad y = b\cos nt \quad\text{...............}(2),$$

whence $\dfrac{x}{a} - \dfrac{y}{b}\cos\epsilon = -\sin nt\sin\epsilon$, $\dfrac{y}{b}\sin\epsilon = \cos nt\sin\epsilon$.

Squaring, and adding, we find

$$\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\epsilon + \frac{y^2}{b^2} = \sin^2\epsilon \quad\text{..................}(3).$$

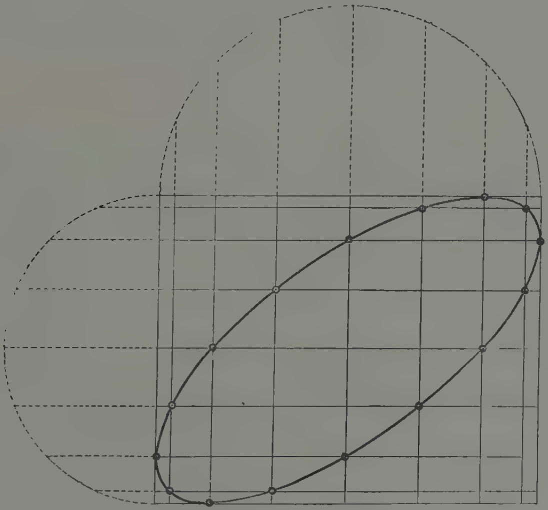

Fig. 83.

This represents an ellipse. In the special case of $\epsilon = 0$ or $\epsilon = \pi$, the ellipse degenerates into a straight line

$$\frac{x}{a} \mp \frac{y}{b} = 0 \quad\text{.........................}(4).$$

If the equality of periods be not quite exact, the figure described may be regarded as an ellipse which gradually changes its form owing to a continuous variation of the relative phase (ϵ) of the two component motions.

When the ellipse (3) is referred to its principal axes, the coordinates of the moving point take the forms

$$x = a\cos(nt+\epsilon), \quad y = b\sin(nt+\epsilon) \;\ldots\ldots\ldots\ldots(5).$$

We identify $nt+\epsilon$ with the 'eccentric angle'; and since this increases uniformly with the time it appears that the point (x, y) moves like the orthogonal projection of a point describing a circle of radius a with a constant velocity na. Since in the transition from the circle to the ellipse any infinitely small chord is altered in the same ratio as the radius parallel to it, we see that in the elliptic motion the velocity at any point P will be $n . CD$, where CD is the semi-diameter conjugate to CP, C being the centre.

The type of motion here considered is called 'elliptic harmonic.'

Ex. 2. If $n' = 2n$, we may write

$$x = a\cos nt, \quad y = b\cos(2nt+\epsilon) \;\ldots\ldots\ldots\ldots(6).$$

Here y goes through its period twice as fast as x, and the point $(0, -b\cos\epsilon)$ is passed through twice as nt increases by 2π. The curve therefore consists in general of two loops.

For $\epsilon = \pm\frac{1}{2}\pi$, the curve is symmetrical with respect to both axes, the algebraic equation being

$$\frac{y^2}{b^2} = 4\frac{x^2}{a^2}\left(1 - \frac{x^2}{a^2}\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

When $\epsilon = 0$, or π, the curve degenerates into an arc of the parabola

$$\frac{y}{b} = \pm\left(2\frac{x^2}{a^2} - 1\right) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

When the relation of the periods is not quite exact the curve oscillates between these two parabolic arcs as extreme forms*.

---

* A method of constructing Lissajous' curves is indicated in Fig. 83, where the vertical and horizontal lines, being drawn through equidistant points on the respective auxiliary circles, mark out equal intervals of time.

There are numerous optical and mechanical contrivances for producing the curves. For a description of these, and for specimens of the curves described, we must refer to books on experimental Acoustics.

136. The Cycloid.

The 'cycloid' is the curve traced by a point on the circumference of a circle which rolls in contact with a fixed straight line. It evidently consists of an endless succession of exactly congruent portions, each of which represents a complete revolution of the circle. The points (such as A in the figure) where the curve is furthest from the fixed straight line or 'base' (BD) are called 'vertices'; the points (D) half-way between successive vertices, where the curve meets the base, are the 'cusps.' A line (AB) through a vertex and perpendicular to the base is called an 'axis' of the curve. It is evidently a line of symmetry.

It is convenient to employ the circle described on an axis AB as diameter as a circle of reference. Let IPT be any other position of the rolling circle, I the point of contact with the base, C the centre, T the opposite extremity of the diameter through I, and let P be the position of the tracing-point. Draw PMN parallel to the base, meeting TI and AB

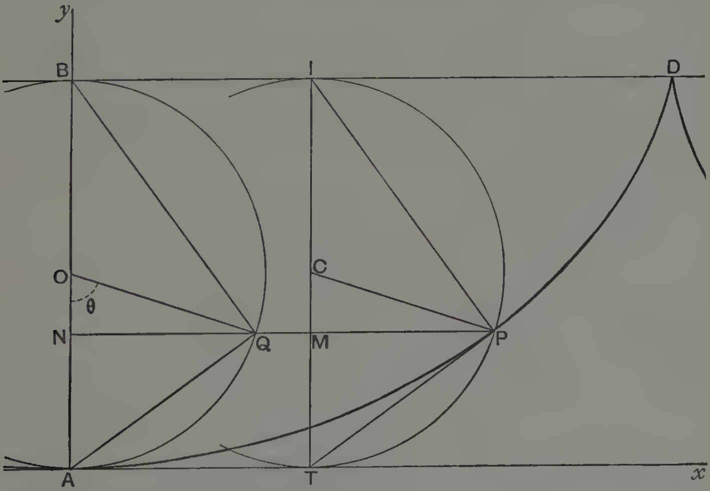

Fig. 84.

in M and N respectively, and the circle of reference in Q. If AT, AB be taken as axes of x and y, the coordinates of P will be

$$x = NP = BI + MP, \quad y = AN = CT - CM.$$

Let a be the radius of the rolling circle, and θ the angle (PCT) through which it turns as the tracing point travels from A to P. We have, then, $BI = a\theta$, $PM = a\sin\theta$, $CM = a\cos\theta$, and therefore

$$\left.\begin{aligned} x &= a(\theta + \sin\theta) \\ y &= a(1 - \cos\theta) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

From these equations all the properties of the curve can be deduced. Thus if ψ denote the inclination of the tangent to AT, or of the normal to BA, we have

$$\tan\psi = \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{\sin\theta}{1 + \cos\theta} = \tan\tfrac{1}{2}\theta,$$

whence $$\psi = \tfrac{1}{2}\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Since the angle TIP is one-half of TCP, it follows that IP is the normal, and PT the tangent, to the curve at P. Cf. Art. 164, below.

Again, to find the arc (s) of the curve, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\{(1 + \cos\theta)^2 + \sin^2\theta\} = 4a^2\cos^2\tfrac{1}{2}\theta,$$

whence, by Art. 109,

$$s = 2a\int\cos\tfrac{1}{2}\theta\, d\theta = 4a\sin\tfrac{1}{2}\theta,$$

or, in terms of ψ,

$$s = 4a\sin\psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

no additive constant being required, if the origin of s be at A. This relation is important in Dynamics.

Since $TP = TI\sin\psi$, we have

$$\text{arc } AP = 2TP = 2 \text{ chord } AQ \ldots\ldots\ldots\ldots (4).$$

In particular, the length of the arc from one cusp to the next is $8a$.

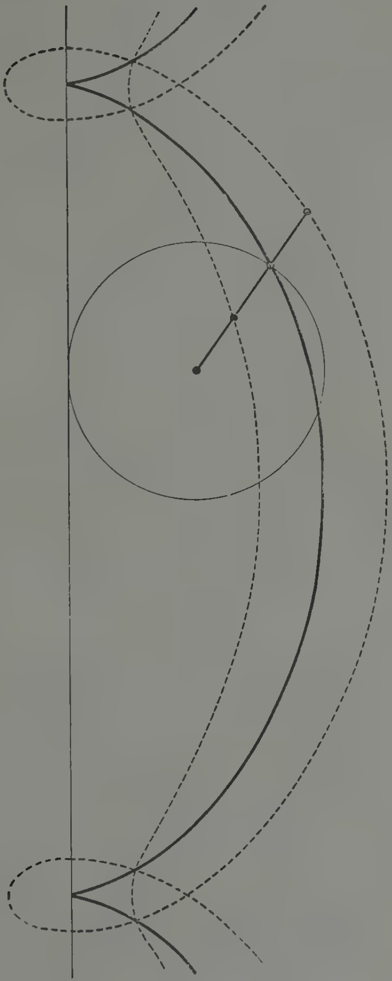

Fig. 85.

If we put

$$y' = IM = a(1 + \cos\theta) \quad\text{..............}\quad (5),$$

the area included between the curve and the base is given by

$$\int y'\,dx = a^2 \int (1 + \cos\theta)^2 d\theta = 4a^2 \int \cos^4 \tfrac{1}{2}\theta\, d\theta = 8a^2 \int \cos^4 \psi\, d\psi.$$

Taking this between the limits $\mp \frac{1}{2}\pi$, we find that the area included between the base and one arch of the curve is three times the area of the generating circle.

The curve traced by *any* point fixed relatively to a circle which rolls on a fixed straight line is called a 'trochoid.'

If, in Fig. 84, the tracing point be in the radius CP, at a distance k from the centre, its coordinates will be

$$\left.\begin{aligned} x &= a\theta + k\sin\theta \\ y &= a - k\cos\theta \end{aligned}\right\} \quad\text{......................}\quad (6).$$

When $k > a$ we have loops, which in the particular case $(k = a)$ of the cycloid degenerate into cusps. When $k < a$, the curve does not meet the base. Fig. 85 shews the cases $k = \frac{1}{2}a$, $k = a$, $k = \frac{3}{2}a$.

It is readily proved from (6) that the normal at any point of the trochoid passes through the corresponding position of the point of contact of the rolling circle. Cf. Art. 164.

137. Epicycloids and Hypocycloids.

The path traced out by a point on the circumference of a circle which rolls in contact with a fixed circle is called an 'epicycloid' or a 'hypocycloid' according as the rolling circle is outside or inside the fixed circle*. Those epicycloids in which the rolling circle surrounds the fixed circle may be referred to, when a distinction is desired, as 'pericycloids.'

Let O be the centre of the fixed circle, C that of the rolling circle in any position, I the point of contact, P the tracing point; and suppose that, initially, the other extremity P' of the diameter PCP' was in contact with A. We take as our standard case that in which each circle is external to the other. Let

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* This is the definition as improved by Proctor in his *Treatise on the Cycloid, etc.* (1878).

$$OA = a, \quad CP = b, \quad \angle IOA = \theta, \quad \angle ICP' = \phi.$$

The inclination of CP to OA will be $\theta + \phi$. Hence if we

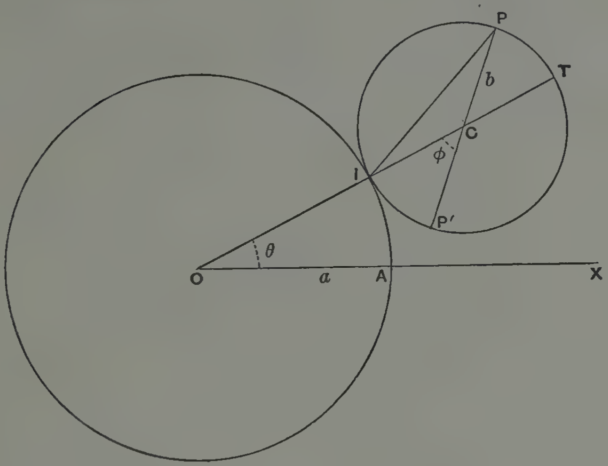

Fig. 86.

take O as origin of rectangular coordinates, and OA as axis of x, we find, by orthogonal projections, that the coordinates of P are

$$\left.\begin{aligned} x &= (a+b)\cos\theta + b\cos(\theta+\phi), \\ y &= (a+b)\sin\theta + b\sin(\theta+\phi) \end{aligned}\right\} \quad \ldots\ldots\ldots (1),$$

or, since

$$a\theta = \text{arc } AI = \text{arc } P'I = b\phi \ \ldots\ldots\ldots\ldots (2),$$

$$\left.\begin{aligned} x &= (a+b)\cos\theta + b\cos\frac{a+b}{b}\theta, \\ y &= (a+b)\sin\theta + b\sin\frac{a+b}{b}\theta \end{aligned}\right\} \quad \ldots\ldots\ldots (3).$$

The epicycloid traced out by P' is found by changing the sign of b in the *coefficient* of the second terms; viz. we have

$$\left.\begin{aligned} x &= (a+b)\cos\theta - b\cos\frac{a+b}{b}\theta, \\ y &= (a+b)\sin\theta - b\sin\frac{a+b}{b}\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (4).$$

This has a cusp at A.

In the above standard case the circles lie on opposite sides of the tangent at I. If they lie on the same side, as in the 'pericycloids' and 'hypocycloids,' we have merely to reverse the sign of b throughout, the formulæ corresponding to (3) being then

$$\left.\begin{aligned} x &= (a-b)\cos\theta - b\cos\frac{a-b}{b}\theta, \\ y &= (a-b)\sin\theta + b\sin\frac{a-b}{b}\theta \end{aligned}\right\} \quad \ldots\ldots\ldots (5).$$

The verification is left to the reader; see Fig. 87. In the hypocycloids we have $a > b$, in the pericycloids $a < b$.

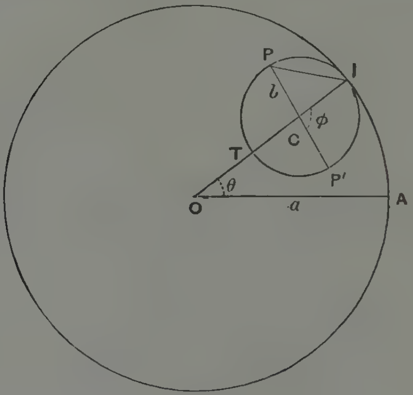

Fig. 87.

Similarly, for the locus of P' we obtain

$$\left.\begin{aligned} x &= (a-b)\cos\theta + b\cos\frac{a-b}{b}\theta, \\ y &= (a-b)\sin\theta - b\sin\frac{a-b}{b}\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (6).$$

To find the tangent at any point of an epicycloid, we have from (1), since $d\phi/d\theta = a/b$,

$$\frac{dy}{dx} = -\frac{\cos\theta + \cos(\theta+\phi)}{\sin\theta + \sin(\theta+\phi)} = -\cot(\theta + \tfrac{1}{2}\phi) \text{....(7)}.$$

On reference to Fig. 86, we see that $\theta + \frac{1}{2}\phi$ is the inclination of IP to OA. Hence IP is normal to the epicycloid at P. A similar result can be deduced, of course, for the pericycloids and hypocycloids, from the equations (5). Cf. Art. 164.

Again, from (1),

$$\left(\frac{dx}{d\phi}\right)^2 + \left(\frac{dy}{d\phi}\right)^2 = \frac{(a+b)^2 b^2}{a^2}(2 + 2\cos\phi) = \frac{4(a+b)^2 b^2}{a^2}\cos^2 \tfrac{1}{2}\phi,$$

or $$\frac{ds}{d\phi} = \frac{2(a+b)b}{a}\cos\tfrac{1}{2}\phi \text{ ....(8)}.$$

Hence $$s = \frac{4(a+b)b}{a}\sin\tfrac{1}{2}\phi \text{ ....(9)},$$

no additive constant being necessary, if $s = 0$ for $\phi = 0$.

If we denote by ψ the inclination of the normal IP to OA, we have

$$\psi = \theta + \tfrac{1}{2}\phi = \frac{a+2b}{2a}\phi \text{ ....(10)},$$

and therefore $$s = \frac{4(a+b)b}{a}\sin\frac{a}{a+2b}\psi \text{ ....(11)}.$$

The formula (9) has a simple interpretation. It appears from Fig. 86 that $TP = 2b\sin\frac{1}{2}\phi$, whence

$$s = 2\,\frac{a+b}{a} \times \text{chord } TP \text{ ....(12)*}.$$

In particular, the length of the curve from one cusp to the next is $8(a+b)b/a$.

The corresponding results for the pericycloids and hypocycloids are easily inferred by changing the sign of b.

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* Cf. Newton, *Principia*, lib. i., prop. xlix.

The curve traced out by a point of the rolling circle which is not on the circumference is called an 'epitrochoid,' or a 'hypotrochoid,' as the case may be. If k denote the distance of the tracing point from the centre of the rolling circle, the expressions for the coordinates x, y in the various cases are obtained by writing k for b in the coefficients (only) of the second terms.

138. Special Cases.

1°. If the radius of the *fixed* circle be infinitely great we fall back on the case of the cycloid. The corresponding equations are easily deduced from Art. 137 (1), writing $x+a$ for x, $a\theta=b\phi$, and (finally) $\theta=0$.

2°. Again, making the radius of the *rolling* circle infinite, we get the path described by a point of a straight line which rolls on a fixed circle. The curve thus defined is called the 'involute of the circle'; see Art. 161. The equations may be obtained as limiting forms of Art. 137 (4), or they may be written down at once from a figure. We find

$$\left.\begin{aligned} x &= a\cos\theta + a\theta\sin\theta, \\ y &= a\sin\theta - a\theta\cos\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

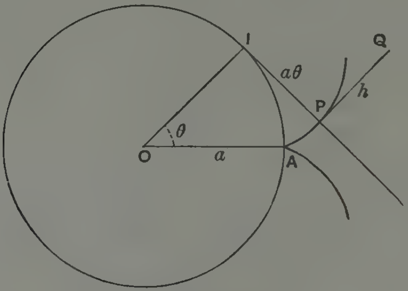

Fig. 88.

The corresponding trochoidal curve is

$$\left.\begin{aligned} x &= (a+h)\cos\theta + a\theta\sin\theta, \\ y &= (a+h)\sin\theta - a\theta\cos\theta \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots(2),$$

where $h=PQ$ in the figure, Q being the tracing point. The particular case of $h=-a$ gives the 'spiral of Archimedes,' see Art. 140.

3°. If the radii a, b be commensurable, then after some exact number of revolutions the tracing point will have returned to its original position, and its subsequent course will be a repetition of the previous path. In such cases the curve is algebraic, since the trigonometrical functions can be eliminated between the expressions for x and y. Sometimes the equation is more conveniently expressed in polar coordinates.

Figs. 89, 90, 91 shew the epi- and hypo-cycloids in which the ratio of the radius of the rolling circle to that of the fixed circle has the values 1, $\frac{1}{2}$, $\frac{1}{3}$, respectively.

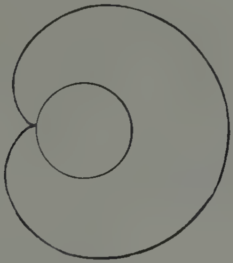

Fig. 89.

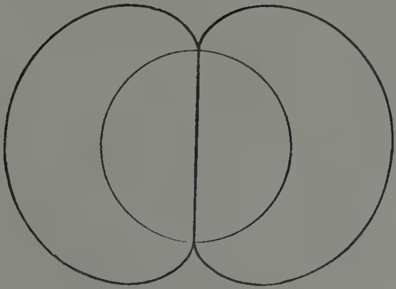

Fig. 90.

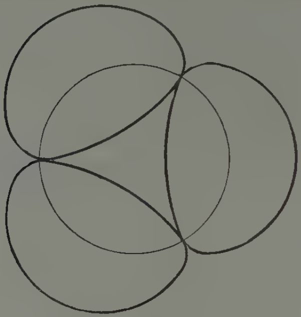

Fig. 91.

We proceed to notice in detail one or two of the cases which have specially important properties.

Ex. 1. The 'cardioid.'

If in Art. 137 (3) we put $b = a$, we get

$$x = 2a \cos \theta + a \cos 2\theta, \quad y = 2a \sin \theta + a \sin 2\theta,$$

whence $\quad x + a = 2a(1 + \cos \theta) \cos \theta, \quad y = 2a(1 + \cos \theta) \sin \theta \ldots (3).$

This shews that the radius vector drawn from the point $(-a, 0)$ as pole is given by

$$r = 2a(1 + \cos \theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

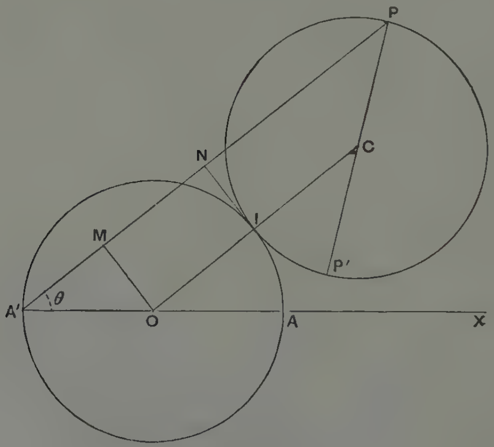

Fig. 92.

This is otherwise evident from Fig, 92, where

$$A'P = 2A'N = 2(OI + A'M).$$

The corresponding trochoids are given by

$$x = 2a\cos\theta + k\cos 2\theta, \quad y = 2a\sin\theta + k\sin 2\theta.$$

Referred to the point $(-k, 0)$ as pole these formulæ are equivalent to

$$r = 2(a + k\cos\theta) \text{ ....}(5),$$

which is the polar equation of the 'limaçon' (Art. 141). This equation, again, is easily obtained geometrically.

Ex. 2. A circle rolls inside another of twice its radius.

If in Art. 137 (6) we put $b = \frac{1}{2}a$, we get

$$x = a\cos\theta, \quad y = 0 \text{ ....}(6);$$

i.e. the tracing point on the circumference of the rolling circle traces out a diameter of the fixed circle.

Again, the corresponding trochoidal curve is given by

$$x = (b + k)\cos\theta, \quad y = (b - k)\sin\theta \text{ ....}(7),$$

and is therefore an ellipse of semi-axes $b \pm k$. Moreover if the rolling circle have a constant angular velocity, the motion of the tracing point is elliptic-harmonic.

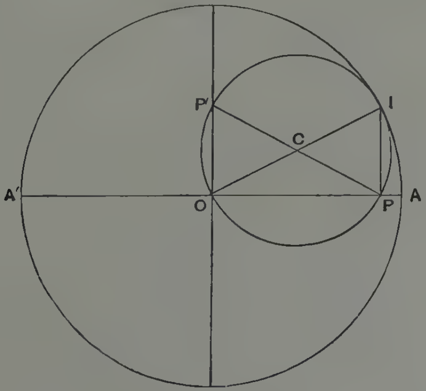

Fig. 93.

These results also follow easily from geometrical considerations. The rolling circle passes always through the centre O of the fixed circle; also, if P be the point of the rolling circle which initially coincides with A, the arc IP is equal to the arc IA. Hence, since the radii are as 1 : 2, the angle which the arc IP subtends at the circumference of its circle must be equal to the angle which the arc IA subtends at the centre of *its* circle; that is, OP and OA coincide in direction, and P describes the fixed diameter OA. Again, since the angle POP' is a right angle, the other extremity of the diameter PP' of the rolling circle describes the diameter of the fixed circle which is perpendicular to OA. Hence PP' is a line of constant length whose extremities move on two fixed straight lines at right angles to one another. It is known that under these circumstances any other point on PP' describes an ellipse. Cf. Art. 163, Ex. 1.

Ex. 3. A circle rolls on the outside of a fixed circle of one-half the radius, which it encloses.

The formulæ (5) of Art. 137 give, for $b = 2a$,

$$x = -a\cos\theta - 2a\cos\tfrac{1}{2}\theta, \quad y = -a\sin\theta - 2a\sin\tfrac{1}{2}\theta,$$

or $$x - a = -2a\left(1 + \cos\tfrac{1}{2}\theta\right)\cos\tfrac{1}{2}\theta, \quad y = -2a\left(1 + \cos\tfrac{1}{2}\theta\right)\sin\tfrac{1}{2}\theta \quad \text{............(8).}$$

If we put $$\theta' = \tfrac{1}{2}\theta + \pi,$$

it appears that the pericycloid, referred to the point $(a, 0)$ as pole, has the equation

$$r = 2a\left(1 - \cos\theta'\right) \quad \text{......................(9),}$$

and is therefore a cardioid.

The connection between this result and that of Ex. 1, above, will appear in Art. 168.

Ex. 4. The 'four-cusped hypocycloid.'

If in Art. 137 (6) we put $b = \frac{1}{4}a$, we get

$$\left.\begin{aligned} x &= \tfrac{3}{4}a\cos\theta + \tfrac{1}{4}a\cos 3\theta = a\cos^3\theta, \\ y &= \tfrac{3}{4}a\sin\theta - \tfrac{1}{4}a\sin 3\theta = a\sin^3\theta \end{aligned}\right\} \quad \text{...........(10),}$$

from which the curve is easily traced. The Cartesian form is

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad \text{........................(11).}$$

This curve is sometimes called the 'astroid.' It possesses the property that the length of the tangent intercepted between the coordinate axes is constant. If, in Fig. 94, P be the tracing-point, TP is the tangent, and it is easily seen that the angle CTP is double the angle AOC. Hence $HK = 2OT = a$. See also Fig. 134, p. 438.

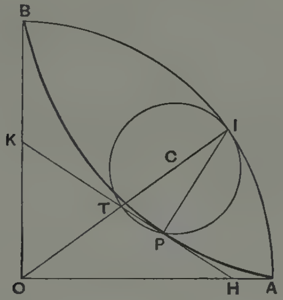

Fig. 94.

139. Superposition of Circular Motions. Epicyclics.

The cycloidal and trochoidal curves discussed in Arts. 136—138 present themselves in another manner, as the paths of points whose motion is compounded of two uniform circular motions.

If an arm OQ revolve about a fixed point O with constant angular velocity n, its projections on rectangular axes through O may be taken to be

$$x = c \cos nt, \quad y = c \sin nt \text{ ....(1)},$$

where $c = OQ$, provided the origin of t be suitably chosen. If another arm OQ' revolve about O with constant angular velocity n', starting simultaneously with OQ from coincidence with the axis of x, the projections of OQ' will be

$$x = c' \cos n't, \quad y = c' \sin n't \text{ ....(2)},$$

where $c' = OQ'$. If we complete the parallelogram $OQPQ'$, the vector OP will represent the geometric sum of OQ and OQ', and the coordinates of P will be

$$x = c\cos nt + c'\cos n't, \quad y = c\sin nt + c'\sin n't \ldots(3)^*.$$

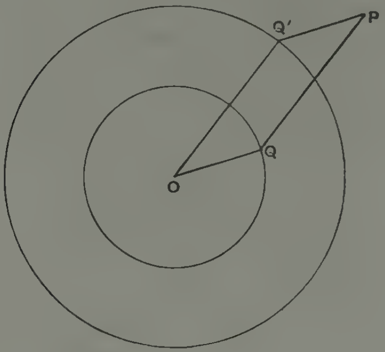

Fig. 95

Ex. 1. If the angular velocities of the component circular motions are equal and opposite ($n' = -n$), we have

$$x = (c + c')\cos nt, \quad y = (c - c')\sin nt \;\ldots\ldots\ldots\ldots(4),$$

so that the resultant motion is elliptic-harmonic. In the particular case of $c' = c$, the ellipse degenerates into a straight line.

This example is of importance in Physical Optics.

Since, in the figure, QP is always equal and parallel to OQ', the path of P is that of a point describing uniformly a circular orbit relatively to a point Q which itself has a uniform circular motion about O. Curves described in this manner are called 'epicyclics.' If the angular velocities n, n' have the same sign, the epicyclic is said to be 'direct'; if they have opposite signs it is said to be 'retrograde.'

---

[*] If the parallelogram $OQPQ'$ consist of four jointed rods, and if OQ, OQ' be made to revolve at the proper rates about O, the distance of P from any fixed line through O will represent the sum of two simple-harmonic motions of periods $2\pi/n$, $2\pi/n'$. This is the principle of Lord Kelvin's 'tidal clock,' which performs mechanically the superposition of the solar and lunar tides.

Figs. 96—99 shew some specimens of direct and retrograde epicyclics*.

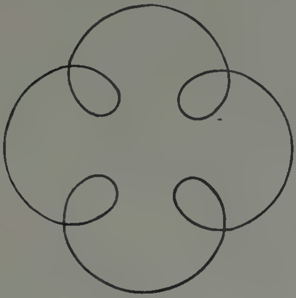

Fig. 96.

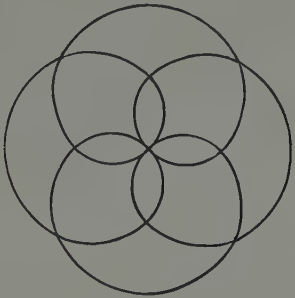

Fig. 97.

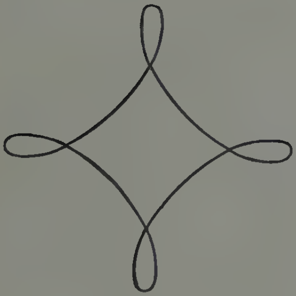

Fig. 98.

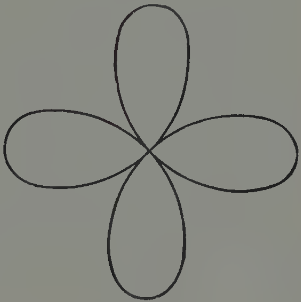

Fig. 99.

Since the path of P may be equally well defined as that of a point moving in a circular orbit relatively to a point Q' which itself is in uniform circular motion about O, we see that every epicyclic can be generated in two distinct ways.

It is evident that every epi- or hypo-cycloid, and (more generally) every epi- and hypo-trochoid, is an epicyclic, since if the rolling circle has a constant angular velocity its centre C will describe a circle uniformly about O, the centre of the fixed circle, whilst the radius CP which contains the tracing point P has a uniform rotation about C. See Figs. 86, 87.

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* The variety of such figures is of course endless. Epicyclics are easily described mechanically with a lathe; a number of very interesting diagrams obtained in this manner are reproduced in Proctor's treatise cited on p. 350 *ante*.

Conversely, it may be proved that every epicyclic is either an epi- or a hypo-trochoid; more particularly that every direct epicyclic is an epitrochoid, and every retrograde epicyclic is a hypotrochoid. This may be shewn by a comparison of (3), above, with the results of Art. 137. A simple geometrical proof will be given later (Art. 168), in connection with the theory of the 'instantaneous centre.'

The connection of the direct and retrograde epicyclics in Figs. 96—99 with the four-cusped epi- and hypo-cycloids will be apparent.

Epicyclics played a great part in ancient Astronomy. If we ignore the eccentricities and inclinations of the planetary orbits, the Sun may be regarded as describing a circle round the Earth, and any other planet describes a circle in the same plane about the Sun. The path of the planet relatively to the Earth is therefore an epicyclic. This was the accepted view of the matter from the time of Ptolemy down to the sixteenth century, when it was gradually superseded by the simpler mode of describing the phenomena discovered by Copernicus.

The relative orbits of the planets have loops, as in Fig. 96. This accounts for the 'stationary points' and 'retrograde motions,' which were in fact the occasion of Ptolemy's invention of epicyclics.

The orbit of the moon relatively to the sun, on the other hand, though an epicyclic, has no loops; it is, moreover, everywhere concave inwards.

Ex. 2. The special form assumed by an epicyclic when $c' = c$ may be noticed.

The equations (3) then become

$$\left.\begin{aligned} x &= 2c\cos\tfrac{1}{2}(n+n')t\,.\cos\tfrac{1}{2}(n-n')t,\\ y &= 2c\sin\tfrac{1}{2}(n+n')t\,.\cos\tfrac{1}{2}(n-n')t \end{aligned}\right\}\quad\ldots\ldots\ldots(5),$$

or

$$x = r\cos\theta,\quad y = r\sin\theta\quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where

$$\theta = \tfrac{1}{2}(n+n')t,\quad r = 2c\cos\frac{n-n'}{n+n'}\theta\quad\ldots\ldots\ldots\ldots(7).$$

The polar equation of the curve is therefore of the form

$$r = a \cos m\theta \quad \text{..........................(8)},$$

where $m \lessgtr 1$, according as the epicyclic is direct or retrograde.

Figs. 97, 99 on p. 361 correspond to the cases $m = \frac{2}{3}$, $m = 2$, respectively.

EXAMPLES. XLI.

1. Prove that, in the catenary $y = c \cosh x/c$,

$$s^2 = y^2 - c^2.$$

2. Prove that the catenary is the only curve in which the perpendicular from the foot of the ordinate on the tangent is of constant length.

3. Of all the catenaries which pass through two given points at the same level, and have their axes vertical, shew that there is one in which the depth of the directrix below the given points is a minimum.

Also prove that in this catenary the tangents at the given points meet on the directrix.

If $2b$ be the distance between the given points, the depth of the directrix is $b \sinh u$, the arc of the curve is $b(\sinh u)/u$, and the inclination to the horizontal at the given points is $\cos^{-1}(\operatorname{sech} u)$, where u is the positive root of $u \tanh u = 1$.

4. The coordinates of any point on the tractrix may be expressed in the forms

$$x = a(u - \tanh u), \quad y = a \operatorname{sech} u,$$

where u is a variable parameter.

5. Prove that, in the tractrix,

$$y = ae^{-s/a},$$

the arc s being measured from the cusp.

6. The volume of the solid generated by the revolution of the tractrix about its asymptote is $\frac{2}{3}\pi a^3$.

The surface of the same solid is $4\pi a^2$.

7. If the coordinates of a moving point be

$$x = a \cosh nt, \quad y = b \sinh nt,$$

where t is the time, the path is a hyperbola, and the velocity varies as the length of the semi-conjugate diameter measured up to its intersection with the conjugate hyperbola.

Also shew that the area swept over by the radius vector increases uniformly with the time.

8. The area of either loop of the Lissajous' curve

$$x = a \sin 2(nt - \epsilon), \quad y = b \cos nt$$

is $\frac{4}{3} ab \cos 2\epsilon$.

9. Prove that the Lissajous' curve

$$x = a \cos nt, \quad y = b \cos 3nt$$

consists of part of the curve

$$\frac{y}{b} = \frac{x}{a}\left(4\frac{x^2}{a^2} - 3\right).$$

Trace this curve.

10. If, in the cycloid, the rolling circle has a constant angular velocity, the velocity of the tracing point P is proportional to the normal IP (see Fig. 84).

11. The volume generated by the revolution of a cycloid about its base is $5\pi^2 a^3$, if a be the radius of the generating circle.

The surface of the same solid is $\frac{64}{3}\pi a^2$.

12. The portion of a cycloid between two consecutive cusps revolves about the tangent at the vertex; prove that the area of the surface generated is $\frac{32}{3}\pi a^2$.

Also prove that the volume included by the above surface and the planes of the circles described by the cusps is $\pi^2 a^3$.

13. The volume generated by the revolution of a cycloid about its axis is $\frac{1}{6}(9\pi^2 - 16)\pi a^3$.

14. The surface of the same solid is $\frac{8}{3}(3\pi - 4)\pi a^2$.

15. The mass-centre of the arc of a cycloid, from cusp to cusp, is at a distance $\frac{4}{3}a$ from the base.

16. The mass-centre of the area included between a cycloid and its base is at a distance $\frac{5}{6}a$ from the base.

17. Prove that, in the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$

the intercepts made by the tangent at any point on the coordinate axes are $a^{\frac{2}{3}}x^{\frac{1}{3}}$, $a^{\frac{2}{3}}y^{\frac{1}{3}}$, respectively.

Hence verify that the length of the tangent intercepted by the axes is constant.

18. Prove, from the equations

$$x = a\cos^3\theta, \quad y = a\sin^3\theta,$$

that, in the astroid,

$$\frac{ds}{d\theta} = 3a\sin\theta\cos\theta,$$

and thence that the whole length of the curve is $6a$.

19. Prove that the area of the astroid is $\frac{3}{8}\pi a^2$.

20. The volume generated by the revolution of the astroid about the line joining two opposite cusps is $\frac{32}{105}\pi a^3$.

21. The length of a quadrant of the curve

$$x = a\cos^3\theta, \quad y = b\sin^3\theta,$$

is

$$(a^2 + ab + b^2)/(a + b).$$

The area enclosed by the same curve is $\frac{3}{8}\pi ab$.

22. The whole perimeter of an n-cusped epi- or hypo-cycloid is

$$\frac{8(n \pm 1)}{n} a,$$

where a is the radius of the fixed circle.

23. Sketch the curve obtained by compounding two uniform circular motions when the radii of the circles are equal, but the periods slightly different, (i) when the rotations are in the same direction, and (ii) when they are in opposite directions.

24. Prove that in an epicyclic the tangent line cannot pass through the centre unless $nc < n'c'$, where c is the greater of the two quantities c, c'. (Art. 139.)

140. Curves referred to Polar Coordinates. The Spirals.

There are several curves of interest whose equations are most conveniently expressed in polar coordinates. We begin with the 'spirals.'

1°. The 'equiangular spiral' is defined by the property that the curve makes a constant angle with the radius vector.

Denoting this angle by α, we have, by Art. 55,

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\alpha \quad\text{.......................}(1),$$

whence, integrating,

$$\log r = \theta \cot\alpha + \text{const.},$$

or

$$r = ae^{\theta\cot\alpha} \quad\text{.......................}(2).$$

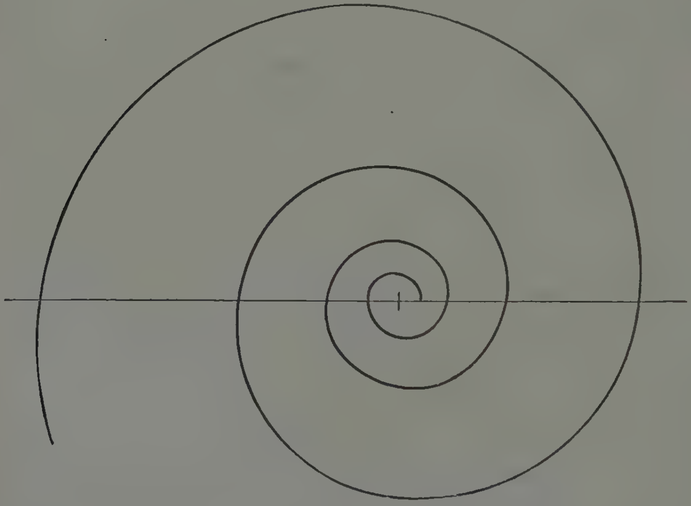

Fig. 100.

As θ ranges from $-\infty$ to $+\infty$, r ranges from 0 to ∞. See Fig. 100.

Since, by Art. 110, we have $dr/ds = \cos\alpha$, it appears that the length of the curve, between the radii r_1, r_2, is

$$\int_{r_1}^{r_2} \frac{ds}{dr}\,dr = (r_2 - r_1)\sec\alpha \quad\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

2°. The 'spiral of Archimedes' is the curve described by a point which travels along a straight line with constant velocity, whilst the line rotates with constant angular velocity about a fixed point in it.

In symbols, $r = ut, \quad \theta = nt,$

whence $$r = a\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

if $a = u/n$.

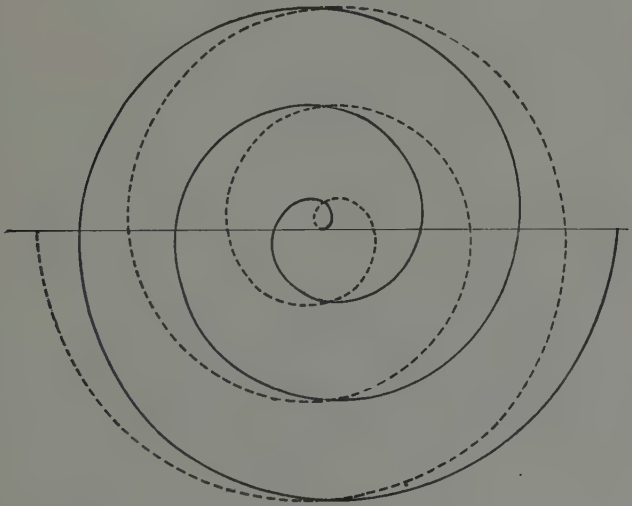

Fig. 101.

Fig. 101 shews the curve. The dotted branch corresponds to negative values of θ.

Another mode of generation of this curve has been explained in Art. 138.

3°. The 'reciprocal spiral' is defined by the equation

$$r = a/\theta \quad\text{.........................}(5).$$

If y be the ordinate drawn to the initial line, we have

$$y = r \sin\theta = a\,\frac{\sin\theta}{\theta}.$$

As θ approaches the value zero, r becomes infinite, but y approaches the finite limit a. Hence the line $y = a$ is an asymptote.

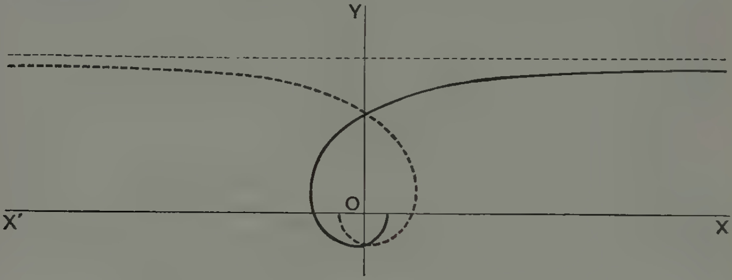

Fig. 102.

The dotted part of the curve in Fig. 102 corresponds to negative values of θ.

141. The Limaçon, and Cardioid.

If a point O on the circumference of a fixed circle of radius $\frac{1}{2}a$ be taken as pole, and the diameter through O as initial line, the radius vector of any point Q on the circumference is given by

$$r = a\cos\theta \quad\text{......................}(1).$$

If on this radius we take two points P, P' at equal constant distances c from Q, the locus of these points is called a 'limaçon.' Its equation is evidently

$$r = a\cos\theta + c \quad\text{....................}(2).$$

This includes the paths both of P and of P', if θ range from 0 to 2π.

If $c < a$, the curve passes through O when

$$\theta = \cos^{-1}(-c/a),$$

and forms a loop. See the curve traced by the points P_2, P_2' in Fig. 103. If $c > a$, r cannot vanish; see the curve traced by P_3, P_3' in the figure.

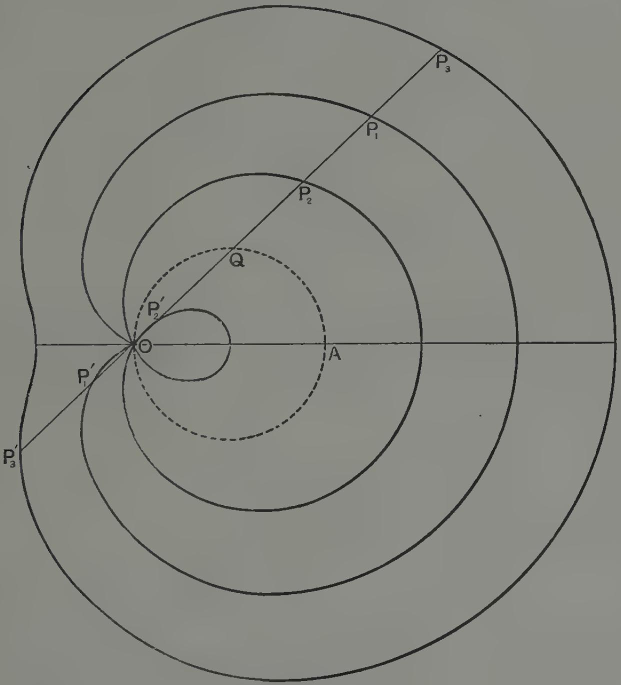

Fig. 103.

In the critical case of $c = a$, the loop shrinks into a cusp. The locus is now called a 'cardioid' or heart-shaped curve. Its equation is

$$r = a(1 + \cos\theta) \text{.....................} (3).$$

See the curve traced by P_1, P_1' in the figure. Also Fig. 89, p. 355.

142. The curves $r^n = a^n \cos n\theta$.

A number of important curves are included in the type

$$r^n = a^n \cos n\theta \quad\text{..................... (1).}$$

Thus if $n = \pm 1$, we have the circle

$$r = a \cos \theta \quad\text{........................ (2)}$$

and the straight line

$$r \cos \theta = a \quad\text{........................(3).}$$

If $n = \pm 2$ we have the 'lemniscate of Bernoulli'

$$r^2 = a^2 \cos 2\theta \quad\text{..................... (4),}$$

and the rectangular hyperbola

$$r^2 \cos 2\theta = a^2 \quad\text{..................... (5).}$$

The equation (4) makes r real for values of θ between $\pm\frac{1}{4}\pi$, imaginary for values between $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$, and so on. Also r^2 is a maximum for $\theta = 0$, $\theta = \pi$, etc. It follows that the lemniscate consists of two loops, with a node at the origin. See Fig. 113, p. 389.

If $n = \pm\frac{1}{2}$, we have the cardioid

$$r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \tfrac{1}{2}\theta, \quad\text{or}\quad r = \tfrac{1}{2}a(1 + \cos\theta) \quad\text{.........(6),}$$

and the parabola

$$r^{\frac{1}{2}} \cos \tfrac{1}{2}\theta = a^{\frac{1}{2}}, \quad\text{or}\quad r = \frac{2a}{1 + \cos\theta} \quad\text{..............(7).}$$

The curves corresponding to equal, but oppositely-signed, values of n, are 'inverse' to one another; see Art. 145.

If we differentiate (1) logarithmically, we find, if ϕ denote the angle between the tangent and the radius vector,

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta \quad\text{..................(8),}$$

or

$$\phi = \tfrac{1}{2}\pi + n\theta \quad\text{........................(9).}$$

The student should examine the meaning of this result in the various special cases mentioned above.

143. Tangential-Polar Equation.

If p be the perpendicular from the origin on any tangent, and r the radius vector of the point of contact, p will in general be a function of r. The equation expressing this relation is called the 'tangential-polar' equation of the curve.

If the ordinary polar equation be given, the tangential-polar equation is to be found by eliminating θ and ϕ between the formulæ

$$p = r \sin \phi, \quad \frac{1}{r}\frac{dr}{d\theta} = \cot \phi \quad \text{.................} \quad (1)$$

(for which see Art. 55) and the given equation.

From (1) we obtain

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 \phi) = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{..........} \quad (2).$$

It is occasionally convenient to employ the reciprocal of the radius vector instead of the radius itself. If we write

$$u = \frac{1}{r}, \quad \text{we have} \quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} \quad \text{...........} \quad (3),$$

and the formula (2) takes the shape

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \text{......................} \quad (4).$$

Ex. 1. In the equiangular spiral, we have

$$p = r \sin \alpha \quad \text{.........................} \quad (5).$$

Ex. 2. In the circle $\quad r = 2a \sin \theta$ ....(6)

we have, as in Art. 55, $\phi = \theta$, and therefore $p/r = r/2a$, or

$$p = r^2/2a \quad \text{...........................} \quad (7).$$

Ex. 3. In the parabola

$$r = a \sec^2 \tfrac{1}{2}\theta \quad \text{.........................} \quad (8),$$

where the focus is the pole, we find $\phi = \frac{1}{2}\pi - \frac{1}{2}\theta$, $p = r \cos \frac{1}{2}\theta$, whence

$$p^2 = ar \quad \text{..............................} \quad (9).$$

This is a well-known property of the curve.

This example, like the preceding, is included in a general result embracing all curves of the type

$$r^n = a^n \cos n\theta \text{ ....}(10).$$

By Art. 142 (9) we have

$$p = r \sin \phi = r \cos n\theta \text{ ....}(11),$$

whence, eliminating θ,

$$p = r^{n+1}/a^n \text{ ....}(12).$$

Thus in the case of the cardioid ($n = \frac{1}{2}$), we have

$$p^2 = r^3/a \text{ ....}(13).$$

Ex. 4. The tangential-polar equations of the central conics may be given here, as they are sometimes employed in Dynamics, although the proofs will not require the use of the Calculus.

First, let the origin be at the centre. The Cartesian equation of the conic being

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \text{ ....}(14),$$

we have, if β be the semi-conjugate diameter,

$$p\beta = ab, \text{ and } \beta^2 \pm r^2 = b^2 \pm a^2 \text{..............}(15),$$

by known properties of central conics. Hence

$$\frac{a^2b^2}{p^2} = b^2 \pm a^2 \mp r^2 \text{ ... .}(16).$$

In the particular case of the rectangular hyperbola we have

$$pr = a^2 \text{ ....}(17),$$

since $\beta = r$. This is also obtained by making $n = -2$ in (12) above.

Ex. 5. Again, taking a focus as pole, let us denote the perpendicular and radius vector corresponding to the other focus by p' and r'. Since the tangent makes equal angles with the two focal radii, we have $p/r = p'/r'$, and therefore

$$\frac{p^2}{r^2} = \frac{pp'}{rr'}.$$

Now $pp' = b^2$, and, in the ellipse, $r + r' = 2a$. Hence, for this curve,

$$p^2/r = b^2/(2a - r),$$

or, if l denote the semi-latus rectum (b^2/a),

$$\frac{l}{p^2} = \frac{2}{r} - \frac{1}{a} \text{ ....}(18).$$

In the hyperbola, we find

$$\frac{l}{p^2} = \pm\frac{2}{r} + \frac{1}{a} \quad\text{.....................(19)},$$

the upper sign relating to the branch nearest to the origin, the lower to the further branch.

It is important, with a view to some applications in Dynamics, to notice that if the tangential-polar equation be given, say

$$p = f(r) \quad\text{.......................(20)},$$

the curve is determinate save as to orientation. For we have

$$r\frac{d\theta}{dr} = \tan\phi = \frac{p}{\sqrt{(r^2 - p^2)}} \quad\text{...........(21)},$$

whence

$$\theta - \alpha = \int \frac{pdr}{r\sqrt{(r^2 - p^2)}} \quad\text{................(22)}.$$

A variation of the additive constant α has merely the effect of turning the curve bodily through an angle about O.

Ex. 6. To find the curve in which

$$p = r^3/a^2 \quad\text{.........................(23)}.$$

Substituting in (21), and integrating, we find

$$\theta - \alpha = \int \frac{rdr}{\sqrt{(a^4 - r^4)}} = \tfrac{1}{2}\sin^{-1}\frac{r^2}{a^2},$$

or

$$r^2 = a^2 \sin 2(\theta - \alpha) \quad\text{...................(24)},$$

a lemniscate.

EXAMPLES. XLII.

1. Prove that all equiangular spirals of the same angle are identically equal.

2. Prove that in an equiangular spiral of angle α the area swept over by the radius vector (r) is

$$\tfrac{1}{4}(r_2^2 - r_1^2)\tan\alpha,$$

where r_1, r_2 are the extreme values of r.

3. Prove that in the spiral of Archimedes the angle (ϕ) between the tangent and the radius vector is given by

$$\cos\phi = \frac{a}{\sqrt{(a^2+r^2)}}.$$

4. Prove that in the reciprocal spiral the area swept over by the radius increases proportionally to the radius.

5. Shew that all chords drawn through the pole of a cardioid are of the same length.

Does the same hold of the limaçon?

6. The area of the cardioid

$$r = a(1+\cos\theta)$$

is $\frac{3}{2}\pi a^2$.

7. Prove that, in the cardioid,

$$\frac{ds}{d\theta} = 2a\cos\tfrac{1}{2}\theta,$$

and thence that the whole perimeter is $8a$.

8. The volume generated by the revolution of the cardioid about its axis is $\frac{8}{3}\pi a^3$.

9. Prove that, in the cardioid, the maximum breadth (perpendicular to the axis) is $\frac{3}{2}\sqrt{3}a$, and that the double tangent cuts the axis at a distance $\frac{1}{4}a$ from the pole.

10. Find the maximum ordinate, and the minimum abscissa, in the limaçon

$$r = a\cos\theta + c.$$

11. The area of the limaçon

$$r = a\cos\theta + c,$$

when $c > a$, is

$$\pi(c^2 + \tfrac{1}{2}a^2).$$

12. Prove geometrically that if two straight lines, touching two fixed circles, make a constant angle with one another, their intersection traces out a limaçon.

13. The whole area of the lemniscate

$$r^2 = a^2\cos 2\theta$$

is a^2.

14. The perimeter of either loop of the same curve is

$$2a\int_0^{\frac{1}{4}\pi}\frac{d\theta}{\sqrt{(1-2\sin^2\theta)}}.$$

Prove that, in the notation of elliptic integrals (Art. 109), this is equal to

$$\sqrt{2}aF_1\left(\frac{1}{\sqrt{2}}\right)$$

15. The mass-centre of the area of either loop of the lemniscate is at a distance $\frac{1}{8}\sqrt{2}\pi a$ from the pole.

16. Shew that the area included by one loop of the epicyclic

$$r=a\sin m\theta$$

is $\pi a^2/4m$.

17. Trace the curve $r^2=a^2\cos\theta$.

18. Prove the following properties of the 'solid of greatest attraction' (viz. the figure generated by the revolution of the curve $r^2=a^2\cos\theta$ about the initial line):

(1) The volume is $\frac{4}{15}\pi a^3$;

(2) The greatest breadth is $1{\cdot}2408a$, at a distance $\cdot4389a$ from the pole;

(3) The mass-centre of the volume is at a distance $\frac{15}{32}a$ from the pole.

19. If the 'polar subtangent' of a curve be defined to be the length intercepted by the tangent, on a perpendicular drawn to the radius vector from the pole, prove that it is equal to $r^2d\theta/dr$.

Prove that in the reciprocal spiral the polar subtangent is constant.

20. The tangential-polar equation of the involute of a circle of radius a is

$$p^2=r^2-a^2,$$

the centre being pole.

21. Shew that in the spiral of Archimedes (Fig. 101)

$$p^2=\frac{r^4}{a^2+r^2}.$$

22. Shew that in the reciprocal spiral (Fig. 102)

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{a^2}.$$

23. Shew that in the curve

$$r = \frac{a}{\cos m\theta},$$

$$\frac{1}{p^2} = \frac{1 - m^2}{r^2} + \frac{m^2}{a^2}.$$

24. Shew that in the curves

$$r = \frac{a}{\cosh m\theta}, \quad r = \frac{a}{\sinh m\theta},$$

$$\frac{1}{p^2} = \frac{1 + m^2}{r^2} \mp \frac{m^2}{a^2},$$

respectively.

25. Prove that, in the epicycloid (Art. 137),

$$r^2 = a^2 + \frac{4(a+b)b}{(a+2b)^2} p^2.$$

What is the corresponding formula for the hypocycloid?

26. Prove the formula

$$s = \int \frac{r dr}{\sqrt{(r^2 - p^2)}}$$

for the arc of a curve whose tangential-polar equation is given.

27. Prove the formula

$$p ds = r^2 d\theta,$$

and give its geometrical interpretation.

Hence shew that if the area swept over by the radius vector of a moving point increase uniformly with the time, the velocity will vary inversely as the perpendicular from the origin on the tangent to the path.

144. Associated Curves. Similarity.

There are several methods of associating with a given curve another curve connected with it by a definite relation.

The simplest relation is that of similarity. Two plane figures are said to be 'similar' when they differ only in scale. More precisely, it is implied that to every point P of one figure corresponds a point P' of the other, and that the distance PQ between any two points of the one bears a constant ratio to the distance $P'Q'$ between the corresponding points in the other.

It follows that if P, Q, R be any three points of the one figure, and P', Q', R' the corresponding points of the other, the triangles PQR and $P'Q'R'$ will be equiangular to one another. Hence straight lines in one figure will correspond to straight lines in the other, and angles in the one figure to equal angles in the other.

If we take rectangular axes OX, OY in the one figure, and the corresponding axes $O'X'$, $O'Y'$ in the other, the coordinates x, y of any point P in the one figure will be connected with the coordinates x', y' of the corresponding point P' of the other by the relations

$$x' = mx,\ y' = my \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

where m is a constant ratio.

If by a displacement of either figure in its own plane $O'X'$ and $O'Y'$ can be made to coincide with OX and OY, respectively, the two figures are said to be 'directly' similar. In the new position, any two corresponding points P, P' are in a straight line with O; moreover $OP' = m \,.\, OP$, and any two corresponding straight lines are parallel. The same statements hold if either figure be turned about O through two right angles, until OX' coincides with XO produced, and OY' with YO produced. In either of these two relative positions the figures are said to be 'similarly situated,' and the origin is called the 'centre of similitude.' If

$$r = f(\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

be the polar equation of any curve in the one figure, the equation of the corresponding curve in the other will be

$$r = \pm\, mf(\theta) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

It may happen, however, that when $O'X'$ is brought into coincidence with OX, the line $O'Y'$ coincides, not with OY

but with *YO* produced. The two figures may be said, in this case, to be 'perversely' similar. To the curve (2) would now correspond

$$r = mf(-\theta) \quad\text{.......................}(4).$$

We shall not have occasion to consider this kind of relation.

Ex. 1. Conics of the same excentricity are similar.

Let the two conics be placed so as to have a common focus, and the perpendiculars on the corresponding directrices coincident in direction. The polar equations will then be of the forms

$$\frac{l}{r} = 1 + e\cos\theta,\quad \frac{l'}{r} = 1 + e\cos\theta \quad\text{..............}(5),$$

whence it follows that radii drawn in the same direction are in the constant ratio $l : l'$.

Ex. 2. All catenaries are similar curves.

For the equation

$$y = c\cosh x/c \quad\text{.......................}(6)$$

is unaltered when x, y, c are all altered in any the same ratio.

145. Inversion.

If from a fixed origin O we draw a radius vector OP to any given curve, and in OP take a point P' such that

$$OP \,.\, OP' = \pm k^2 \quad\text{....................}(1),$$

where k is a given constant, the locus of P' is said to be the 'inverse' of that of P. The point O is called the 'centre,' and $\pm k^2$ is called the 'constant,' of inversion.

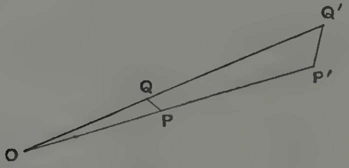

Fig. 104.

A curve and its inverse make supplementary angles with the radius vector. For if P, Q be consecutive points of a curve, and P', Q' the corresponding points on the inverse curve, we have $OP \,.\, OP' = OQ \,.\, OQ'$, and therefore

$$OP : OQ = OQ' : OP' \text{..................(2).}$$

Hence the triangles POQ, $Q'OP'$ are similar, and the angles OPQ, $OP'Q'$ are supplementary. In the limit, when Q is infinitely close to P, these are the angles which the respective tangents make with the radius vector.

The curves obtained from a given curve, with the same centre but different constants of inversion, are similar. For

if $$rr_1 = \text{const.},\ rr_2 = \text{const.} \text{ ....(3),}$$

we have $$r_1 : r_2 = \text{const.} \text{ ....(4).}$$

Ex. 1. The inverse of a straight line is a circle through the centre of inversion, and *vice versâ.*

First let the constant of inversion be equal to the square of the perpendicular distance OA of the origin O from the given straight line. If P be any point on the line, and if OP meet the circle on OA as diameter in P', we have

$$OP \,.\, OP' = OA^2 = k^2 \text{ ....(5),}$$

that is, the straight line inverts into the circle.

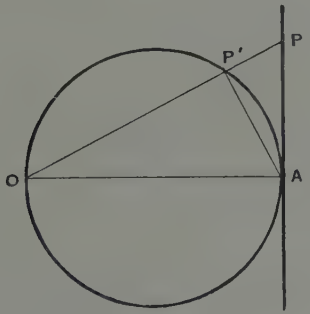

Fig. 105.

If the constant of inversion be changed we get a similar curve, which will still be a circle through the centre of similitude O.

Ex. 2. More generally, the inverse of any circle is a circle.

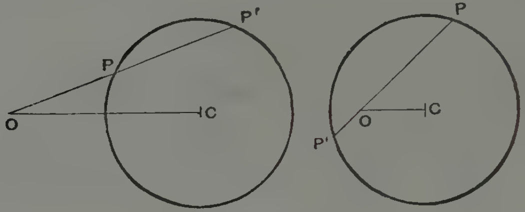

Fig. 106.

Let O be the centre of inversion, C the centre of the given circle, a its radius; and let

$$k^2 = OC^2 - a^2 \quad\text{.......................}\quad (6).$$

If, then, we draw any chord OPP' through O, it is known from Geometry that

$$OP \, . \, OP' = OC^2 - a^2 = k^2 \quad\text{..................}\quad (7)$$

Hence P' traces out the inverse of the locus of P; *i.e.* the circle inverts into itself.

And by changing the constant of inversion we get a similar curve, and therefore a circle.

If, as in the right-hand figure, O be within the given circle, the constant in (7) is negative. This means that P' and P are now on opposite sides of O.

146. **Mechanical Inversion.**

There are various devices by which the inverse of a given curve can be traced mechanically.

1°. *Peaucellier's Linkage.*

This consists of a rhombus $PAQB$ formed of four rods freely jointed at their extremities, and of two equal bars connecting two opposite corners A, B to a fixed pivot at O.

It is evident that, whatever shape and position the linkage assumes, the points P, Q will always be in a straight line with O. If N be the intersection of the diagonals of the rhombus, we have

$$OP \,.\, OQ = ON^2 \sim PN^2 = OA^2 \sim AP^2 = \text{const.} \dots (1).$$

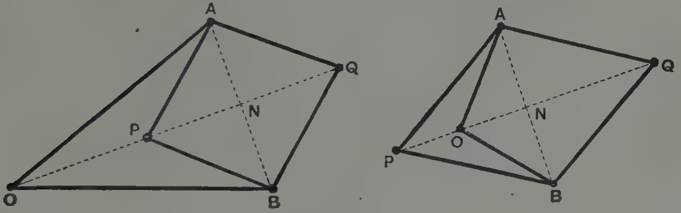

Fig. 107.

Hence if P (or Q) be made to describe any curve, Q (or P) will describe the inverse curve with respect to O.

In particular if, by a link, P be pivoted to a fixed point S, such that $SO = SP$, the locus of P is a circle through O, and consequently the locus of Q will be a straight line perpendicular to OS. This gives an exact solution of the important mechanical problem of converting circular into rectilinear motion by means of link-work.

2°. *Hart's Linkage.*

This consists of a 'crossed parallelogram' $ABCD$ formed

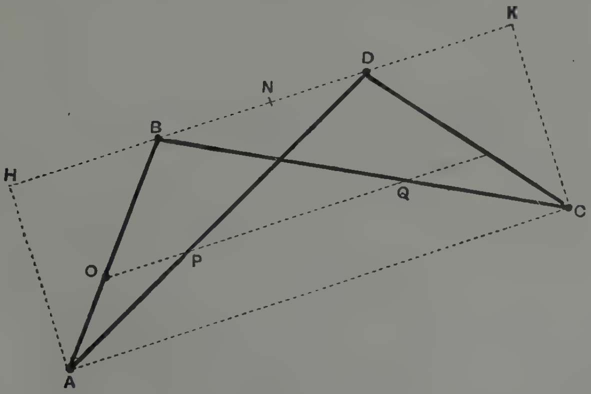

Fig. 108.

of four rods jointed at their extremities, the alternate sides being equal. A point O in one side AB is made a fixed pivot, and P, Q are points in AD and BC such that

$$AP : PD = CQ : QB = AO : OB, = m : n, \text{ say.}$$

Evidently O, P, Q will lie in a straight line parallel to AC and BD. If H, K be the orthogonal projections of A, C on BD, and if N be the middle point of BD, we have

$$AC . BD = 2NH . 2NB = DH^2 - BH^2 = AD^2 - AB^2.$$

Now $$OP : BD = AO : AB = m : m+n,$$

and $$OQ : AC = BO : AB = n : m+n.$$

Hence $$OP . OQ = \frac{mn}{(m+n)^2}(AD^2 - AB^2) = \text{const.} \text{ ...... (2).}$$

Hence P and Q describe inverse curves with respect to O.

As before, by connecting P to a fixed pivot S by a link PS equal to SO, we can convert circular into rectilinear motion.

147. Pedal Curves.

If a perpendicular OZ be drawn from a fixed point O on the tangent to a curve, the locus of the foot Z of this perpendicular is called the 'pedal' of the original curve with respect to the origin O.

Thus: the pedal of a parabola with respect to the focus is the tangent at the vertex. The pedal of an ellipse or hyperbola with respect to either focus is the 'auxiliary circle.'

If $OZ = p$, and if ψ be the angle which OZ makes with any fixed straight line, then p, ψ may be taken to be the polar coordinates of Z with respect to O as pole. Hence if the relation between p and ψ can be found, the polar equation of the pedal can be at once written down.

Ex. 1. If the origin be at the centre of the conic

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \text{ ....(1),}$$

and ψ be the angle which p makes with Ox, it is shewn in books on Conic Sections that

$$p^2 = a^2 \cos^2 \psi \pm b^2 \sin^2 \psi \text{.....................(2).}$$

Hence the polar equation of the pedal is

$$r^2 = a^2 \cos^2\theta \pm b^2 \sin^2\theta \quad\text{.....................(3).}$$

In the case of the rectangular hyperbola

$$x^2 - y^2 = a^2 \quad\text{..........................(4)}$$

the pedal is the lemniscate

$$r^2 = a^2 \cos 2\theta \quad\text{........................(5).}$$

Ex. 2. In the case of a circle of radius a, the pole O being at a distance c from the centre C, and the line OC being the origin of ψ, we have at once from a figure

$$p = a + c\cos\psi \quad\text{........................(6).}$$

Hence the pedal is the limaçon

$$r = a + c\cos\theta \quad\text{........................(7).}$$

If O be on the circumference, we have $c = a$, and the pedal is the cardioid

$$r = a(1 + \cos\theta) \quad\text{.......................(8).}$$

The angle which the tangent makes with the radius vector at corresponding points is the same for a curve and its pedal. For let OZ, OZ' be the perpendiculars from O on two consecutive tangents PZ, PZ', and let OU be drawn perpendicular to ZZ' produced. The points Z, Z' lie on the circle described on OP as diameter. Hence the exterior angle OZU of the quadrilateral $OZZ'P$ is equal to the interior and opposite OPZ'. In the limit these are the angles which OZ and OP make with the tangent to the pedal, and with the tangent to the original curve, respectively.

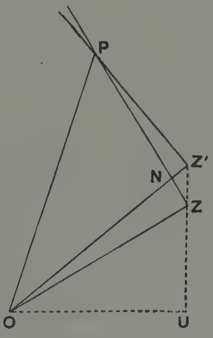

Fig. 109.

Also, by similar triangles, we have

$$OU : OZ = OZ' : OP \quad\text{..................(9).}$$

Hence if r be the radius vector of the original curve, p the perpendicular from O on the tangent, and p' the perpendicular from O on the tangent to the pedal, we have, ultimately,

$$p'/p = p/r, \text{ or } p' = p^2/r \qquad (10).$$

Again, if OZ' meet PZ in N, we may write,

$$OZ = p, \quad OZ' = p + \delta p, \quad \angle ZOZ' = \angle ZPZ' = \delta\psi.$$

Neglecting small quantities of the second order, we have

$$\delta p = NZ' = PZ'\delta\psi.$$

Hence, proceeding to the limit, when PZ' coincides with PZ, we obtain an expression for the projection of the radius vector on the tangent to a curve, viz.

$$PZ = \frac{dp}{d\psi} \qquad (11).$$

This result enables us easily to solve the problem of 'negative pedals,' viz. to find the curve having a given pedal. Taking O as origin, and the initial line of ψ as axis of x, the coordinates of the point of contact P are given by

$$x = OZ\cos\psi - ZP\sin\psi, \quad y = OZ\sin\psi + ZP\cos\psi,$$

or

$$\left.\begin{aligned} x &= p\cos\psi - \frac{dp}{d\psi}\sin\psi, \\ y &= p\sin\psi + \frac{dp}{d\psi}\cos\psi \end{aligned}\right\} \qquad (12).$$

Ex. 3. To find the curve whose pedal is the cardioid

$$r = a(1 + \cos\theta) \qquad (13).$$

Writing

$$p = a(1 + \cos\psi) \qquad (14),$$

the formulæ (12) make

$$x = a\cos\psi + a, \quad y = a\sin\psi,$$

whence

$$(x - a)^2 + y^2 = a^2 \qquad (15),$$

a circle through the origin.

148. Reciprocal Polars.

The locus of the pole of the tangent to a curve S, with respect to a fixed conic Σ, is called the 'reciprocal polar' of

S. It is proved in books on Conics that if S' be the locus of the poles of the tangents to S then S is the locus of the poles of the tangents to S'. This explains the use of the word 'reciprocal.'

We shall here only notice the case where the fixed conic Σ is a circle. If O be the centre of this circle, and k denote its radius, the pole P' of any tangent to the curve S is found

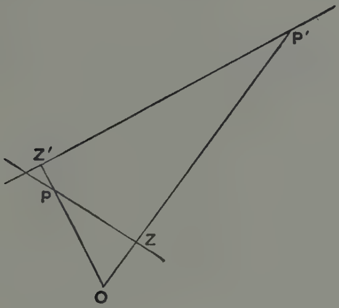

Fig. 110.

by drawing OZ perpendicular to this tangent, and by taking in OZ a point P' such that

$$OZ \,.\, OP' = k^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Hence the reciprocal polar is in this case the *inverse of the pedal* of the given curve, with respect to the point O.

By the reciprocal property above cited, the original curve must be the inverse of the pedal of the locus of P'. This is easily verified; for if P be the point of contact of the tangent to the original curve, and if OP meet the tangent to the locus of P' in Z', the angles $OP'Z'$ and OPZ will be equal, by Art. 147. Hence $OZ'P'$ is a right angle, and Z' traces out the pedal of P'. And, since $PZP'Z'$ is a cyclic quadrilateral, we have

$$OP \,.\, OZ' = OZ \,.\, OP' = k^2 \ldots\ldots\ldots\ldots\ldots (2).$$

Hence P describes the inverse of the locus of Z'.

Ex. 1. The reciprocal polar of a circle with respect to any origin is a conic having the origin as focus.

As in Art. 147, Ex. 2, the formula for the pedal of the circle is

$$p = a + c\cos\psi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Writing θ for ψ, and k^2/r for p, we get the equation of the reciprocal polar in the form

$$\frac{k^2}{r} = a + c\cos\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which represents a conic, having its focus at the origin, of eccentricity c/a. Hence the conic is an ellipse, parabola, or hyperbola, according as the origin is inside, on, or outside the circle.

Ex. 2. The pedal of the conic

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

with respect to the centre, is given by

$$p^2 = a^2\cos^2\psi \pm b^2\sin^2\psi \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Hence the reciprocal polar is

$$\frac{k^4}{r^2} = a^2\cos^2\theta \pm b^2\sin^2\theta \quad\ldots\ldots\ldots\ldots\ldots\ldots (7),$$

or

$$a^2x^2 \pm b^2y^2 = k^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8),$$

a concentric conic.

149. Bipolar Coordinates.

A curve may be defined by a relation between the distances (r, r') of any point P on it from two fixed points, or foci, S, S'; thus

$$f(r, r') = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

If we denote the angles PSS', $PS'S$ by θ, θ', respectively, and the angles which the radii r, r' make with the tangent by ϕ, ϕ', we have, as in Art. 110,

$$\left.\begin{aligned} \frac{dr}{ds} &= \cos\phi, & \frac{dr'}{ds} &= \cos\phi', \\ r\frac{d\theta}{ds} &= \sin\phi, & r'\frac{d\theta'}{ds} &= \sin\phi' \end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots (2).$$

We have, in addition, the relations

$$r\sin\theta = r'\sin\theta',\quad r\cos\theta + r'\cos\theta' = 2c \text{ ......} (3),$$

where $c = \frac{1}{2}SS'$.

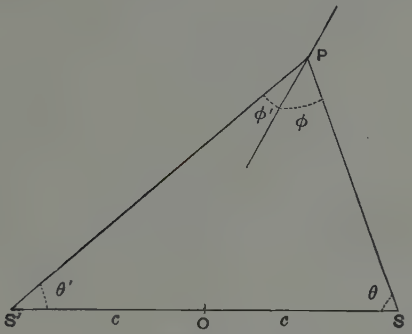

Fig. 111.

Ex. 1. In the ellipse we have

$$r + r' = 2a \text{ ....} (4),$$

and therefore

$$\frac{dr}{ds} + \frac{dr'}{ds} = 0,$$

that is, $\cos\phi + \cos\phi' = 0,\quad$ or $\quad \phi' = \pi - \phi$ .... (5).

The focal distances therefore make supplementary angles with the curve.

Similarly, in the hyperbola

$$r \sim r' = 2a \text{ ....} (6),$$

we find

$$\cos\phi = \cos\phi' \text{ ....} (7),$$

or, the focal distances make equal angles with the curve on opposite sides.

Ex. 2. To find the form which a reflecting or refracting surface must have in order that incident rays whose directions pass through a fixed point S may be reflected or refracted in directions passing through a fixed point S'.

The case of *reflection* is merely the converse of Ex. 1. The surface must have the form generated by the revolution of an ellipse or hyperbola about the line joining the foci (S, S').

In the case of *refraction*, we have, if μ and μ' be the refractive indices of the two media,

$$\mu \sin \chi = \mu' \sin \chi' \qquad (8),$$

where

$$\chi = \pm (\tfrac{1}{2}\pi - \phi), \quad \chi' = \pm (\tfrac{1}{2}\pi - \phi') \qquad (9).$$

Hence

$$\mu \cos \phi \pm \mu' \cos \phi' = 0 \qquad (10),$$

or

$$\frac{d}{ds}(\mu r \pm \mu' r') = 0 \qquad (11).$$

Integrating, we have

$$\mu r \pm \mu' r' = \text{const.} \qquad (12).$$

These curves, in which the sum (or difference) of given multiples of the two radii is constant, are called 'Cartesian ovals,' after Descartes, by whom the optical problem was first discussed.

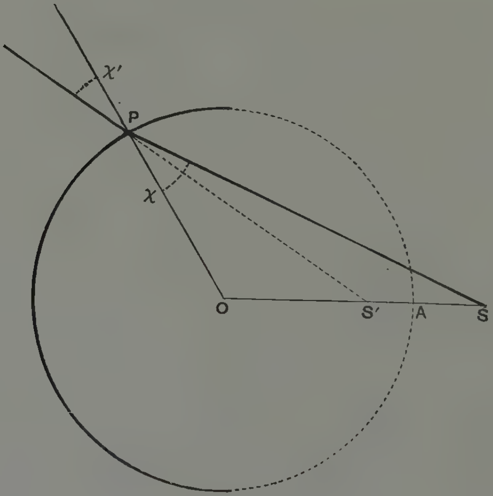

Fig. 112.

When the lower sign in (12) is taken, the family includes the *circle*

$$r/r' = \mu/\mu' \qquad (13).$$

See Fig. 112.

Ex. 3. The 'ovals of Cassini' are defined by

$$rr' = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

k being a given constant. Since for a point P in SS' the greatest value of rr' is c^2, it follows that the curve will consist of two detached ovals surrounding S, S', respectively, or of a single oval embracing both points, according as $k \lesseqgtr c$.

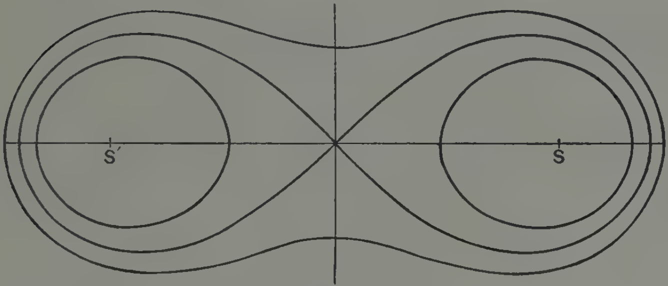

Fig. 113

In the critical case of $k = c$ the curve is known as the 'lemniscate of Bernoulli'; this presents itself in various mathematical problems. If O, the middle point of $S'S$, be taken as pole, and OS as initial line, of a system of coordinates r_1, θ_1, we have $\quad r^2 = r_1{}^2 + c^2 - 2cr_1 \cos\theta_1,\ \ r'^2 = r_1{}^2 + c^2 + 2cr_1 \cos\theta_1$;

the equation of the lemniscate is therefore

$$(r_1{}^2 + c^2)^2 - 4c^2 r_1{}^2 \cos^2\theta_1 = c^4,$$

which reduces to

$$r_1{}^2 = 2c^2 \cos 2\theta_1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Cf. Art. 142.

Ex. 4. The magnetic curves.

If S, S' be the N. and S. poles of a magnet, the forces at any point P may be represented by μ/r^2 along SP, and μ/r'^2 along PS'. A 'line of force' is a line drawn from point to point always in the direction of the resultant force. Expressing that the total force at right angles to the line is zero, we have

$$\frac{\mu}{r^2}\sin\phi + \frac{\mu}{r'^2}\sin\phi' = 0,$$

or
$$\frac{1}{r}\frac{d\theta}{ds}+\frac{1}{r'}\frac{d\theta'}{ds}=0 \text{ .... } (16).$$

Hence, since $r \sin\theta = r' \sin\theta'$, we have

$$\sin\theta\frac{d\theta}{ds}+\sin\theta'\frac{d\theta'}{ds}=0,$$

or
$$\cos\theta+\cos\theta'=\text{const. } \text{.................. } (17).$$

An 'equipotential line' is a line such that no work is done on a magnetic pole describing it. Expressing that the total force in the direction of the line is zero, we find

$$\frac{\mu}{r^2}\cos\phi-\frac{\mu}{r'^2}\cos\phi'=0,$$

or
$$\frac{1}{r^2}\frac{dr}{ds}-\frac{1}{r'^2}\frac{dr'}{ds}=0 \text{ .... } (18),$$

whence
$$\frac{1}{r}-\frac{1}{r'}=\text{const.} \text{ .... } (19).$$

The equipotential lines will necessarily cut the lines of force at right angles.

EXAMPLES. XLIII.

1. The inverse of an equiangular spiral with respect to the pole is an equal spiral.

2. The inverse of a hyperbola with respect to the centre has a node at the centre.

3. The inverse of a rectangular hyperbola with respect to the centre is a lemniscate of Bernoulli.

4. Prove by means of the polar equations that the inverse of a straight line is a circle through the pole of inversion, and conversely.

5. Prove by means of the polar equation that the inverse of a circle is a circle.

6. The inverse of a parabola with respect to the focus is a cardioid.

The inverse of any conic with respect to a focus is a limaçon.

7. Prove that the inverse of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with respect to the centre is the curve

$$(x^2+y^2)^2=k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right).$$

Also shew that the curve, where it cuts the axis of y, will be concave or convex to the origin according as $b^2 \lesseqgtr 2a^2$.

8. From the fact that the cardioid is the inverse of a parabola with respect to the focus, or otherwise, prove that the normals at the extremities of any chord through the cusp are at right angles, and that the line joining their intersection to the cusp is perpendicular to the chord.

9. Prove by inversion, or otherwise, that the cardioids

$$r=a\,(1+\cos\theta),\quad r=b\,(1-\cos\theta)$$

cut one another at right angles.

10. If ds, ds' be corresponding elements of a curve and its inverse,

$$ds : ds' = r^2 : k^2 = k^2 : r'^2,$$

where r, r' are the radii.

11. The pedal of a parabola with respect to its vertex is the cissoid (Art. 133 (16)).

12. If two tangents to a curve make a constant angle with one another, the locus of their intersection (P) touches the circle through P and the two points of contact.

13. Prove that the area of a pedal curve is given by the formula

$$\tfrac{1}{2}\int p^2 d\psi.$$

14. Prove that the arc of a pedal curve is expressed by

$$\int r\,d\psi.$$

15. The area of the pedal of an ellipse, the centre being pole, is

$$\tfrac{1}{2}\pi\,(a^2+b^2),$$

where a, b are the semi-axes.

16. The pedal of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

with respect to the centre consists of two loops, each of area

$$\tfrac{1}{2}ab+\tfrac{1}{2}(a^2-b^2)\tan^{-1}\frac{a}{b}.$$

17. If p_0, p_1 be perpendiculars on the tangent to a curve from the origin of (rectangular) coordinates, and from the point (x_1, y_1) respectively, prove that

$$p_1=p_0-x_1\cos\psi-y_1\sin\psi,$$

where ψ is the inclination of the perpendiculars to the axis of x.

18. If A_0, A_1 be the areas of the pedals of a closed oval curve with respect to the origin O and with respect to the point (x_1, y_1), both these points being within the curve, prove that

$$A_1=A_0-x_1\int_0^{2\pi}p_0\cos\psi d\psi-y_1\int_0^{2\pi}p_0\sin\psi d\psi+\tfrac{1}{2}\pi(x_1^2+y_1^2).$$

19. Prove that the locus of a point such that the pedal of a given closed oval curve with respect to it as pole has a given constant area is a circle; and that the circles corresponding to different values of the constant are concentric.

Also that, if O be the common centre, the area of the pedal with respect to any other point P exceeds the area of the pedal with respect to O by the area of the circle whose radius is OP.

20. The negative pedal of the parabola

$$y^2=4ax$$

with respect to the vertex is the curve

$$27ay^2=(x-4a)^3.$$

21. In what case is

$$p=a\cos\psi\,?$$

22. Prove that the curve for which

$$p=a\sin\psi\cos\psi$$

is the astroid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

23. State what property follows by differentiation with respect to the arc (s) from the equation

$$r^2 + r'^2 = k^2,$$

and verify the result geometrically.

24. Prove the following construction for the normal at any point P of a Cassini's oval: In PS, PS' take points Q, Q', respectively, such that $PQ = PS'$, and $PQ' = PS$; the line joining P to the middle point of QQ' is the required normal.

25. A system of parallel rays is to be reflected so as to pass through a fixed point; prove that the reflecting curve must be a parabola.

26. A system of parallel rays is to be refracted so that their directions pass through a fixed point; prove that the refracting curve must be a conic, and that the eccentricity of the conic will be equal to the ratio of the refractive indices.

27. Prove that the equation of a Cartesian oval, referred to either focus as pole, is of the form

$$r^2 - 2(a + b\cos\theta)\,r + c^2 = 0.$$

28. Prove that a Cartesian oval is necessarily closed, if we except the case where the curve is a branch of a hyperbola.

CHAPTER X.

CURVATURE.

150. Measure of Curvature.

As regards the applications of the Calculus to the theory of plane curves we have so far been concerned chiefly with the direction of the tangent at various points. We have not considered specially the manner in which this direction varies from point to point.

The subject of curvature, to which we now proceed, can be treated from several independent stand-points, and although all the methods lead to identically the same formulæ, it is important for the student to observe that they are in their foundations logically distinct.

In the first of these methods*, we begin by defining the 'total' or 'integral' curvature of an arc of a curve as the angle ($\delta\psi$) through which the tangent turns as the point of contact travels from one end of the arc to the other.

The 'mean curvature' of the arc is defined as the ratio of the total curvature to the length (δs) of the arc; it is therefore equal to

$$\frac{\delta\psi}{\delta s}.$$

The 'curvature at a point' P of a curve is defined as the mean curvature of an infinitely small arc terminated by that point. In conformity with the previous notation it is denoted by

---

* Other methods are explained in Arts. 153, 154.

$$\frac{d\psi}{ds} \text{..........................} (1).$$

In a circle of radius R we have $\delta s = R\delta\psi$, and therefore

$$\frac{d\psi}{ds} = \frac{1}{R},$$

i.e. the curvature of a circle is measured by the reciprocal of its radius. Hence, if ρ be the radius of the circle which has the same curvature as the given curve at the point P, we have

$$\rho = \frac{ds}{d\psi} \text{..........................} (2).$$

A circle of this radius, having the same tangent at P, and its concavity turned the same way, as in the given curve, is called the 'circle of curvature,' its radius is called the 'radius of curvature,' and its centre the 'centre of curvature.'

The length intercepted by this circle on a straight line drawn through P in any specified direction is called the 'chord of curvature' in that direction. If θ be the angle which the direction makes with the normal, the length (q) of the chord is given by

$$q = 2\rho \cos\theta \text{........................} (3).$$

If ξ, η be the rectangular coordinates of the centre of curvature, we have by orthogonal projections

$$\xi = x - \rho \sin\psi, \quad \eta = y + \rho\cos\psi \text{............} (4),$$

provided the zero of ψ be when the tangent is parallel to the axis of x.

The centre of curvature is the intersection of two consecutive normals to the given curve. For if PC, $P'C$ be the normals at two consecutive points, including an angle $\delta\psi$, and if δs be the arc PP', then drawing the chord PP' we have (see Fig. 114)

$$\frac{CP}{PP'} = \frac{\sin CP'P}{\sin\delta\psi},$$

or

$$CP = \sin CP'P \cdot \frac{PP'}{\delta s} \cdot \frac{\delta\psi}{\sin\delta\psi} \cdot \frac{\delta s}{\delta\psi}.$$

When P' is taken infinitely near to P, the limiting value of each factor on the right hand, except the last, is unity. Hence, ultimately, $CP = ds/d\psi = \rho$.

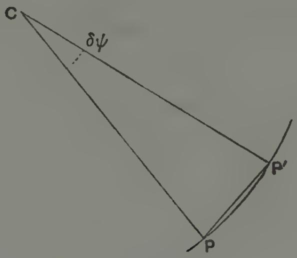

Fig. 114.

In modern geometry a curve is regarded as generated in a two-fold way, first as the locus of a point, and secondly as the envelope of a straight line (see Art. 158). Considering any continuous succession of these associated elements, the straight line is at any instant rotating about the point, and the point is travelling along the straight line; and the curvature $d\psi/ds$ expresses the relation between these two motions.

If at any point the curvature is zero, the rotation of the tangent is momentarily arrested, and we have what is called a 'stationary tangent.' The simplest instance of this is at a point of inflexion (Art. 68), where the direction of the rotation of the tangent is reversed after the stoppage.

If at any point the radius of curvature ($ds/d\psi$) vanishes, the motion of the point along the line is momentarily arrested, and we have a 'stationary point.' The simplest instance of this is at a 'cusp' such as we have met with in Figs. 75, 79, 84, 88, etc. The direction of motion of the point is in such cases reversed after the stoppage. In the examples of Art. 133 a cusp was regarded as due to the evanescence of a loop: this shews in another way why the radius of curvature should vanish there.

The consideration of curvature is of importance in numerous dynamical and physical problems. For example, in Dynamics, if the force acting on a moving particle be resolved into two components, along the tangent and normal to the path, respectively, the former component affects the velocity, and the latter the direction of motion. If from a fixed origin we draw a vector OV to represent the velocity at any instant, the polar coordinates of V may be taken to be v, ψ, where $v = ds/dt$. Hence the radial and transverse velocities of V will (see Art. 110 (8)) be

$$\frac{dv}{dt} \text{ and } v\frac{d\psi}{dt} \quad\quad\quad\quad (5),$$

respectively. These are the rates of change of the velocity estimated in the direction of the tangent and normal to the path of the particle. Since

$$v\frac{d\psi}{dt} = v\frac{d\psi}{ds}\frac{ds}{dt} = \frac{v^2}{\rho} \quad\quad\quad\quad (6),$$

the latter component is equal to the product of the curvature into the square of the velocity.

151. Intrinsic Equation of a Curve.

The formula

$$\rho = \frac{ds}{d\psi} \quad\quad\quad\quad (1)$$

is of course most immediately applicable when the relation between s and ψ for the curve in question is given in the form

$$s = f(\psi) \quad\quad\quad\quad (2).$$

This is called the 'intrinsic' equation of the curve, for the reason that its form does not depend materially on space-elements extraneous to the curve. The only arbitrary elements are the origin of s and the origin of ψ, and a change in either of these origins merely adds a constant to the corresponding variable.

Ex. 1. In the catenary we have

$$s = a \tan \psi \quad\quad\quad\quad (3),$$

whence

$$\rho = a \sec^2 \psi = y \sec \psi \quad\quad\quad\quad (4),$$

the notation being as in Art. 134. On reference to the figure there given it appears that the radius of curvature is equal to the normal PG.

Ex. 2. In the cycloid (Art. 136) we have

$$s = 4a \sin\psi \text{ ....(5)},$$

and therefore

$$\rho = 4a \cos\psi \text{ ....(6)}.$$

Hence in Fig. 84, p. 347, we have $\rho = 2PI$, or the radius of curvature is double the normal.

Ex. 3. Again, in the epicycloid we have (Art. 137 (11))

$$s = \frac{4(a+b)b}{a} \sin\frac{a}{a+2b}\psi \text{ ....(7)},$$

and therefore

$$\rho = \frac{4(a+b)b}{a+2b} \cos\frac{a}{a+2b}\psi = \frac{4(a+b)b}{a+2b} \cos\tfrac{1}{2}\phi \text{ ......(8)}.$$

Hence, on reference to Fig. 86, p. 351, it appears that

$$\rho = \frac{2(a+b)}{a+2b} PI \text{ ....(9)},$$

where PI is the length of the normal between the tracing point and the fixed circle.

If the intrinsic equation be not known, we may employ one or other of the formulæ of Art. 152; or we may, in particular cases, have recourse to special artifices.

Ex. 4. In the parabola $y^2 = 4ax$ we have, by Art. 53 (9),

$$y = 2a \cot\psi \text{ ....(10)},$$

whence

$$\sin\psi = \frac{dy}{ds} = -\frac{2a}{\sin^2\psi}\frac{d\psi}{ds},$$

or

$$\rho = -\frac{2a}{\sin^3\psi} \text{ ....(11)},$$

the negative sign representing the fact that ψ diminishes as s increases.

Ex. 5. If the ellipse

$$x = a\cos\phi,\ y = b\sin\phi \text{ ....(12)}$$

be supposed derived by homogeneous strain (or by orthogonal projection) from the circle

$$x = a\cos\phi,\ y = a\sin\phi \text{ ....(13)},$$

we have
$$\frac{ds}{d\phi}=\beta \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14),$$
where β is the semi-conjugate diameter. For the element of arc is altered from $a\delta\phi$ to δs, and the parallel radius from a to β. Also since $\frac{1}{2}\beta^2\delta\psi$ and $\frac{1}{2}a^2\delta\phi$ represent corresponding elements of area, we have
$$\beta^2\delta\psi=\frac{b}{a}\times a^2\delta\phi,$$
or
$$\frac{d\phi}{d\psi}=\frac{\beta^2}{ab}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$
Hence
$$\rho=\frac{ds}{d\psi}=\frac{ds}{d\phi}\frac{d\phi}{d\psi}=\frac{\beta^3}{ab}\;\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

If p be the perpendicular from the centre on the tangent-line, we have $p\beta=ab$, so that our result may also be written
$$\rho=\frac{\beta^2}{p},\quad \text{or}\quad \rho=\frac{a^2b^2}{p^3}\;\ldots\ldots\ldots\ldots\ldots\ldots (17).$$
Since $\quad p^2=a^2\cos^2\psi+b^2\sin^2\psi=a^2(1-e^2\sin^2\psi),$
the last form is equivalent to
$$\rho=a\frac{1-e^2}{(1-e^2\sin^2\psi)^{\frac{3}{2}}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

This formula leads to an important result in Geodesy. The figure of the Earth being taken to be an ellipsoid of revolution, the expression for the radius of curvature in terms of the latitude ψ, is, if we neglect e^4,
$$\frac{ds}{d\psi}=a(1-e^2+\tfrac{3}{2}e^2\sin^2\psi)=a(1-\tfrac{1}{2}\epsilon-\tfrac{3}{2}\epsilon\cos 2\psi)\ldots(19),$$
where $\epsilon=(a-b)/a=\frac{1}{2}e^2$; that is, ϵ denotes the 'ellipticity' of the meridian. Integrating (19) we find, for the length of an arc of the meridian, from the equator to latitude ψ,
$$s=a(1-\tfrac{1}{2}\epsilon)\psi-\tfrac{3}{4}a\epsilon\sin 2\psi\ldots\ldots\ldots\ldots\ldots(20).$$

Ex. 6. In the equiangular spiral (Art. 140), we have
$$\psi=\theta+\alpha\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (21),$$
whence
$$d\psi/ds=d\theta/ds=(\sin\alpha)/r,$$
or
$$\rho=\frac{r}{\sin\alpha}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22).$$
Hence the radius of curvature subtends a right angle at the origin.

152. Formulæ for the Radius of Curvature.

The expression $d\psi/ds$ for the curvature is easily translated into a variety of other forms.

1°. In rectangular Cartesian coordinates, we have

$$\tan\psi = \frac{dy}{dx} \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and therefore

$$\sec^2\psi \frac{d\psi}{ds} = \frac{d}{ds}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}\frac{dx}{ds} = \cos\psi \frac{d^2y}{dx^2},$$

whence

$$\frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} \quad\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

This form shews, again, that the curvature vanishes at a point of inflexion.

Ex. 1. In the catenary

$$y = a\cosh x/a \quad\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

we have

$$\frac{dy}{dx} = \sinh\frac{x}{a}, \quad \frac{d^2y}{dx^2} = \frac{1}{a}\cosh\frac{x}{a}, \quad 1+\left(\frac{dy}{dx}\right)^2 = \cosh^2\frac{x}{a},$$

whence

$$\rho = a\cosh^2 x/a = y^2/a \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Since $y = a\sec\psi$, this agrees with Art. 151, Ex. 1.

When dy/dx is a small quantity the formula (2) gives, approximately,

$$\frac{1}{\rho} = \frac{d^2y}{dx^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

the proportional error being of the *second* order.

The form (5) is an obvious transcript of $d\psi/ds$, since when ψ is small we may write dy/dx ($=\tan\psi$) for ψ, and d/dx for d/ds.

The approximate formula (5) has many important practical applications, *e.g.* to the theory of flexure of bars (see Art. 130). If the axis of x be parallel to the length, and if y be the lateral deflection at any point, the 'bending moment,' or 'flexural couple,' is

$$EAk^2\frac{d^2y}{dx^2} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

2°. It was proved in Art. 147 that the projection (t) of the radius on the tangent is given by

$$t=\frac{dp}{d\psi} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

If OU, OU' be the perpendiculars from the origin on two consecutive normals PC, $P'C$, and if OU' meet PC in N, we have, ultimately,

$$OU'-OU=U'N=CN\delta\psi, \quad\text{or}\quad \delta t=CN\delta\psi.$$

The limiting value of CU or CN is therefore $dt/d\psi$, whence

$$\rho=CP=OZ+CU=p+\frac{dt}{d\psi}=p+\frac{d^2p}{d\psi^2} \quad\ldots\ldots (8).$$

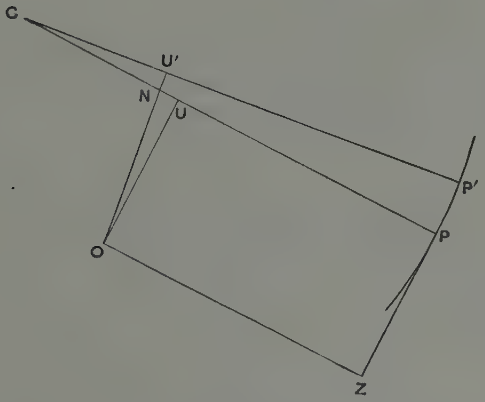

Fig. 115.

3°. With the notation of Arts. 110, 147 we have

$$\frac{t}{r}=\cos\phi=\frac{dr}{ds} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

Since $$\frac{ds}{d\psi}=\frac{ds}{dr}\frac{dr}{dp}\frac{dp}{d\psi}=t\frac{ds}{dr}\frac{dr}{dp},$$

this gives $$\rho = r\frac{dr}{dp} \qquad\qquad (10),$$

a form which is very convenient of application when the tangential-polar equation (Art. 143) is given.

Ex. 2. In the parabola
$$r = p^2/a \qquad\qquad (11)$$
we have $$\rho = r\frac{dr}{dp} = \frac{2p^3}{a^2} = \frac{2r^{\frac{3}{2}}}{a^{\frac{1}{2}}} \qquad\qquad (12).$$

Ex. 3. In the central conics we have (Art. 143, Ex. 4)
$$\frac{a^2b^2}{p^2} = b^2 \pm a^2 \mp r^2 \qquad\qquad (13),$$
and therefore $$\rho = \pm\frac{a^2b^2}{p^3} \qquad\qquad (14).$$
Cf. Art. 151, Ex. 5.

153. Newton's Method.

In another method of treating curvature, employed by Newton*, a circle is described touching the given curve at P, and passing through a neighbouring point Q on it, and we investigate the limiting value of the radius of this circle when Q is taken infinitely near to P.

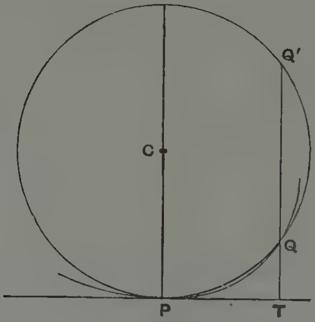

Fig. 116.

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* *Principia*, lib. i., prop. vi., cor. 3.

We can easily shew that in the limit the circle becomes identical with the 'circle of curvature' at P, as defined in Art. 150. For if C be the centre, then, since $CP = CQ$, there will be some point (P') on the curve, between P and Q, such that its distance from C is a maximum or minimum, and therefore* such that CP' is normal to the curve. In the limit P' approaches P indefinitely, and C, being the intersection of consecutive normals, will coincide with the 'centre of curvature' (Art. 150).

Newton's method leads to a very simple formula for the radius of curvature. Let $Q'QT$ be drawn perpendicular to the tangent at P, meeting the circle again in Q', and the tangent in T. Since

$$TP^2 = TQ \,.\, TQ',$$

we have

$$2\rho = \lim TQ' = \lim \frac{TP^2}{TQ} \qquad \text{(1)}.$$

If $Q'QT$ be drawn at a definite inclination to the normal at P, instead of parallel to this normal, the limiting value of the same fraction gives the chord of curvature in the corresponding direction. It occasionally happens that the chord of curvature in some particular direction can be found with special facility; the radius of curvature can then be inferred by the formula (3) of Art. 150.

Ex. 1. In the parabola, let QR be a chord drawn parallel to the tangent at P, to meet the diameter through P in V; see Fig. 117. We have, then, from the geometry of the curve,

$$QV^2 = 4SP \,.\, PV,$$

where S is the focus. Hence, for the chord of curvature (q) parallel to the axis,

$$q = \lim \frac{QV^2}{PV} = 4SP \qquad \text{(2)}.$$

If θ be the angle which the normal at P makes with the axis, we have $\cos\theta = SZ/SP$, where SZ is the perpendicular from the focus on the tangent at P. Hence

$$\rho = \tfrac{1}{2} q \sec\theta = 2\frac{SP^2}{SZ} = 2\frac{SP^{\frac{3}{2}}}{SA^{\frac{1}{2}}} \qquad \text{(3)},$$

since $SZ^2 = SA \,.\, SP$, A being the vertex.

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* See Art. 55, Ex. 2.

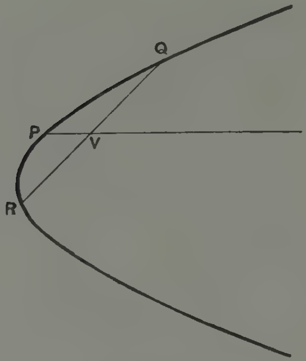

Fig. 117.

Ex. 2. In the ellipse (or hyperbola), if QR, drawn parallel to the tangent at either extremity of the diameter PCP', meet this diameter in V, we have

$$QV^2 : PV \,.\, VP' = CD^2 : CP^2,$$

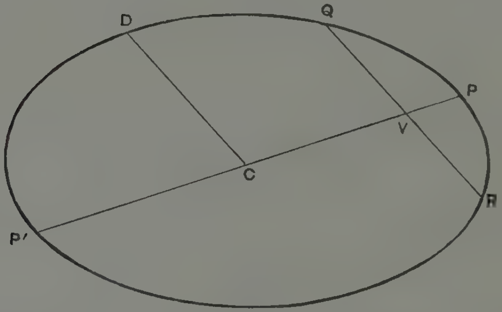

Fig. 118.

where CD is the semi-diameter conjugate to CP. Hence, for the chord of curvature (q) through the centre,

$$q = \lim \frac{QV^2}{PV} = \lim \frac{CD^2}{CP^2} \,.\, VP' = 2\,\frac{CD^2}{CP} \qquad \text{...........(4)}.$$

If CZ be the perpendicular from the centre on the tangent at P, and θ the angle which CP makes with the normal, we have $\cos\theta = CZ/CP$, and therefore

$$\rho = \tfrac{1}{2}q \sec\theta = \frac{CD^2}{CZ} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

in agreement with Art. 151 (17).

Again, if θ' be the inclination of either focal distance to the normal at P, it is known that $\cos\theta' = CZ/CA$, where A is an extremity of the major axis. The chord of curvature (q') through either focus is therefore given by

$$q' = 2\rho\cos\theta' = 2\,\frac{CD^2}{CA} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Ex. 3. To find the radius of curvature (ρ_0) at the vertex of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta) \quad\ldots\ldots\ldots\ldots (7).$$

We have

$$\frac{x^2}{2y} = a(\theta + \sin\theta)^2 \div 4\sin^2\tfrac{1}{2}\theta = a\left(1 + \frac{\sin\theta}{\theta}\right)^2 \div \left(\frac{\sin\frac{1}{2}\theta}{\frac{1}{2}\theta}\right)^2,$$

whence

$$\rho_0 = \lim_{\theta=0} \frac{x^2}{2y} = 4a \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

Newton's method, combined with the result of Art. 66, 2°, leads to a general formula for the chord of curvature parallel to the axis of y, and thence to the Cartesian expression for the radius of curvature. Denoting the chord in question by q, we have, in Fig. 43, p. 155,

$$\frac{1}{q} = \lim \frac{QV}{PV^2} = \lim \frac{QV}{AH^2}\,.\,\cos^2\psi = \tfrac{1}{2}\phi''(a)\cos^2\psi \ldots (9),$$

where ψ is the inclination of the tangent at P to the axis of x. Since

$$q = 2\rho\cos\psi, \quad \tan\psi = \phi'(a),$$

it follows that

$$\frac{1}{\rho} = \phi''(a)\cos^3\psi = \frac{\phi''(a)}{[1 + \{\phi'(a)\}^2]^{\frac{3}{2}}} \quad\ldots\ldots\ldots (10).$$

This is identical, except as to notation, with the formula (2) of Art. 152.

154. Osculating Circle.

A slightly different way of treating the matter is based on the notion of the 'osculating circle.' If Q and R be two neighbouring points of the curve, one on each side of P, we consider the limiting value of the radius of the circle PQR, when Q and R are taken infinitely close to P.

We can shew that if the curvature of the given curve be continuous at P, this circle coincides in the limit with the 'circle of curvature.' For if C be the centre of the circle PQR, there will be a point P', between P and Q, such that CP' is normal to the given curve, and a point P'', between P and R, such that CP'' is normal to the curve. Let $P'C$ and $P''C$ meet the normal at P in the points C' and C'', respectively. Under the condition stated, C' and C'' will ultimately coincide with the centre of curvature at P, and, since $CC' < C'C''$, C will *à fortiori* ultimately coincide with the same point.

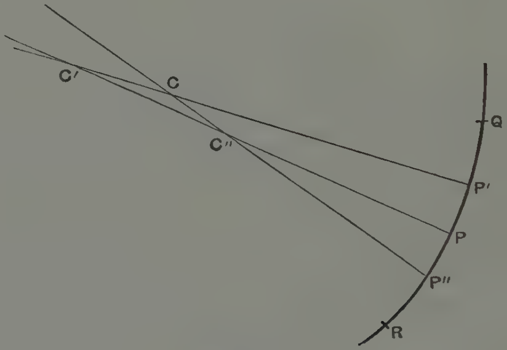

Fig. 119.

Since, before the limit, the circle PQR crosses the given curve three times in the neighbourhood of P, it appears that the osculating circle will in general cross the curve at the point of contact. See Fig. 123, p. 422.

Ex. If in Fig. 117, p. 404, the circle PQR meet PV in W, we have $QV \cdot VR = PV \cdot VW$, and therefore $VW = 4SP$. Hence the chord of curvature parallel to the axis of the parabola is $4SP$.

A similar argument may be used to find the chord of curvature through the centre, in the case of the ellipse (Fig. 118, p. 404).

If in Fig. 42, p. 153, QV meet the circle through P, Q, P' again in W, we have

$$VW = PV^2/QV,$$

and therefore, for the chord of curvature of the curve $y = \phi(x)$, parallel to the axis of y,

$$\frac{1}{q} = \lim \frac{QV}{PV^2} = \lim \frac{QV}{AH^2} \cos^2 \psi = \tfrac{1}{2}\phi''(a) \cos^2 \psi,$$

as in Art. 153 (9).

EXAMPLES. XLIV.

1. Prove that the circle is the only curve whose curvature is constant.

2. Prove that the intrinsic equation of an equiangular spiral is of the form

$$s = ae^{\psi \cot \alpha}.$$

3. Prove that the intrinsic equation of the tractrix may be written

$$s = a \log \operatorname{cosec} \psi.$$

Prove that in the tractrix the curvature varies as the normal.

4. By differentiation of the formulæ

$$\frac{dx}{ds} = \cos \psi, \quad \frac{dy}{ds} = \sin \psi,$$

prove that

$$\frac{1}{\rho} = -\frac{d^2x}{ds^2} \bigg/ \frac{dy}{ds} = \frac{d^2y}{ds^2} \bigg/ \frac{dx}{ds},$$

and

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

5. If a curve be defined by the equations

$$x = F(t), \quad y = f(t),$$

prove that

$$\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{\frac{3}{2}}},$$

where the accents denote differentiations with respect to t.

6. Apply the preceding formula to the cases of the ellipse

$$x = a \cos \phi, \quad y = b \sin \phi,$$

and the hyperbola

$$x = a \cosh u, \quad y = b \sinh u.$$

7. Prove that the curve whose intrinsic equation is

$$s = k \sin \psi$$

is a cycloid. (Use the method of Art. 134 (3).)

8. Given that in the 'catenary of equal strength'

$$\rho = k \sec \psi,$$

where ψ is the inclination to the horizontal, prove that if the origin be at the lowest point

$$x = k\psi, \quad y = k \log \sec \psi,$$

the axes of x and y being horizontal and vertical.

9. Given that the intrinsic equation of a curve is

$$s = k \sin^2 \psi,$$

deduce the Cartesian equation

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = (\tfrac{2}{3}k)^{\frac{2}{3}}.$$

10. If the coordinates x, y of a point on a curve be given functions of t, prove that

$$\frac{d^2x}{dt^2} = \frac{d^2s}{dt^2} \cos \psi - \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \sin \psi,$$

$$\frac{d^2y}{dt^2} = \frac{d^2s}{dt^2} \sin \psi + \frac{1}{\rho}\left(\frac{ds}{dt}\right)^2 \cos \psi,$$

and give the kinematical interpretation of these results.

Hence shew that

$$\frac{1}{\rho^2} = \left\{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 - \left(\frac{d^2s}{dt^2}\right)^2\right\} \div \left(\frac{ds}{dt}\right).$$

11. Shew how to express the coordinates x, y of a point on a curve whose Cartesian equation is given in terms of the inclination (ψ) of the tangent, and prove that

$$\rho = \sqrt{\left\{\left(\frac{dx}{d\psi}\right)^2 + \left(\frac{dy}{d\psi}\right)^2\right\}}.$$

12. Prove that, in the astroid

$$x = a\cos^3\theta, \quad y = a\sin^3\theta,$$

$$\psi = \pi - \theta,$$

and thence shew that

$$\rho = 3a\sin\theta\cos\theta.$$

13. Prove from the Cartesian formula of Art. 152 (2) that in the rectangular hyperbola $xy = k^2$

$$\rho = \frac{(x^2 + y^2)^{\frac{3}{2}}}{2k^2}.$$

14. Also that, in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\rho = \frac{(a^2 - e^2x^2)^{\frac{3}{2}}}{ab}.$$

15. Also that, in the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$$\rho = \frac{(e^2x^2 - a^2)^{\frac{3}{2}}}{ab}.$$

16. Also that, in the parabola $y^2 = 4ax$,

$$\rho = \frac{2(a + x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

17. Also that, in the semi-cubical parabola $ay^2 = x^3$,

$$\rho = \frac{(4a + 9x)^{\frac{3}{2}}x^{\frac{1}{2}}}{6a}.$$

18. Also that, in the cubical parabola $a^2y = x^3$,

$$\rho = \frac{a^2}{6x}\left(1 + 9\frac{x^4}{a^4}\right)^{\frac{3}{2}}.$$

19. Also that, in the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}},$$
$$\rho = -3\,(axy)^{\frac{1}{3}}.$$

20. Shew by differentiating the expression

$$(x-\xi)^2 + (y-\eta)^2$$

for the square of the distance of a variable point (x, y) of a curve from a fixed point (ξ, η) that when this distance is stationary the point (x, y) must be at the foot of a normal from (ξ, η) to the curve.

Also that the distance is then a minimum or maximum according as the point (ξ, η) is nearer to or further from the curve than the centre of curvature.

21. The curvature at any point of an ellipse is

$$\frac{a \cos \frac{1}{2}\phi}{rr'},$$

when r, r' are the focal distances, and ϕ is the angle between them.

22. In the rectangular hyperbola $r^2 \cos 2\theta = a^2$,

$$\rho = r^3/a^2.$$

23. In the lemniscate $r^2 = a^2 \cos 2\theta$,

$$\rho = a^2/3r.$$

24. In the curve $r^m = a^m \cos m\theta$,

$$\rho = \frac{r^2}{(m+1)\,p} = \frac{a^m}{(m+1)\,r^{m-1}}.$$

25. Apply the formula $\rho = r\,dr/dp$ to find the radius of curvature at any point of an epicycloid. (See Ex. 25, p. 376.)

Examine the case of the involute of a circle.

26. If the equation of a curve be given in the form $r = f(p)$, the chord of curvature through the pole is

$$2p\,\frac{dr}{dp}.$$

Prove that the chord of curvature through the pole of a cardioid is $1\frac{1}{3}$ times the radius vector.

27. Prove that the chord of curvature, through the pole, at any point of the curve $r^m = a^m \cos m\theta$ is $2r/(m+1)$.

28. Prove that the curvature of the pedal of a curve $r = f(p)$ with respect to the origin is

$$\frac{2}{r} - \frac{p}{r^3}\rho,$$

where r, p, ρ refer to the original curve.

29. Prove that the curvature at any point of the pedal of an ellipse of semi-axes a, b with respect to the centre is equal to

$$\frac{3}{r} - \frac{a^2+b^2}{r^3},$$

where r is the radius vector of the corresponding point of the ellipse.

30. Prove the formula

$$\frac{1}{\rho} = \left\{\frac{1}{r} - \frac{1}{r}\left(\frac{dr}{ds}\right)^2 - \frac{d^2r}{ds^2}\right\} \div \left\{1 - \left(\frac{dr}{ds}\right)^2\right\}^{\frac{1}{2}};$$

and apply it to deduce the conclusions of Ex. 20.

31. Prove that in polar coordinates the condition for a stationary tangent is

$$\frac{d^2u}{d\theta^2} + u = 0,$$

where $u = 1/r$.

32. From the formula

$$\psi = \theta + \phi = \theta + \cot^{-1}\frac{1}{r}\frac{dr}{d\theta}$$

deduce the formula for curvature in polar coordinates:

$$\frac{1}{\rho} = \left\{r^2 - r\frac{d^2r}{d\theta^2} + 2\left(\frac{dr}{d\theta}\right)^2\right\} \div \left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}$$

$$= \left(\frac{d^2u}{d\theta^2} + u\right) \div \left\{1 + \left(\frac{1}{u}\frac{du}{d\theta}\right)^2\right\}^{\frac{3}{2}},$$

where $u = 1/r$.

33. With the same notation, prove that the chord of curvature through the origin is

$$2\left\{1 + \left(\frac{1}{u}\frac{du}{d\theta}\right)^2\right\} \div \left(\frac{d^2u}{d\theta^2} + u\right).$$

34. The radius of curvature of the curve

$$ay^2 = (x - \alpha)(x - \beta)^2$$

at the point $(\alpha, 0)$ is $(\alpha - \beta)^2/2a$.

35. Prove by Newton's method that the radius of curvature at the vertex of the catenary

$$y = a \cosh x/a$$

is equal to a.

36. The radius of curvature of the curve

$$y^2 = a^2(a + x)/x$$

at the point $(-a, 0)$ is $\frac{1}{2}a$.

37. The radius of curvature of the 'witch'

$$y^2 = a^2(a - x)/x$$

at its vertex is $\frac{1}{2}a$.

38. The radii of curvature of the trochoid

$$x = a\theta + k \sin\theta, \quad y = a - k\cos\theta$$

at the points where it is nearest to and furthest from the base are

$$(a \pm k)^2/k.$$

39. Prove that in the meridian-curve $(r^2 = a^2 \cos\theta)$ of the 'solid of greatest attraction' (see Ex. 18, p. 375) the radii of curvature at the extremities of the axis are ∞ and $\frac{2}{3}a$, respectively.

40. Prove that by Newton's method the radius of curvature at either vertex of the lemniscate $r^2 = a^2 \cos 2\theta$ is $\frac{1}{3}a$.

41. Apply Newton's method to shew that the radii of curvature of the epicyclic

$$x = a_1 \cos n_1 t + a_2 \cos n_2 t, \quad y = a_1 \sin n_1 t + a_2 \sin n_2 t,$$

at the points nearest to and furthest from the centre, are

$$\frac{(n_1 a_1 \pm n_2 a_2)^2}{n_1^2 a_1 \pm n_2^2 a_2}.$$

Infer the condition that an epicyclic, at the points of nearest approach to the centre, should be concave to the centre (as in the case of the orbit of the Moon relative to the Sun).

42. If a curve be referred to polar coordinates r, θ, and if the pole be on the curve, and the initial line be the tangent at the pole, prove that the diameter of curvature at the pole $= \lim r/\theta$.

Find the radius of curvature at the pole of the curve

$$r = a \cos m\theta.$$

43. If P be a point of a curve where the curvature, but not the direction of the tangent, is discontinuous, and if Q, R be neighbouring points on opposite sides of P, prove that the curvature of the circle PQR is ultimately equal to

$$\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2},$$

where ρ_1, ρ_2 are the radii of curvature of the given curve on the two sides of P, and m_1, m_2 are the limiting values of the ratios PQ/QR and PR/QR, respectively.

44. The acute angle which a chord PQ of a curve makes with the tangent at P, when Q is taken infinitely close to P, is ultimately equal to $\frac{1}{2}\delta s/\rho$, where δs is the arc PQ and ρ is the radius of curvature at P.

45. Prove that if the tangents at the extremities of an infinitely small arc PQ meet in T, then TP and TQ are ultimately in a ratio of equality.

Why does it not follow that the line joining T to the middle point of PQ will be ultimately perpendicular to PQ?

46. Assuming that the radius of the circumcircle of a triangle ABC is equal to $\frac{1}{2}a/\sin A$, shew that it follows from Ex. 44 that the osculating circle coincides with the circle of curvature.

47. Prove that when the resultant force on a particle is in the direction of motion the tangent to the path is 'stationary.'

155. **Envelopes.**

Suppose that we have a singly-infinite system, or family, of curves differing from one another only in the value assigned to some constant which enters into their specification. Two distinct curves of the system will in general intersect; and we consider here, more particularly, the limiting positions of the intersections when the change in the constant (or 'parameter,' of the system, as it is sometimes called), as we pass from one curve to the other, is infinitely small. On each curve we have then, in general, one or more points of 'ultimate intersection' with the consecutive curve of the system. The locus of these points of ultimate intersection is called the 'envelope' of the system.

Ex. 1. A system of circles of given radius, having their centres on a given straight line. The parameter here is the coordinate of the centre.

If C, C' be the centres of two circles of the system, the line joining their intersections bisects CC' at right angles. Hence the points of ultimate intersection of any circle with the consecutive circle are the extremities of the diameter which is perpendicular to the line of centres. The envelope therefore consists of two straight lines parallel to the line of centres, at a distance equal to the given radius.

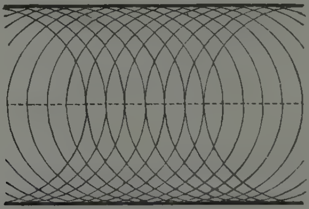

Fig. 120.

Ex. 2. A straight line including, with the coordinate axes, a triangle of constant area (k^2).

If AB, $A'B'$ be two positions of the line, intersecting in P, the triangles APA', BPB' will be equal, whence

$$PA \,.\, PA' = PB \,.\, PB'.$$

Hence, ultimately, when AA' is infinitely small, P will be the middle point of AB. If x, y be the coordinates of P, and ω the inclination of the axes, we have, then, $OA = 2x$, $OB = 2y$, and therefore

$$2xy \sin \omega = k^2.$$

The envelope is therefore a hyperbola having the coordinate axes as asymptotes. Fig 121 illustrates the case of $\omega = \frac{1}{2}\pi$.

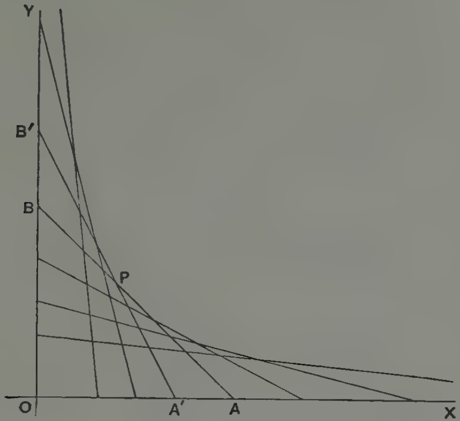

Fig. 121.

156. General Method of finding Envelopes.

The equation of any curve of the system being

$$\phi(x, y, \alpha)=0 \quad \ldots\ldots\ldots\ldots (1),$$

where α is the parameter, then at the intersection with another curve

$$\phi(x, y, \alpha')=0 \quad \ldots\ldots\ldots\ldots (2),$$

we have, evidently,

$$\frac{\phi(x, y, \alpha')-\phi(x, y, \alpha)}{\alpha'-\alpha}=0 \quad \ldots\ldots\ldots (3).$$

When the variation $\alpha'-\alpha$ of the parameter is infinitely small, this last equation takes the form

$$\frac{\partial}{\partial\alpha}\phi(x, y, \alpha)=0 \quad \ldots\ldots\ldots\ldots (4),$$

where $\partial/\partial\alpha$ is the symbol of partial differentiation with respect to α. See Art. 45.

The coordinates of the point, or points, of ultimate intersection are determined by (1) and (4) as simultaneous equations, and the locus of the ultimate intersections is to be found by elimination of α between these equations.

Ex. 1. The circles considered in Art. 155, Ex. 1 may be represented by

$$(x-a)^2+y^2=a^2 \qquad \text{.......................(5)}.$$

Differentiating with respect to a, we find

$$x-a=0 \qquad \text{.......................(6)}.$$

Eliminating a between (5) and (6) we get

$$y=\pm a \qquad \text{.......................(7)},$$

the envelope required.

Ex. 2. If a particle be projected from the origin at an elevation θ, with the velocity 'due to' a height h, the equation of the parabolic path is

$$y=x\tan\theta-\tfrac{1}{4}\frac{x^2}{h}\sec^2\theta \qquad \text{..................(8)},$$

where the axes of x, y are respectively horizontal and vertical. Writing a for $\tan\theta$, we get

$$y=ax-\tfrac{1}{4}\frac{x^2}{h}(1+a^2) \qquad \text{....................(9)}.$$

To find the envelope of the paths for different elevations, and therefore for different values of a, we differentiate (9) with respect to a, and find

$$x-\tfrac{1}{2}\frac{ax^2}{h}=0 \qquad \text{.......................(10)}.$$

This is satisfied either by $x=0$, or by $ax=2h$. The former makes $y=0$, and shews that the origin is part of the locus, as is otherwise obvious. The alternative result leads, on elimination of a, to

$$x^2=4h(h-y) \qquad \text{.......................(11)},$$

a parabola having its axis vertical, its focus at the origin, and its vertex at an altitude h*.

157. Algebraical Method.

If in the equation

$$\phi(x, y, \alpha)=0 \qquad \text{....................(1)},$$

ϕ be a rational integral function of α, the rule of the preceding Art. may be investigated otherwise as follows.

---

* This problem is interesting historically as being the first instance in which the envelope of a family of *curved* lines was obtained (Bernoulli). The general method of finding envelopes appears to be due to Leibnitz.

If we assign any particular values to x, y, the equation determines α, that is, it determines what curves of the system pass through the given point (x, y). If the equation be of the nth degree in α, the number of these curves (real or imaginary) will be n, and these n curves will in general be distinct. But if the point in question be at the intersection of two consecutive curves, two of the values of α will be coincident. Now it was shewn in Art. 49 that the condition for a double root of the equation in α is

$$\frac{\partial}{\partial\alpha}\phi(x, y, \alpha)=0 \quad\text{.....................}\quad (2).$$

The ultimate intersections are therefore determined as before by (1) and (2) as simultaneous equations, and the envelope by elimination of α between them.

If the equation (1) be of the *first* degree in α, only one curve of the system passes through any assigned point, and there is of course no envelope.

Examples of this are furnished by the parallel lines

$$lx+my=a \quad\text{..........................}\quad (3),$$

and by the concentric circles

$$x^2+y^2=a \quad\text{..........................}\quad (4).$$

If (1) be a *quadratic* in α, say

$$P\alpha^2+2Q\alpha+R=0 \quad\text{...................}\quad (5),$$

where P, Q, R are given functions of x and y, the condition for equal roots is

$$PR=Q^2 \quad\text{.........................}\quad (6).$$

This is therefore the equation of the envelope.

Ex. 1. If the straight line

$$\frac{x}{\alpha}+\frac{y}{\beta}=1 \quad\text{.........................}\quad (7)$$

include with the coordinate axes a triangle of constant area k^2, we have

$$\alpha\beta\sin\omega=2k^2 \quad\text{.......................}\quad (8),$$

where ω is the inclination of the axes. Hence, eliminating β, the equation of the variable line is found to be

$$\alpha^2 y \sin\omega - 2\alpha k^2 + 2k^2 x = 0 \qquad (9).$$

Expressing that this quadratic in α has equal roots, we find for the envelope

$$2xy \sin\omega = k^2 \qquad (10),$$

as in Art. 155, Ex. 2.

Ex. 2. One leg of a right angle passes through a fixed point, and the vertex describes a fixed straight line; to find the envelope of the other leg.

If the fixed straight line be the axis of y, and the fixed point be at $(a, 0)$, the equation of the second leg is easily seen to be

$$y = mx + \frac{a}{m} \qquad (11),$$

where m is the tangent of the inclination to the axis of x. Writing the equation in the form

$$m^2 x - my + a = 0 \qquad (12),$$

we see that the envelope is the parabola

$$y^2 = 4ax \qquad (13).$$

158. Contact-Property of Envelopes.

The examples already given will have prepared the student for the following theorem:

The envelope of a system of curves touches (in general) at each of its points the corresponding curve of the system.

The equations

$$\phi(x, y, \alpha) = 0 \qquad (1),$$

and

$$\frac{\partial}{\partial\alpha}\phi(x, y, \alpha) = 0 \qquad (2),$$

determine x, y as functions of α, say

$$x = F(\alpha), \quad y = f(\alpha) \qquad (3),$$

and the latter pair of equations define the envelope. If we substitute from (3) on the left-hand side of (1) we obtain a function of α which must vanish identically, and the result of differentiating this function with respect to α must also be zero. Hence, by the rule of Art. 62, 1°, we must have

$$\frac{\partial\phi}{\partial x}\frac{dx}{d\alpha}+\frac{\partial\phi}{\partial y}\frac{dy}{d\alpha}+\frac{\partial\phi}{\partial\alpha}\frac{d\alpha}{d\alpha}=0 \;\ldots\ldots\ldots\ldots\; (4),$$

which reduces, in virtue of (2), to

$$\frac{\partial\phi}{\partial x}\frac{dx}{d\alpha}+\frac{\partial\phi}{\partial y}\frac{dy}{d\alpha}=0 \;\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

or

$$\frac{\dfrac{dy}{d\alpha}}{\dfrac{dx}{d\alpha}}=-\frac{\dfrac{\partial\phi}{\partial x}}{\dfrac{\partial\phi}{\partial y}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

Now, by Art. 54, the left-hand side of this equality is the value of dy/dx for the envelope; and the right-hand side is, by Art. 62 (10), the value of dy/dx for the curve (1). Hence at the point of ultimate intersection the curve (1) and the envelope have a common tangent line.

The geometrical basis of the theorem may be indicated as follows:

Let the figure represent portions of two curves of the system, corresponding to values α_0, α_1 of the parameter α,

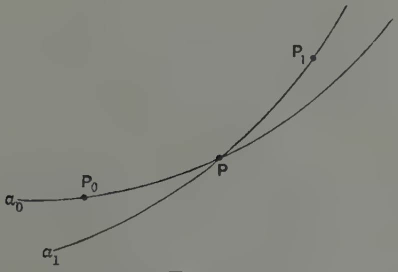

Fig. 122.

and intersecting in P. Let P_0 and P_1 be the corresponding points on the envelope; viz. P_0 is the limiting position of P when, α_0 being fixed, α_1 is taken infinitely nearly equal to α_0;

and P_1 is the limiting position of P when, α_1 being fixed, α_0 is taken infinitely nearly equal to α_1. Since these variations of α are in opposite senses, and since the coordinates of P are as a rule symmetric functions of α_0, α_1, the corresponding displacements of P, viz. PP_0 and PP_1, will in general, when $|\alpha_1 - \alpha_0|$ is very small, be in nearly opposite directions, and P_0PP_1 will be a very obtuse-angled triangle. Hence, ultimately, when $|\alpha_1 - \alpha_0|$ is infinitely small, the chords P_0P_1 and P_0P will coincide in direction; *i.e.* the tangent to the envelope is identical with the tangent to the variable curve

The foregoing investigations break down in certain cases. As regards the analytical proof, it is plain that no inference can be drawn from (5) whenever at the point in question we have

$$\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

simultaneously; *i.e.* when the value of dy/dx for the curve (1) is not uniquely determinate. This peculiarity occurs at a 'singular point,' whether it be of the nature of a node, a cusp, or an isolated point (see Art. 133). It appears that the locus of the singular points of the given family, when such a locus exists, is included in the result of eliminating a between (1) and (2), but this locus does not in general 'touch' the given curves, in any proper sense of the word. The full investigation of this matter is beyond our limits*, but a simple example may be given. Consider the family

$$a(y-a)^2 = x^2(x+b) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

It appears from Art. 133 that there is a node, a cusp, or an isolated point at $(0, a)$, according as b is positive, zero, or negative. The process for finding the envelope gives $y - a = 0$ and therefore

$$x^2(x+b) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

The line $x = 0$ gives the locus of the singular points, and does not touch the original curves; the line $x = -b$ on the other hand does so (unless $b = 0$).

In the geometrical view of the matter it was assumed that there is no other intersection of the curves α_0, α_1 in the immediate neighbourhood of P. In the case of a node we have usually *two* adjacent intersections, whose x-coordinates (for instance) are of the forms $f(\alpha_0, \alpha_1)$ and $f(\alpha_1, \alpha_0)$, respectively; but $f(\alpha_0, \alpha_1)$ is not a symmetric function of α_0, α_1. The argument does not therefore apply to the node-locus. Again, in the case of a cusp the displacement of the point P in Fig. 122, due to an infinitesimal variation of α_0 or α_1, is found not to be of the first order; and the points P_0, P_1 are as a rule on the *same* side of P. In the neighbourhood of an isolated point there is no real intersection of consecutive curves.

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* It is usually given in books on Differential Equations, under the head of 'Singular Solutions.'

159. Evolutes.

The 'evolute' of a curve is the locus of its centre of curvature. Since the centre of curvature is (Art. 150) the intersection of two consecutive normals, the evolute is also the envelope of the normals to the given curve. Hence the normals to the original curve are tangents to the evolute*.

Ex. 1. In the parabola

$$y^2 = 4ax \text{ .... } (1),$$

we have

$$x = a\cot^2\psi, \quad y = 2a\cot\psi \text{..................} (2),$$

and (by Art. 151, Ex. 4)

$$\rho = -2a/\sin^3\psi \text{ .... } (3).$$

The coordinates of the centre of curvature are therefore

$$\left.\begin{aligned}\xi &= x - \rho\sin\psi = 3x + 2a,\\ \eta &= y + \rho\cos\psi = -y^3/4a^2\end{aligned}\right\} \text{ .... } (4).$$

Hence

$$\eta^2 = y^6/16a^4 = 4x^3/a = \tfrac{4}{27}(\xi - 2a)^3/a.$$

The evolute is therefore the semi-cubical parabola

$$ay^2 = \tfrac{4}{27}(x - 2a)^3 \text{ .... } (5).$$

Otherwise: it is shewn in books on Conics that the equation of the normal is of the form

$$y = m(x - 2a) - am^3 \text{.....................} (6).$$

To find the envelope of this we differentiate partially with respect to m, and obtain

$$x - 2a = 3am^2, \quad y = 2am^3 \text{..................} (7).$$

The elimination of m leads again to the result (5).

The curve is shewn in Fig. 123.

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[*] It being evident that the exceptional cases noted at the end of Art. 158 cannot present themselves in the envelope of a *straight line*.

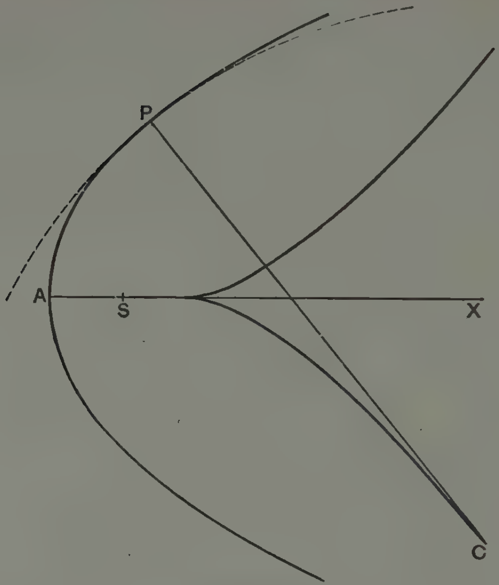

Fig. 123.

Ex. 2. The normal at any point of the ellipse

$$x = a\cos\phi, \quad y = b\sin\phi \quad \text{..................(8)}$$

is

$$\frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2 \quad \text{......................(9).}$$

Differentiating with respect to ϕ we find

$$\frac{ax}{\cos^3\phi} = -\frac{by}{\sin^3\phi}, \ = \lambda, \text{ say} \quad \text{............... (10).}$$

Substituting in (9), we have

$$\lambda = a^2 - b^2 \quad \text{...................... (11).}$$

Hence the coordinates of the centre of curvature are

$$x = \frac{a^2 - b^2}{a}\cos^3\phi, \quad y = -\frac{a^2 - b^2}{b}\sin^3\phi \quad \text{.........(12);}$$

and the evolute is

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}} \quad \text{.................. (13).}$$

This curve, which may be obtained by homogeneous strain from the astroid, is shewn in Fig. 124.

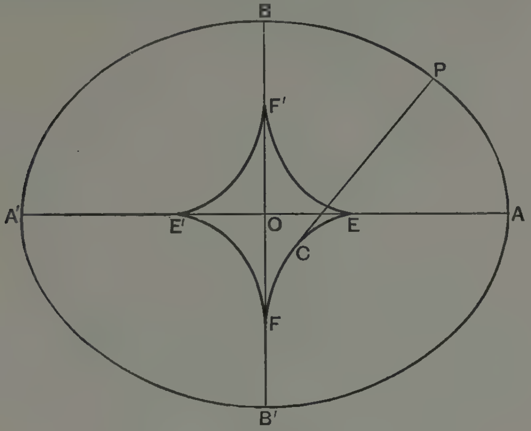

Fig. 124.

The centres of curvature at the points A, B, A', B' are E, F, E', F', respectively.

Ex. 3. To find the evolute of a cycloid.

At any point P on the cycloid APB (Fig. 125), we have, by Art. 151, Ex. 2,

$$\rho = 2PI \text{ ....(14).}$$

Let the axis AB be produced to D', so that $BD' = AB$; and produce TI to meet a parallel to BI, drawn through D', in I'. If a circle be described on II' as diameter, and PI be produced to meet the circumference in P', we have $P'I = PI$, so that P' is the centre of curvature of the cycloid at P. And since the arc $P'I'$ is equal to the arc TP, and therefore to BI or $D'I'$, the locus of P' is evidently the cycloid generated by the circle $IP'I'$, supposed to roll on the under side of $D'I'$, the tracing point starting from D'. That is, the evolute is a cycloid equal to the original cycloid, and having a cusp at D'.

It appears, further, from Art. 136 (4), that the cycloidal arc $P'D$ is equal to $2IP'$, or $P'P$. Hence

$$\text{arc } D'P' + P'P = \text{const. .... (15).}$$

The lower cycloid in Fig. 125 is therefore an 'involute' (Art. 161) of the upper one*.

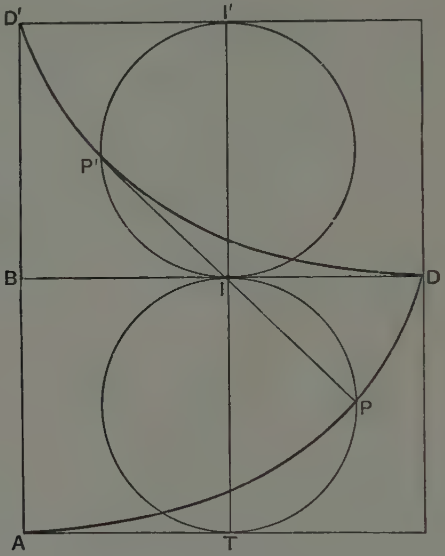

Fig. 125.

It may be noted that whenever a curve is defined by a relation between p and ψ, say

$$p = f(\psi) \quad \text{.....................} (16),$$

the evolute is given by

$$p = f'(\psi) \quad \text{.....................} (17),$$

provided that in (17) the origin of ψ be supposed moved forwards through a right angle. This is seen at once on reference to Fig. 115, p. 401, since OU, the perpendicular from the origin on the tangent to the evolute, is equal to PZ, or $dp/d\psi$, when the symbols refer to the original curve.

---

* This example is interesting historically in connection with the theory of the cycloidal pendulum. The results are due to Huyghens (1673).

Ex. 4. To find the evolute of an epi- or hypo-cycloid.

If in Fig. 86, p. 351, a perpendicular p be drawn from O to TP, the tangent to the epicycloid at P, we have

$$p = OT \cos PIO = (a + 2b) \cos \tfrac{1}{2}\phi,$$

or

$$p = (a + 2b) \cos \frac{a}{a + 2b} \psi \dots\dots\dots\dots\dots (18).$$

If the origin of ψ correspond to a cusp instead of to a vertex, the cosine of the angle must be replaced by the sine.

Hence, for the evolute, we have

$$p = - a \sin \frac{a}{a + 2b} \psi \dots\dots\dots\dots\dots\dots (19),$$

which can be brought to the same form as (18) by an adjustment of the origin of ψ. The evolute is therefore a similar epicycloid in which the dimensions are reduced in the ratio $a/(a + 2b)$.

For a hypocycloid we have merely to change the sign of b*.

160. Arc of an Evolute.

The difference of the radii of curvature at any two points of a curve is equal to the arc between the corresponding points of the evolute.

To prove this, let the normals at two neighbouring points P_0, P_1 of the curve meet in C; and let C_0, C_1 be the corresponding centres of curvature. As in Art. 158, C_1CC_0 is in general an obtuse-angled triangle; and when P_0, P_1 are taken infinitely close to one another, $C_1C + CC_0$ is ultimately in a ratio of equality to C_1C_0. Also, since the triangle P_1CP_0 has the angles at P_0 and P_1 infinitely nearly equal to right angles, the difference between CP_1 and CP_0 is ultimately

---

* It appears on examination that the equation

$$p = c \cos m\psi, \quad \text{or} \quad p = c \sin m\psi,$$

represents an epi- or a hypo-cycloid according as $m \lesseqgtr 1$, provided we include the pericycloids among the epicycloids, in accordance with the definition of Art. 137.

The pedal of an epi- or a hypo-cycloid with respect to its centre is therefore an epicyclic of the special type referred to in Art. 139, Ex. 2. Thus Fig. 97 represents the pedal of a four-cusped epicycloid, and Fig. 99 that of a four-cusped hypocycloid.

of the *second order* of small quantities. Hence, neglecting

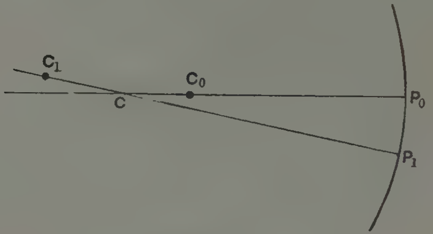

Fig. 126.

this difference, we have

$$C_1P_1 - C_0P_0 = C_1C + CC_0 = C_1C_0.$$

It follows that if ρ be the radius of curvature of the original curve, and σ the arc of the evolute, we have $\delta\rho = \delta\sigma$, ultimately, or

$$\frac{d\rho}{d\sigma} = 1 \quad \text{.........................} \quad (1).$$

Hence, integrating,

$$\rho = \sigma + C \quad \text{......................} \quad (2),$$

where C is an arbitrary constant depending on the origin of measurement of σ.

Otherwise: by differentiation of the equations

$$\xi = x - \rho \sin\psi, \quad \eta = y + \rho\cos\psi \quad \text{..............} \quad (3)$$

of Art. 150, we find, since

$$dx/ds = \cos\psi, \quad dy/ds = \sin\psi, \quad d\psi/ds = 1/\rho,$$

$$\frac{d\xi}{ds} = -\frac{d\rho}{ds}\sin\psi, \quad \frac{d\eta}{ds} = \frac{d\rho}{ds}\cos\psi \quad \text{..............} \quad (4).$$

Hence $$\frac{d\eta}{d\xi} = -\cot\psi \quad \text{.......................} \quad (5),$$

which shews that the tangent to the evolute is normal to the original curve, and

$$\frac{d\sigma}{ds} = \sqrt{\left\{\left(\frac{d\xi}{ds}\right)^2 + \left(\frac{d\eta}{ds}\right)^2\right\}} = \frac{d\rho}{ds} \quad \text{..............} \quad (5),$$

which gives on integration the result (2).

For the case of the cycloid, this property has been already obtained in Art. 159 (15).

Ex. The radii of curvature of an ellipse of semi-axes a, b, at the extremities of these axes, are b^2/a and a^2/b, respectively. Hence the length of any one of the four portions into which the evolute is divided (see Fig. 124) by its cusps is

$$a^2/b - b^2/a \text{ or } (a^3 - b^3)/ab.$$

161. Involutes, and Parallel Curves.

If a curve A be the evolute of a curve B, then B is said to be an 'involute' of A.

We say *an* involute because any given curve has an infinity of involutes. To obtain an involute we take any fixed point O on the curve, and along the tangent at a variable point P measure off a length PQ in the direction *from* O, so that

$$\text{arc } OP + PQ = \text{const.} \quad\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

It is easily shewn, by an inversion of the argument of Art. 160, that the tangents to the given curve are normals to the locus of Q, so that this locus fulfils the above definition of an involute. And, by varying the 'constant,' we obtain a series of involutes of the same curve.

As a concrete example we may imagine a string to be wound on a material arc of the given shape, being attached to a fixed point on it. The curve traced out by *any* point on the free portion of the string will be an involute. This is in fact the origin of the term.

Ex. 1. The tractrix is an involute of the catenary; see Art. 134.

Ex. 2. In an involute of a circle of radius a we have, evidently,

$$\frac{ds}{d\psi} = \rho = a\psi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

if the origin of ψ be properly chosen. Hence, integrating,

$$s = \tfrac{1}{2}a\psi^2 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

no additive constant being required, if s be measured from the cusp ($\psi = 0$).

In this particular case (of the circle) it is evident that all the involutes are identically equal. It is therefore customary to speak of *the* involute of a circle. The curve is shewn in Fig. 127.

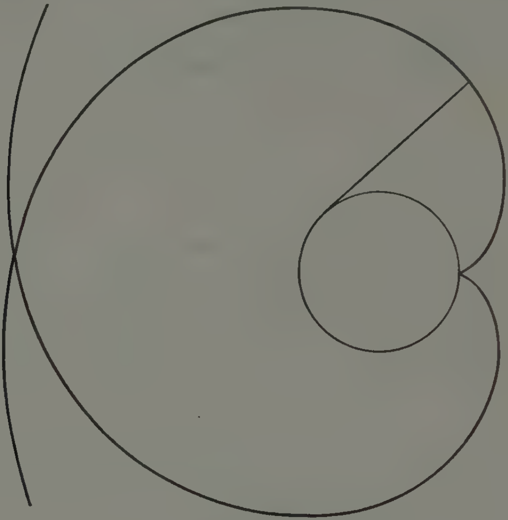

Fig. 127.

If a constant length be measured along the normal to a given curve, from the curve, the locus of the point thus determined is called a 'parallel' to the given curve.

If CP, CP' be two consecutive normals to the given curve, and Q, Q' the corresponding points of a parallel curve, we have

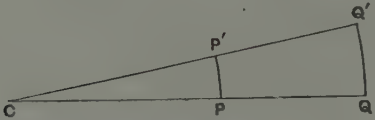

Fig. 128.

$$PQ = P'Q'.$$

Since the difference between CP and CP' is of the second order of small quantities, it follows that the same holds of the difference between CQ and CQ', and thence that the angles at Q and Q' in the triangle CQQ' are ultimately right angles. Hence CQ, CQ' are normals to the parallel curve.

Hence two parallel curves have the same normals, and therefore the same evolute; in other words, parallel curves are involutes of the same curve.

Conversely, it is evident that the various involutes of any curve form a system of parallel curves.

EXAMPLES. XLV.

1. The envelope of the parabolas

$$y^2 = 4\alpha (x - \alpha),$$

where α is the parameter, is a pair of straight lines.

2. The envelope of the parabolas

$$\alpha y^2 = a^2 (x - \alpha),$$

where α is the parameter, is the curve

$$ay^2 = \tfrac{4}{27} x^3.$$

3. Circles are described on the radii vectores of a curve as diameters; prove geometrically that their envelope is the pedal of the given curve with respect to the origin.

4. Find the envelope of the circles described on the focal radii of a conic as diameters.

5. Chords of a circle are drawn through a fixed point on the circumference; prove that the envelope of the circles described on these chords as diameters is a cardioid.

6. The envelope of the circles described on the central radii of a rectangular hyperbola as diameters is a lemniscate of Bernoulli.

7. Prove that the envelope of the curves

$$P \cos \alpha + Q \sin \alpha = R,$$

where P, Q, R are given functions of x, y, and α is a variable parameter, is $$P^2 + Q^2 = R^2.$$

8. Find the envelope of the circles

$$x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = c^2,$$

and interpret the result.

9. A system of ellipses of constant area have the same centre and their axes coincident in direction; prove that the envelope consists of two conjugate rectangular hyperbolas.

10. A straight line moves so that the product of the perpendiculars on it from the fixed points $(\pm c, 0)$ is constant $(= b^2)$; prove that the envelope is the ellipse

$$\frac{x^2}{b^2 + c^2} + \frac{y^2}{b^2} = 1,$$

or the hyperbola

$$\frac{x^2}{c^2 - b^2} - \frac{y^2}{b^2} = 1,$$

according as the two perpendiculars are on the same or on opposite sides of the variable line.

11. Circles are described on the double ordinates of the parabola $y^2 = 4ax$ as diameters; prove that the envelope is the parabola

$$y^2 = 4a(x + a).$$

12. Circles are described on the double ordinates of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

as diameters: prove that the envelope is the ellipse

$$\frac{x^2}{a^2 + b^2} + \frac{y^2}{b^2} = 1.$$

13. A straight line moves so that the sum of the squares of the perpendiculars on it from the fixed points $(\pm c, 0)$ is constant $(= 2k^2)$; prove that the envelope is the conic

$$\frac{x^2}{k^2 - c^2} + \frac{y^2}{k^2} = 1,$$

and examine the various cases.

14. A straight line moves so that the difference of the squares of the perpendiculars on it from two fixed points is constant; prove that the envelope is a parabola.

15. Find the envelope of the ellipses

$$x = a \sin(\theta - \alpha), \quad y = b \cos\theta,$$

where α is the parameter.

16. The envelope of the catenaries

$$y = c \cosh x/c,$$

where c is the variable parameter, consists of two straight lines.

17. The envelope of the ellipses

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

where $\alpha + \beta = k,$

is the 'astroid' $x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}}.$

18. The envelope of the straight line which makes on the coordinate axes intercepts whose sum is k is the parabola

$$\sqrt{x} + \sqrt{y} = \sqrt{k}.$$

19. From any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

perpendiculars are drawn to the coordinate axes; prove that the envelope of the straight line joining the feet of these perpendiculars is the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1.$$

20. Find the locus of ultimate intersections of the curves

$$ay^2 = x(x + a)^2,$$

where a is the parameter; and examine the result.

21. If a circle of constant radius has its centre on a given curve, the envelope of the circle consists of two parallel curves.

22. If a circle of given radius touch a given curve, its envelope consists of two parallel curves.

23. If the equation of a curve be given in the form

$$r^2 = f(p),$$

that of any parallel curve is of the form

$$r^2 = f(p - c) + 2cp - c^2.$$

24. Prove that the problem of negative pedals (Art. 147) is equivalent to finding the envelope of the straight line

$$x \cos \psi + y \sin \psi = p,$$

where p is a given function of the parameter ψ.

Verify that this leads to the formulæ (12) of Art. 147.

25. Shew that the negative pedal of the parabola

$$y^2 = 4ax$$

with respect to the vertex is the curve

$$ay^2 = \tfrac{1}{27}(x - 4a)^3.$$

26. Prove by the method of envelopes that the negative pedal of a circle is an ellipse or hyperbola according as the pole is inside or outside the circle.

27. Prove geometrically that the radius of curvature at any point of an equiangular spiral subtends a right angle at the pole.

28. The evolute of an equiangular spiral is an equiangular spiral of the same angle.

29. The coordinates of the centre of curvature at any point of the curve

$$ay^2 = x^3$$

are

$$\xi = -x - \tfrac{9}{2}\frac{x^2}{a}, \quad \eta = 4y + \tfrac{4}{3}\frac{ay}{x}.$$

Shew that near the origin the evolute has the form of the parabola

$$y^2 = -\tfrac{16}{9}ax.$$

30. Shew that if a curve has a point of inflexion the evolute has an asymptote.

Shew that the part of the evolute of the curve

$$a^2 y = x^3$$

which corresponds to the part of the curve near the origin may be represented approximately by the hyperbola

$$xy = \tfrac{1}{12}a^2.$$

31. The evolute of the hyperbola

$$x = a \cosh u, \quad y = b \sinh u$$

is

$$(ax)^{\frac{2}{3}} - (by)^{\frac{2}{3}} = (a^2 + b^2)^{\frac{2}{3}}.$$

32. If rays emanating from a point O be reflected at any given curve, the reflected rays are all normal to a curve which is similar to the pedal of the given curve with respect to O, but of double the dimensions.

33. Hence shew that the caustic by reflection at a circle will be the evolute of a limaçon; and that in the particular case where the luminous point is on the circumference of the given circle the caustic is a cardioid.

34. Prove that the caustic by reflection at any curve is the evolute of the envelope of a system of circles described with the various points of the curve as centres, and all passing through the luminous point.

What is the corresponding theorem for the case of refraction?

162. Displacement of a Figure in its own Plane. Centre of Rotation.

Suppose that we have two plane figures, each of invariable form, and that a curve fixed in one rolls without sliding on a curve fixed in the other. Any point of either figure will then describe a curve relatively to the other; a curve so described is called a 'roulette.'

The cases where the rolling curves are circles have been considered in Arts. 136—138.

The general theory of roulettes is of some importance in Geometry and in Kinematics, owing to the fact that any continuous motion whatever of a figure in its own plane may be regarded as consisting in the rolling of a certain curve fixed relatively to the figure on a certain curve fixed in space. This theorem will be proved in Art. 167; but it is necessary, as a preliminary, to recall one or two propositions of elementary Kinematics, which are, moreover, of independent interest.

In the first place, any displacement whatever of a figure in its own plane is equivalent to a rotation about some finite or infinitely distant point.

If A, B be any two points of the figure in its first position, and A', B' the same points in the second position, the new position P' of any third point originally at P is found by constructing the triangle $A'P'B'$ congruent with

APB. Hence the positions of *two* points are sufficient to determine the position of the moveable figure.

Now, considering any point whatever of the figure, let P be its initial and Q its final position; and let R be the final position of that point of the figure which was originally at Q. Since PQ and QR are two positions of the same line, they are equal. Hence if I be the centre of the circle PQR, the triangles PIR, QIR are congruent; that is, I represents the same point in the two positions. The displacement is therefore equivalent to a rotation about I. This point is called the 'centre of rotation.'

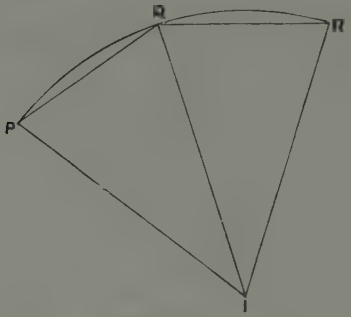

Fig. 129.

It may happen that PQ, QR are in a straight line. The displacement is then equivalent to a *translation* of the figure, without rotation; or, we may say, the centre of rotation is at infinity.

163. Instantaneous Centre.

Next considering any continuous motion of a plane figure in its own plane, let us fix our attention on two consecutive positions. The figure may be brought from the first of these to the second by a rotation about the proper centre. The limiting position of this point, when the two positions are taken infinitely close to one another, is called the 'instantaneous centre.'

If P, P' be consecutive positions of any the same point of the figure, and $\delta\theta$ the corresponding angle of rotation, the centre of rotation (I) is on the line bisecting PP' at right angles, and the angle PIP' is equal to $\delta\theta$. Hence, ultimately, the infinitesimal displacement of any point P at a finite distance from I is at right angles to IP and equal to $IP \, . \, \delta\theta$.

If we introduce the consideration of time, and denote by δt the interval that elapses between the two positions, the limiting value of $\delta\theta/\delta t$, viz. $d\theta/dt$, is called the 'angular velocity' of the figure. The velocity of that point of the figure which coincides with the instantaneous centre I is zero, that of any other point P is at right angles to IP, and equal to $IP \, . \, d\theta/dt$.

The fact that in any motion of a plane figure (of invariable form) the normals to the paths of the various points all pass through the instantaneous centre is often useful in geometrical questions. If we know the directions of infinitesimal displacement of two points of the figure, the instantaneous centre is determined as the intersection of the normals at these points to the respective directions. We can thence assign the directions, and relative magnitudes, of the displacements of all other points.

Ex. 1. The extremities of a straight line AB of constant length describe two straight lines OX, OY at right angles to one another.

It is known that any point P of the line describes an ellipse whose principal axes are along OX, OY*. The above theorem now gives us a construction for the normal to this ellipse at P; viz. if we draw AI, BI perpendicular to OX, OY, respectively, I is the instantaneous centre, and IP the required normal.

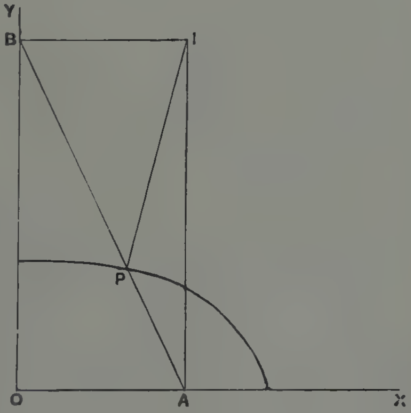

Fig. 130.

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* Viz. if $BP=a$, $AP=b$, $\angle OAB=\phi$, the coordinates of P are

$$x=a\cos\phi, \quad y=b\sin\phi.$$

Ex. 2. An arm OQ revolves about one extremity O; a bar QP is hinged to it at Q; and P is constrained to move in a straight line passing through O. It is required to find the relations between the velocities of the points P and Q. This arrangement is that of the crank and connecting rod of a steam-engine with a *fixed* cylinder.

The instantaneous centre I is at the intersection of OQ produced with the perpendicular to the fixed straight line at P.

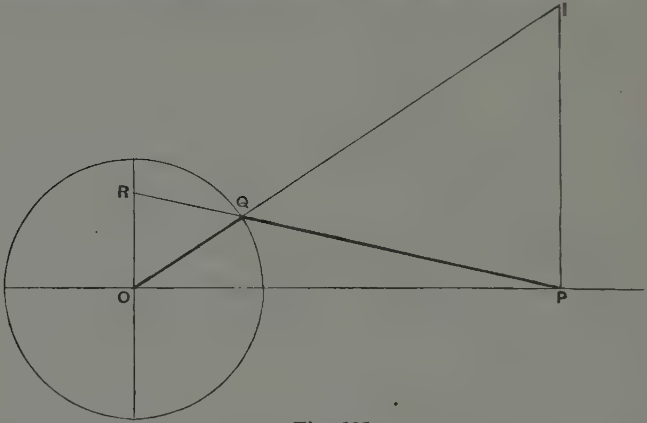

Fig. 131.

Hence the velocity of P is to that of Q as IP to IQ. Let PQ, produced if necessary, meet the perpendicular through O to the line of motion of P in the point R. Then if ω be the angular velocity of OQ,

$$\text{velocity of } P = \omega \, . \, OQ \times \frac{IP}{IQ} = \omega \, . \, OQ \times \frac{OR}{OQ} = \omega \, . \, OR \ldots (1).$$

Ex. 3. In an engine with an *oscillating* cylinder, the piston rod, being connected directly with the crank OQ, has its direction always passing through a fixed point C (on the pivot line of the cylinder).

The instantaneous axis is now at the intersection of OQ produced with the perpendicular to the piston rod at C. Hence, if ON be the perpendicular from O on CQ, produced if necessary, the velocity of the point of the rod which coincides with C is

$$\omega \, . \, OQ \times \frac{IC}{IQ} = \omega \, . \, OQ \times \frac{ON}{OQ} = \omega \, . \, ON \ldots\ldots\ldots\ldots (2).$$

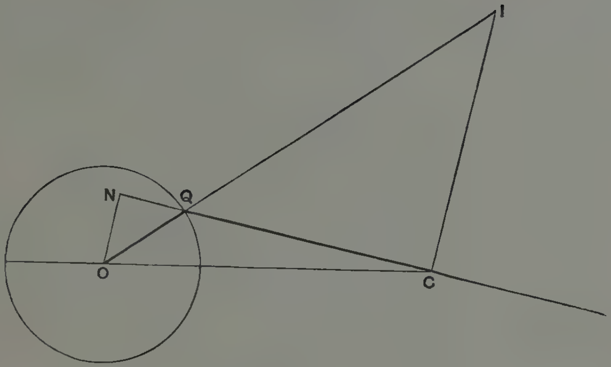

Fig. 132.

Again, considering any line (straight or curved) in the moving figure, it is evident that the point or points of ultimate intersection of this line with a consecutive position are the feet of the normals drawn to it from the instantaneous centre. For any other point of the line is moving in a direction making a finite angle with it.

Ex. 4. Thus in the case of a line AB of constant length moving (as in Art. 163, Ex. 1) with its extremities on the axes

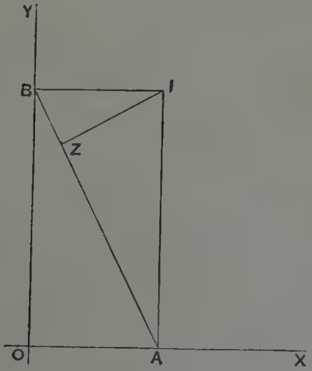

Fig. 133.

of coordinates, the point of ultimate intersection with a consecutive position is at the foot Z of the perpendicular from the instantaneous centre I. Now if

$$AB = k, \quad \angle OAB = \phi,$$

the coordinates of Z are given by

$$\left.\begin{aligned} x &= BZ\cos\phi = BI\cos^2\phi = k\cos^3\phi, \\ y &= AZ\sin\phi = AI\sin^2\phi = k\sin^3\phi \end{aligned}\right\} \quad\text{............}(3)$$

and the envelope of AB is therefore the astroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = k^{\frac{2}{3}} \quad\text{..........................}(4);$$

see Fig. 134. Cf. Art. 138, Ex. 4.

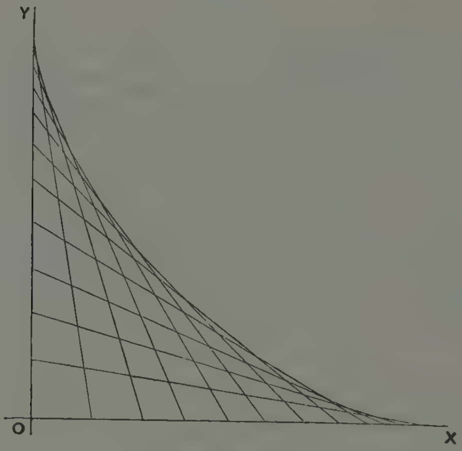

Fig. 134.

164. Application to Rolling Curves.

When one plane curve rolls upon another, which is regarded as fixed, the instantaneous centre is at the point of contact.

We will suppose, in the figure, that it is the *lower* curve which is fixed. Let A be the point of contact, and let equal infinitely small arcs AP, AP' ($= \delta s$, say) be measured off along the two curves. Let the normals at P and P' meet the common normal at A in the points O and O'. Then ultimately we have

$$OA = R, \quad O'A = R',$$

where R, R' are the radii of curvature of the two curves at A. After an infinitely small displacement, $P'O'$ will come into the same straight line with OP, the two curves being then in contact at P. Hence the angle ($\delta\theta$) through which the rolling curve will turn, being equal to the acute angle between OP and $P'O'$, is equal to the sum of the angles at O and O', so that

$$\delta\theta = \frac{\delta s}{R} + \frac{\delta s}{R'} \text{ .... (1)},$$

ultimately. Again, the chords AP, AP' are ultimately equal, and they include an infinitely small angle at A. Hence the distance PP' is ultimately of the *second order* in δs. It follows that when δs is indefinitely diminished the limiting position of the centre of rotation (I) coincides with A, for if it were at a finite distance from this point, the displacement of P', being equal to $IP' . \delta\theta$, by Art. 163, would be of the *first* order in δs.

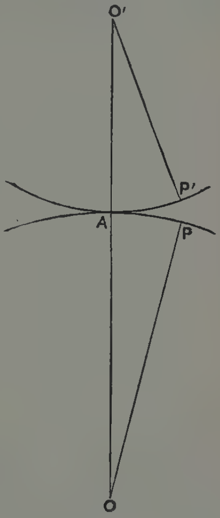

Fig. 135.

It follows that when a curve rolls upon a fixed curve, the normals to the paths of all points connected with the moving curve pass through the point of contact. We have already had instances of this result in the cycloidal and trochoidal curves discussed in Arts. 136, 137. Again if a straight line roll on a curve, it is normal to the path traced out by any of its points (Art. 161).

Further, if we consider any line (straight or curved) which is carried with the rolling curve, the points of ultimate intersection of the carried curve with its consecutive position are the feet of the normals drawn to it from the point of contact. And the envelope of the carried line is the locus of these feet.

Ex. 1. If a circle roll on a fixed straight line, any diameter envelopes a cycloid.

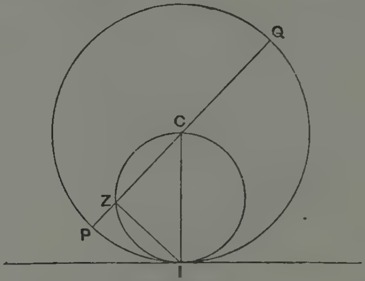

Fig. 136.

Let C be the centre of the rolling circle, I the point of contact, IZ the perpendicular on the diameter PQ. Since Z is on the circle whose diameter is CI, it is easily seen that if this circle be supposed to roll always with twice the angular velocity of the large circle, it will always have the same point of contact with the fixed line, and the point Z will move as if it were carried by the small circle. Its locus is therefore a cycloid.

Ex. 2. Similarly if a circle (A) roll on a fixed circle (B), the envelope of any diameter of A is an epi- or hypo-cycloid which would be generated by the rolling of a circle of half the size of A on the circumference of B.

165. **Curvature of a Point-Roulette.**

To investigate the curvature of any point P fixed relatively to the rolling curve, let I be the point of contact, and let I' be a consecutive point of contact, P' the corresponding position of P. Since the displacement of the point of the rolling curve which comes to I' is of the second order of small quantities, the angle through which the figure has turned is

$$\delta\theta = \angle PI'P' \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

ultimately. Let the normals to the path of P, viz. PI and

$P'I'$, be produced to meet in C. If $\delta\psi$ be the inclination of these normals, we have

$$\delta\psi = \angle ICI' = \frac{\delta s \cos\phi}{CI} \quad\ldots\ldots\ldots\ldots\ldots (2),$$

if ϕ be the angle which IP makes with the normal at I.

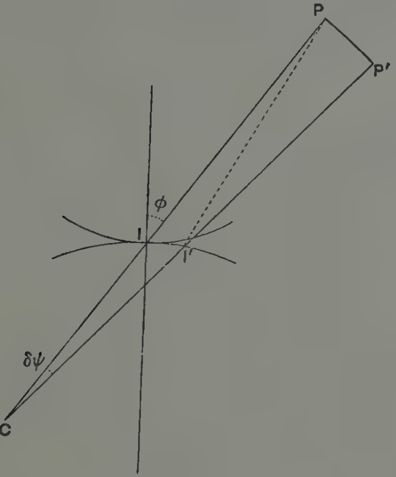

Fig. 137.

Also, from the figure

$$\begin{aligned}\delta\psi &= \angle PI'P' - \angle IPI' \\ &= \delta\theta - \frac{\delta s \cos\phi}{PI} \\ &= \delta s\left(\frac{1}{R} + \frac{1}{R'} - \frac{\cos\phi}{PI}\right) \quad\ldots\ldots\ldots\ldots\ldots (3),\end{aligned}$$

by Art. 164 (1), if R and R' be the radii of curvature of the fixed and rolling curves. Equating (2) and (3), we find

$$\cos\phi\left(\frac{1}{CI} + \frac{1}{IP}\right) = \frac{1}{R} + \frac{1}{R'} \quad\ldots\ldots\ldots\ldots (4).$$

This gives the limiting position of C, *i.e.* the centre of curvature of the path of P. The radius of curvature (ρ) is then found from

$$\rho = CP = CI + IP \quad\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

We have taken as our standard case that in which the two curves are convex to one another, as in Figs. 135, 137. Any other case may be included by giving proper signs to R and R'.

Ex. 1. In the cycloid, if a be the radius of the generating circle, we have

$$R=\infty,\quad R'=a,\quad IP=2a\cos\phi \qquad (6).$$

Substituting in (4), we find

$$CI=2a\cos\phi=IP \qquad (7),$$

and therefore

$$\rho=2IP \qquad (8).$$

Ex. 2. In the epicycloid (Art. 137) we have

$$R=a,\quad R'=b,\quad IP=2b\cos\phi \qquad (9),$$

whence

$$CI=\frac{2ab}{a+2b}\cos\phi=\frac{a}{a+2b}\,.\,IP \qquad (10),$$

$$\rho=\frac{2(a+b)}{a+2b}\,.\,IP \qquad (11).$$

We note that if $b=-\frac{1}{2}a$, we have $\rho=\infty$; cf. Art. 138, Ex. 2.

The result contained in (4) and (5) may be put in a simple geometrical form as follows. On the normal at I mark off a length IH such that

$$\frac{1}{IH}=\frac{1}{R}+\frac{1}{R'} \qquad (12),$$

and describe a circle on IH as diameter. Let IP meet this circle in Q. We have then

$$\frac{1}{IQ}=\frac{1}{IH\cos\phi}=\left(\frac{1}{R}+\frac{1}{R'}\right)\sec\phi,$$

and the relation (4) takes the form

$$\frac{1}{CI}+\frac{1}{IP}=\frac{1}{IQ} \qquad (13).$$

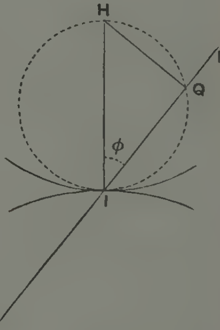

Fig. 138.

This shews that if P coincide with Q, CI is infinite; *i.e.* any point of the moving figure which lies on the circle just defined is at a point of inflexion of its path. For this reason, the circle in question is called the 'circle of inflexions.'

From (13) and (5) we find

$$CI = \frac{IP \, . \, IQ}{QP}, \quad \rho = \frac{IP^2}{QP} \quad \text{..............} (14).$$

The latter result shews that ρ changes sign with QP; that is, the paths of the various points of the moving figure are concave or convex to I, according to the side of the circle of inflexions on which they lie. In the standard case represented in the figures, the paths are concave or convex according as P is outside or inside the circle.

An example is furnished by the trochoidal curves figured on p. 349. The circle of inflexions has in this case half the size of the rolling circle.

The preceding theory has an important application to the problem of 'rocking stones' in Statics. When one rough body rests on another, with a single point of contact, its centre of gravity must be vertically above this point. And for stability of equilibrium it is necessary that the path of the centre of gravity, in any possible rolling displacement, should be concave upwards.

166. **Curvature of a Line-Roulette.**

The curvature of a line-roulette, *i.e.* of the envelope of a straight line carried by the rolling curve, can be found still more simply. The perpendiculars IZ, $I'Z'$ let fall on two consecutive positions of the line, from the corresponding positions (in space) of the instantaneous centre, are normals to the envelope, and the angle which they make with one another at their intersection (C) is equal to the angle of rotation $\delta\theta$. Hence if ϕ be the angle which IZ makes with the normal to the rolling curve at I, and $II' = \delta s$, we have ultimately

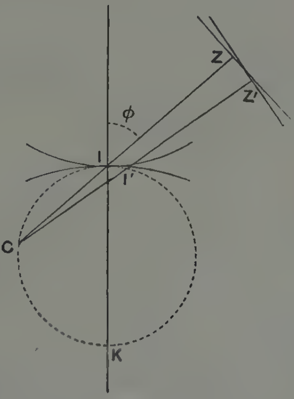

Fig. 139.

$$\delta s \cos \phi = CI \, . \, \delta\theta \quad \text{.....................} (1).$$

Hence, substituting the value of $\delta\theta$ from Art. 164 (1), we have

$$\frac{\cos\phi}{CI} = \frac{1}{R} + \frac{1}{R'} \quad\text{..................... (2).}$$

The radius of curvature of the envelope is then given by

$$\rho = CI + IZ \quad\text{...................... (3)*.}$$

If, along the normal to the rolling curve at I, but in the direction opposite to that chosen in the preceding Art., we measure off a length IK such that

$$\frac{1}{IK} = \frac{1}{R} + \frac{1}{R'} \quad\text{..................... (4),}$$

and describe a circle on this line as diameter, it appears from (2) that C lies on this circle; in other words, the locus of the centres of curvature of all line-roulettes, in any given position of the rolling curve, is a circle. Also, when the carried line passes through K, Z coincides with C, and C is a 'stationary point' (Art. 150) on the envelope. The aforesaid circle is therefore called the 'circle of cusps.'

Ex. 1. Regarding a cycloid as the envelope of the diameter of a circle which rolls on a fixed straight line (Art. 164, Ex. 1), we infer that the radius of curvature is double the normal.

Ex. 2. If an epicycloid be generated as the envelope of the diameter of a circle rolling on a fixed circle, then, to conform with the notation of Art. 137, we must write $R = a$, $R' = 2b$, and therefore, from (2),

$$CI = \frac{2ab}{a+2b}\cos\phi = \frac{a}{a+2b} \, . \, IZ,$$

in agreement with Art. 165, Ex. 2.

167. Continuous Motion of a Figure in its own Plane.

Consider now any continuous series of positions of a plane figure moveable in its own plane. We may suppose that any one of these positions can be specified by the value assigned to some independent variable t (*e.g.* the time). The instantaneous centre will have a certain locus in space, and also a certain locus in the figure. The curves so defined are called 'centrodes'; the former is distinguished as the 'space-centrode,' and the latter as the 'body-centrode.' The theorem referred to in Art. 162, and now to be proved, is that the given motion of the figure can be represented as due to the rolling of the body-centrode, without slipping, on the space-centrode.

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* For the method of finding the curvature of the envelope of *any* carried curve reference may be made to Williamson, *Diff. Calc.* § 291.

Considering, in the first place, any number of successive positions assumed by the moving figure, let us distinguish these by the numerals 1, 2, 3,…; and let the corresponding values of the independent variable be t_1, t_2, t_3,…. Let O_{12}, O_{23}, O_{34},… be the positions (in space) of the centres of rotation by which the figure can be brought from position 1 to position 2, from position 2 to position 3, and so on, respectively. Further, in the position 1, let O_{12}, O'_{23}, O'_{34},… be the positions *in the figure* of these successive centres of rotation. It is then plain that the given series of positions 1, 2, 3,… will be assumed in succession by the moving figure, if we imagine this to rotate about O_{12} until O'_{23} comes into coincidence with O_{23}, then about O_{23} until O'_{34} comes into coincidence with O_{34}, and so on. In other words, the figure

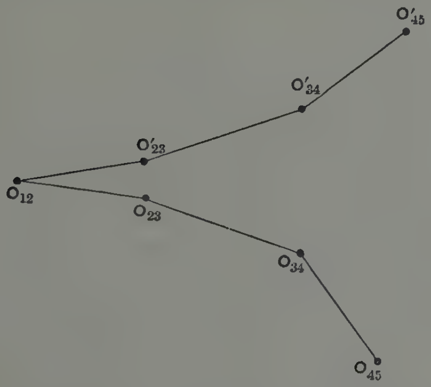

Fig. 140.

will pass through the actual series of positions 1, 2, 3,... if we imagine the polygon $O_{12}O'_{23}O'_{34}\ldots$, supposed fixed relatively to the figure, to roll on the polygon $O_{12}O_{23}O_{34}\ldots$, supposed fixed in space. The intermediate positions of the figure, in this process, will of course be different from those assumed in the actual motion; and the path in space of any point P of the figure will consist of a succession of circular arcs instead of a continuous curve. It is evident, however, that by taking the positions 1, 2, 3,... sufficiently close to one another, the path of any point P can be made to deviate from the true path as little as we please.

It remains to compare the polygons $O_{12}O_{23}O_{34}\ldots$ and $O_{12}O'_{23}O'_{34}\ldots$ with the corresponding centrodes. Let $I_1, I_2, I_3,\ldots$ be the positions in space of the instantaneous centre corresponding to the values $t_1, t_2, t_3,\ldots$ of the independent variable. Then if, keeping t_1 fixed, we make $|t_2 - t_1|$ smaller and smaller, the point O_{12} approaches, and ultimately coincides with I_1.

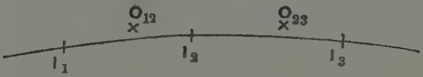

Fig. 141.

Again if, keeping t_2 fixed, we make $|t_2 - t_1|$ smaller and smaller, O_{12} will move into coincidence with I_2. Since these variations of t are equal and opposite, and since the coordinates of O_{12} are as a rule symmetric functions of t_1, t_2, the displacements of O_{12} will in general be ultimately equal and opposite; that is, the triangle $I_1O_{12}I_2$ is ultimately isosceles, with its equal angles infinitely small. It follows that when the positions 1, 2, 3,... are taken closer and closer, the length of any finite portion of the polygon $O_{12}O_{23}O_{34}\ldots$ is ultimately equal to that of the corresponding arc of the space-centrode, and that any side of the polygon is ultimately a tangent to this centrode. Similar relations can be proved in the same manner to exist between the polygon $O_{12}O'_{23}O'_{34}\ldots$ and the body-centrode. It follows that in each position of the moving figure the two centrodes are in contact, and that the lengths of their arcs between corresponding points are equal. This proves the theorem.

Ex. 1. A straight line AB of constant length moves with its extremities on two fixed straight lines OX, OY.

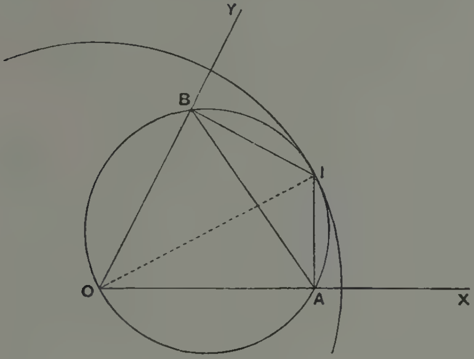

Fig. 142.

The instantaneous centre I is at the intersection of perpendiculars to OX, OY at the points A, B respectively. The points A, B lie on the circle described on OI as diameter; and since in this circle the chord AB, of given length, subtends a constant angle AOB at the circumference, the diameter is determinate. Hence the space-locus of I is a circle with centre O. Again, since the angle AIB is constant, the locus of I relative to AB is a circle whose diameter is equal to the constant value of OI. Hence the motion is equivalent to the rolling of a circle on the inside of a fixed circle of twice its size. This kind of motion has been considered in Art. 138, Ex. 2, and it has been shewn that any point P fixed relatively to AB will describe an ellipse, which in certain cases, viz. when P is on the circumference of the rolling circle, degenerates into a straight line.

Ex. 2. In the linkage known as the 'crossed parallelogram' (see Art. 146, 2°), if the bar AD be held fixed, the instantaneous centre I for the opposite bar BC is at the intersection of AB and CD. Also it is evident from symmetry that the sums

$$AI + ID \quad \text{and} \quad BI + IC$$

are constant, being each equal to AB or CD. Hence the locus of

I relative to AD is an ellipse with A, D as foci, and the locus of I relative to BC is an equal ellipse with B, C as foci. The motion of BC relative to AD is therefore represented by the rolling of an ellipse on an equal ellipse.

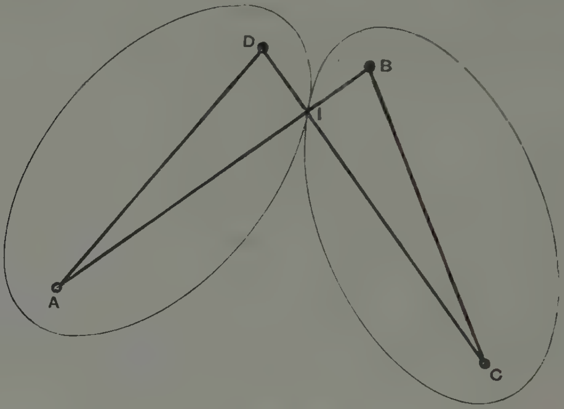

Fig. 143.

If the bar AB be supposed fixed, the relative motion of CD will be represented by the rolling of a hyperbola with C, D as foci on an equal hyperbola with A, B as foci.

168. Double Generation of Epicyclics as Roulettes.

As a further example we return to the mechanical method of compounding uniform circular motions, by means of a jointed parallelogram $OQPQ'$, referred to in Art. 139.

We will suppose for definiteness that the angular velocities n, n', of the bars OQ, OQ', have the same sign.

The instantaneous centre (I) of the bar QP will be a point in QO such that

$$n' . QI = n . OQ \qquad (1).$$

For the velocity of any point rigidly attached to QP will be made up of a translation $n . OQ$ at right angles to OQ, and a rotation with angular velocity n' relatively to Q. Hence under the above condition the velocity of the point attached to QP which at the instant under consideration is at I will be zero. The two centrodes for the motion of the bar QP are therefore the circles described, with O and Q as centres, to pass through I.

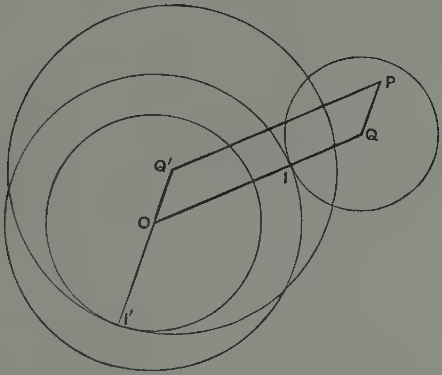

Fig. 144.

For a similar reason, the instantaneous centre (I') of the bar $Q'P$ will be a point in $Q'O$, such that

$$n \, . \, Q'I' = n' \, . \, OQ' \quad \text{.....................(2)*}.$$

Hence, for the motion of the bar $Q'P$, the two centrodes are the circles described, with O and Q' as centres, to pass through I'.

Since P is a point on each of the bars QP, $Q'P$, we see that any direct epicyclic can be described in two ways as an epitrochoid.

In the particular case where $QP = QI$, it follows from (1) and (2) that

$$QP : OQ = QI : OQ = n : n' = OQ' : Q'I' = QP : Q'I',$$

whence $$Q'P = OQ = Q'I'.$$

---

* The figure corresponds to the case of $n' > n$. If $n' < n$, I will lie in QO produced, and I' between Q' and O.

The path of P is in this case an epicycloid, and we learn that any epicycloid can be generated in two ways, viz. by the rolling of either of two determinate circles on the outside of the same fixed circle[*].

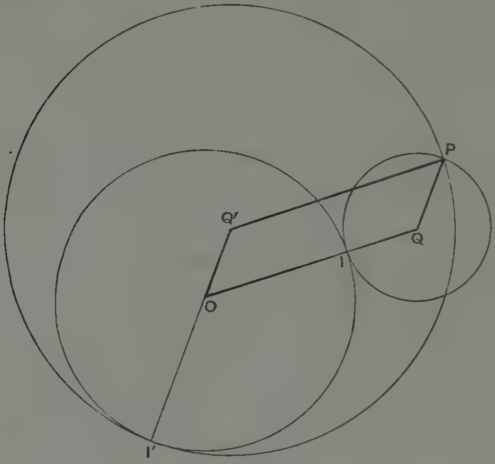

Fig. 145.

As an instance, we have the double generation of the cardioid explained in Art. 138, Exs. 1, 3.

The case where the angular velocities n, n' have opposite signs may be left to the reader to examine. It will appear that any retrograde epicyclic can be generated in two distinct ways as a hypotrochoid. And, in particular, any hypocycloid can be generated in two ways by the rolling of either of two determinate circles on the inside of the same fixed circle[*].

169. Teeth of Wheels.

The theory explained in Arts. 167, 168 has an interesting application in practical Mechanics, viz. to the determination of the proper shapes to be given to the teeth of wheels.

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[*] This proposition is due to Euler (1781).

The kinematical problem is to determine the relations between the forms of two cylindrical surfaces Ω, Ω', which are free to rotate about fixed parallel axes, so that, when either drives the other by sliding contact, the rotations may be in a constant ratio.

There are two methods of solving this problem; they may be distinguished as the method of 'envelopes' and the method of 'roulettes.'

Considering the section made by a plane perpendicular to the generating lines, let O, O' represent the fixed axes. We take as the standard case that in which the directions of the angular velocities (n, n') are opposite, as indicated in Fig. 146. If we divide OO' in I, so that

$$n \, . \, OI = n' . \, O'I,$$

the point I will have the same velocity whether we regard it as belonging to the first cylinder or the second. The circles (C, C') described through I, with O, O' as centres, are called the 'pitch-circles' of the respective cylinders. It is evident that in the desired motion these circles will roll on one another without slipping.

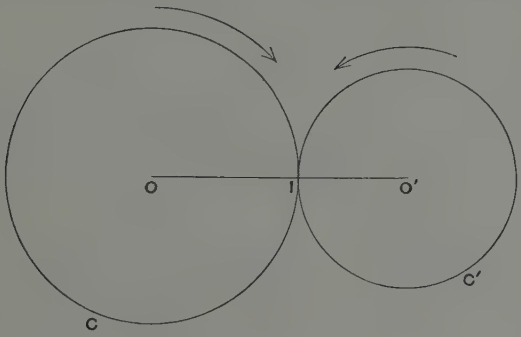

Fig. 146.

The *relative* motion of the cylinders will be unaltered if we impress on the whole system an angular velocity about O, equal and opposite to that of the first cylinder. The pitch-circle C' will then roll with angular velocity $n+n'$ on the pitch-circle C, which is at rest. Evidently, the required condition is that, in the motion as thus modified, the section of the cylinder Ω' shall constantly touch that of the cylinder Ω, which is fixed. In other words, the section of Ω must be the envelope of the successive positions of Ω'. Hence, subject to limitations of a practical kind, we may adopt any curve we please as the section of Ω'; the corresponding form of the section of Ω is then determined as the envelope obtained when the circle C' rolls on C, carrying the assigned curve with it.

For example, if the section of Ω' be a straight line through O', the envelope will be the epicycloid generated by a point on the circumference of the circle described on $O'I$ as diameter (see Art. 164, Ex. 2). This design is very frequent, the profile of the 'flanks'* of the teeth being straight, and that of the 'faces'† epicycloidal.

Again, if the section of Ω' reduce to a point on the pitch-circle C', the envelope is simply the path traced by this point as C' rolls on C, *i.e.* an epicycloid. If the section of Ω' be a circle of finite size, having its centre on the pitch-circle C', the envelope is a curve parallel to the epicycloid traced by the centre, and therefore having the same evolute (Art. 161). It has been shewn (Art. 159) that the evolute of an epicycloid is a similar epicycloid; the envelope in question is therefore an involute of an epicycloid. This case occurs in practice when a toothed wheel is driven by a lantern-pinion, or trundle, or by a row of pins projecting from the plane face of a circular disk.

Another interesting example is that of 'involute teeth.' Two smaller circles, called 'base-circles,' are drawn concentric with the pitch-circles, and having their radii in proportion; and involutes of the base-circles are traced. In any position in which these involutes are in contact, the common normal will be a tangent to both the base-circles, and will therefore go through the point of contact I of the pitch-circles, owing to the proportionality of the radii. Hence the involutes fulfil the condition for sliding contact with a constant ratio of angular velocities. It is evident that any two involute wheels will work together with a constant velocity-ratio, equal to the ratio of the radii of the base-circles, and therefore independent of the particular distance between the axes O, O'.

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* That is, the portions inside the respective pitch-circles.

† That is, the portions outside the pitch-circles.

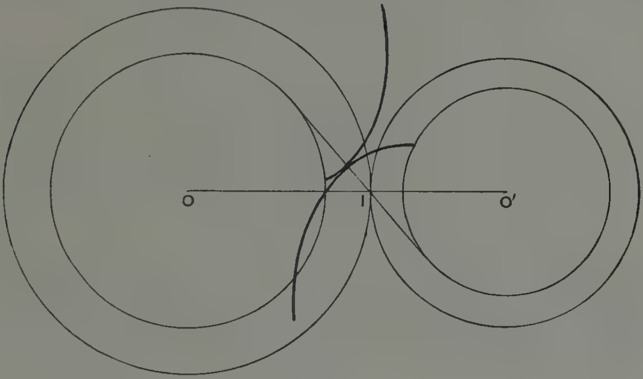

Fig. 147.

In the second method of solving the kinematical problem, we imagine that as the circle C' rolls on C another curve (Σ), chosen at will, also rolls on C, but so as to have always the same point of contact I. Any point P carried by Σ will then trace out a fixed curve, which may be taken as the contour of the cylinder Ω, and also another curve relatively to the pitch-circle C', which may be taken as the contour of the cylinder Ω'. The two curves are in fact roulettes obtained by rolling Σ on C and C' respectively; and to see that they touch at P we need only remark that the absolute motion of P, and the motion of that point of the second figure which instantaneously coincides with P, are both at right angles to IP, so that the motion of P relatively to the circle C' is also at right angles to IP.

In practice, the auxiliary curve Σ is usually a circle, and the tracing point P is taken on its circumference. The roulettes are then (in the standard case) an epicycloid, and a hypocycloid, respectively. As a particular case, Σ may be a circle passing always through O, and having its diameter therefore equal to OI. The hypocycloid then reduces to a straight line (Art. 138, Ex. 2), and we fall back on the case of radial flanks obtained above by the envelope method.

When we have a wheel in gear with a straight rack the radius of one of the pitch-circles is infinite. The further examination of this case is left to the reader.

EXAMPLES. XLVI.

1. A lamina moves in any manner in its own plane; prove that parallel straight lines in the lamina envelope parallel curves.

2. A straight line moves so as always to pass through a fixed point O, whilst a point Q on it describes a circle passing through O. Prove that the instantaneous centre is at the other extremity of the diameter through Q, and determine the two centrodes.

Deduce a construction for the normal to a limaçon; and infer that in a cardioid the normals at the extremity of any chord through the cusp meet at right angles on the perpendicular to the chord at this point.

3. A plane figure moves so that two straight lines in it touch two fixed circles; determine the two centrodes.

4. If a circle roll on a fixed circle of half the size, which it surrounds, every straight line carried by the rolling circle will envelope a circle.

5. Prove that if a plane figure move so that a straight line in it rolls on a fixed circle, the envelope of any other straight line in the figure is an involute of a circle.

6. The radius of curvature of the envelope of the straight line

$$\alpha x + \beta y = 1,$$

where α, β are given functions of a parameter t, is

$$\pm \frac{(\alpha^2 + \beta^2)^{\frac{3}{2}} (\alpha''\beta' - \alpha'\beta'')}{(\alpha\beta' - \alpha'\beta)^3},$$

the accents denoting differentiations with respect to t.

7. If the curve whose tangential-polar equation is $r=f(p)$ roll on a fixed straight line, the curvature of the path of the pole is

$$-\frac{d}{dp}\left(\frac{p}{r}\right),$$

where r is the radius vector to the point of contact.

8. Prove that if a parabola roll on a fixed straight line the path of the focus is a catenary.

9. Prove that if a conic roll on a fixed straight line the path of either focus is a curve such that

$$\frac{1}{\rho}+\frac{1}{n}=\frac{1}{c},$$

where ρ is the radius of curvature, n is the normal, and c is a constant.

10. If an equiangular spiral roll on a fixed straight line, the path of the pole is a straight line.

11. If the reciprocal spiral $r=a/\theta$ roll on a straight line, the path of the pole is a tractrix.

12. If any one of the Cotes' spirals

$$\frac{1}{p^2}=\frac{A}{r^2}+B$$

rolls on a straight line, the pole traces out a curve such that the curvature varies as the normal.

13. A curve rolls on a fixed straight line; prove that the arc of the roulette traced by any carried point O is equal to the corresponding arc of the pedal of the given curve with respect to O. (Steiner.)

14. A closed oval curve rolls on a fixed straight line; prove that in a complete revolution the area swept over by the variable line which joins the point of contact to any internal carried point O is double the area of the pedal of the given curve with respect to O. (Steiner.)

15. Prove that if two wheels with epicycloidal teeth work together about fixed axes, the locus of the point of contact of any two teeth is an arc of a circle.

16. Prove that in the case of involute teeth the locus of the point of contact is a straight line.

CHAPTER XI.

DIFFERENTIAL EQUATIONS OF THE FIRST ORDER.

170. Formation of Differential Equations.

Any relation between an independent variable x, a dependent variable y, and one or more of the derived functions

$$\frac{dy}{dx},\quad \frac{d^2y}{dx^2},\quad \frac{d^3y}{dx^3},\ \ldots$$

is called a 'differential equation*.'

The 'order' of the equation is fixed by that of the highest differential coefficient which occurs in it. Thus a differential equation of the 'first order' is a relation between x, y, and dy/dx.

Before proceeding to methods of solution, it is instructive to consider one manner in which differential equations may arise.

If we are given a relation between the variables x, y, and an arbitrary constant C, then by differentiation we obtain an equation involving x, y, dy/dx, and C. By elimination of C between this and the original equation we obtain a differential equation of the first order.

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* More particularly, it is called an 'ordinary' as distinguished from a 'partial' differential equation, *i.e.* one which involves partial derivatives of a function of two or more independent variables.

More generally, given a relation between the variables x, y, and n arbitrary constants $C_1, C_2, \ldots C_n$, then if we differentiate n times in succession with respect to x, we have altogether $n+1$ equations between which the n arbitrary constants can be eliminated. The result is a differential equation of the nth order.

From this point of view the original equation is called the 'primitive.'

Ex. 1. If the primitive be

$$y = mx + C \qquad (1),$$

where C is arbitrary, the differential equation is

$$\frac{dy}{dx} = m \qquad (2).$$

Ex. 2. From the primitive

$$y^2 = 4ax + C \qquad (3)$$

we deduce

$$y\frac{dy}{dx} = 2a \qquad (4).$$

Ex. 3. If the primitive be

$$x \cos \alpha + y \sin \alpha = a \qquad (5),$$

where α is arbitrary, we deduce

$$\cos \alpha + \frac{dy}{dx} \sin \alpha = 0.$$

These give

$$\left(y - x\frac{dy}{dx}\right) \sin \alpha = a, \qquad \left(y - x\frac{dy}{dx}\right) \cos \alpha = -a\frac{dy}{dx},$$

whence, squaring and adding,

$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left\{1 + \left(\frac{dy}{dx}\right)^2\right\} \qquad (6).$$

Ex. 4. If the primitive be

$$y = Ax + B \qquad (7),$$

where both constants A, B are to be eliminated, we find

$$\frac{d^2y}{dx^2} = 0 \qquad (8).$$

Ex. 5. From the primitive

$$(x-\alpha)^2+(y-\beta)^2=a^2 \;\;\ldots\ldots\ldots\ldots\ldots\ldots(9),$$

where α, β are to be eliminated, we obtain

$$a^2\left(\frac{d^2y}{dx^2}\right)^2=\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^3 \;\;\ldots\ldots\ldots\ldots(10).$$

The details of the work are given in Art. 206.

The above processes admit of a geometrical interpretation. The equations obtained by varying the arbitrary constants in the primitive represent a certain system or family of curves; the differential equation (in which these constants do not appear) expresses some property common to all these curves.

Thus in Ex. 2, above, the primitive represents a system of equal parabolas having their axes coincident with the axis of x, but their vertices at different points of it. The differential equation (4) expresses a property common to all these curves, viz. that the subnormal (Art. 53) has a given constant value $2a$.

Again in Ex. 5, if we vary α, β in the primitive we get a doubly-infinite system of circles of given radius a, having their centres anywhere in the plane xy. The differential equation expresses that the radius of curvature has everywhere the constant value a. See Art. 152.

Other illustrations may be taken from Dynamics.

Ex. 6. If, in the primitive

$$x=\tfrac{1}{2}gt^2+At+B \;\;\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

we vary A and B, we get a certain group or class of rectilinear motions. The differential equation

$$\frac{d^2x}{dt^2}=g \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12)$$

expresses a property common to the group, viz. that the acceleration has the constant value g.

Ex. 7. Again, if the primitive be

$$x=A\cos nt+B\sin nt \;\;\ldots\ldots\ldots\ldots\ldots\ldots(13),$$

we find

$$\frac{d^2x}{dt^2}=-n^2x \;\;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

This asserts that in the whole group of motions represented by the primitive the acceleration is towards the origin of x, and varies (in a given ratio n^2) as the distance from this origin. See also Art. 65, Ex. 2.

The preceding examples will suffice to illustrate the derivation of a differential equation from a primitive relation between x and y involving one or more arbitrary constants. In practice we are more usually confronted with the inverse problem, viz. to ascertain the most general form of relation between the variables which satisfies a given differential equation. Thus in Geometry, or in Dynamics, some general property may be propounded, whose expression takes the form of a differential equation, and it is required to determine the whole system of curves, or group of motions, which possess the property.

The process of passing from a given differential equation to the general relation between the variables which it implies is called 'solving,' or 'integrating' the equation; and the result is called the 'general solution,' or the 'complete primitive,' although the latter name is hardly appropriate from this point of view. A 'particular solution' is any relation between the variables which happens to satisfy it.

EXAMPLES. XLVII.

1. If
$$y = Ax^2 + B,$$
prove that
$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0.$$

2. If
$$y = Ax^2 + Bx,$$
prove that
$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + \frac{2y}{x^2} = 0.$$

3. If
$$y = Ae^{kx} + Be^{-kx},$$
prove that
$$\frac{d^2y}{dx^2} - k^2y = 0.$$

4. If
$$y = Ae^{\alpha x} + Be^{\beta x},$$
prove that
$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0.$$

5. If
$$y = (A + Bx)\,e^{kx},$$
prove that
$$\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0.$$

6. If
$$x = e^{-\frac{1}{2}kt}(A\cos nt + B\sin nt),$$
prove that
$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + (n^2 + \tfrac{1}{4}k^2)\,x = 0.$$

7. If
$$V = \frac{A}{r} + B,$$
prove that
$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0.$$

8. If
$$V = A\log r + B,$$
prove that
$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} = 0.$$

9. If
$$\phi = \frac{Ae^{kr} + Be^{-kr}}{r},$$
prove that
$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} - k^2\phi = 0.$$

10. If
$$\phi = \frac{A\cos kr + B\sin kr}{r},$$
prove that
$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + k^2\phi = 0.$$

11. If
$$y = (A + Bx)\cos kx + (C + Dx)\sin kx,$$
prove that
$$\frac{d^4y}{dx^4} + 2k^2\frac{d^2y}{dx^2} + k^4y = 0.$$

12. If
$$y = A\cosh kx + B\sinh kx + C\cos kx + D\sin kx,$$
prove that
$$\frac{d^4y}{dx^4} = k^4y.$$

13. If

$$y=\left(A\cosh\frac{kx}{\sqrt{2}}+B\sinh\frac{kx}{\sqrt{2}}\right)\cos\frac{kx}{\sqrt{2}}+\left(C\cosh\frac{kx}{\sqrt{2}}+D\sinh\frac{kx}{\sqrt{2}}\right)\sin\frac{kx}{\sqrt{2}},$$

prove that

$$\frac{d^4y}{dx^4}+k^4y=0.$$

14. If

$$y=A\sin^{-1}x+B,$$ ✓

prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=0.$$

15. If

$$y=(\sin^{-1}x)^2+A\sin^{-1}x+B,$$

prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2.$$

16. If

$$y=A\cos(\log x)+B\sin(\log x),$$ ✓

prove that

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=0.$$

17. If $y=A\{x+\sqrt{(x^2-1)}\}^n+B\{x-\sqrt{(x^2-1)}\}^n$,

prove that

$$(x^2-1)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-n^2y=0.$$

18. Shew that the primitive

$$y=mx+\frac{a}{m},$$

where m is arbitrary, leads to

$$x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}+a=0.$$

19. If

$$2cy+c^2=x^2,$$

where c is arbitrary, prove that

$$x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}-x=0.$$

20. The differential equation of all parabolas having their axes parallel to the axis of y is

$$\frac{d^3y}{dx^3}=0.$$

21. The differential equation of all parabolas having their axis of symmetry coincident with the axis of x is

$$y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0.$$

22. The differential equation of all conics having their principal axes coincident with the coordinate axes is

$$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0.$$

23. Prove that the differential equation of all circles touching the axis of x at the origin is

$$\frac{dy}{dx}=\frac{2xy}{x^2-y^2}.$$

24. Prove that the differential equation of all conics touching the axis of y at the origin, and having their centres on the axis of x, is

$$x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0.$$

171. Equations of the First Order and First Degree.

The general type of a differential equation of the first order may be written

$$\phi\left(x,\,y,\,\frac{dy}{dx}\right)=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The equation implies that y is to be a differentiable function of x, and that dy/dx is to be continuous.

The mode of derivation of a differential equation of the first order from a primitive involving an arbitrary constant, explained in Art. 170, may suggest that the general solution of (1) will in all cases consist of a relation between x and y involving an arbitrary constant. With some qualification, due to the occurrence of 'singular' solutions (Art. 180), this is in fact the case. The rigorous proof, however, is difficult, and may be passed over here without inconvenience, since in almost all cases for which practical methods of integration have been discovered the process itself contains the demonstration that the solution is of the kind indicated.

In such problems as ordinarily arise, either the left-hand side of (1) is a rational integral algebraic function of dy/dx, or the equation can be transformed so that this shall be the case. The 'degree' of the equation is then fixed by that of the highest power of dy/dx which occurs in it.

The general equation of the first degree may be written

$$M + N\frac{dy}{dx} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

or

$$Mdx + Ndy = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where M, N are given functions of x and y. The form (2) is also equivalent to

$$\frac{dy}{dx} = -\frac{M}{N} = \phi(x, y) \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If $\phi(x, y)$ be real and single-valued for all values of x and y, then corresponding to any point in the plane xy we have a definite direction, assigned by the equation (4). If we imagine a point, starting from any position in the plane, to move always in the direction thus indicated, it will trace out a curve, which constitutes a particular solution, or primitive, of the proposed equation. And the whole assemblage of such curves will constitute the complete solution. It appears that, in the present case, no two curves of the primitive system will intersect.

We proceed to give an account of the methods which have been devised for the solution of the equation (4) in various cases.

172. Methods of Solution. One Variable absent.

1°. The form

$$\frac{dy}{dx} = f(x) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

where y does not appear explicitly, requires merely an ordinary integration. Thus

$$y = \int f(x)\,dx + C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

where C is an arbitrary constant.

2°. The equation

$$\frac{dy}{dx}=f(y) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

in which x does not appear explicitly, may be written

$$\frac{dy}{f(y)}=dx,$$

whence

$$\int\frac{dy}{f(y)}=x+C \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

Ex. To find the curves whose subtangent has a given constant value a. We have (Art. 53)

$$y\div\frac{dy}{dx}=a,$$

or

$$\frac{dy}{y}=\frac{dx}{a} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Hence

$$\log y=\frac{x}{a}+C,$$

or

$$y=be^{x/a} \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where $b, =e^C$, is arbitrary.

173. Variables Separable.

A more general form is

$$F(x)+f(y)\frac{dy}{dx}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

or

$$F(x)\,dx+f(y)\,dy=0 \quad\ldots\ldots\ldots\ldots\ldots(2).$$

If an equation can be brought to this form the variables are said to be 'separable.' The solution obviously is

$$\int F(x)\,dx+\int f(y)\,dy=C \quad\ldots\ldots\ldots\ldots\ldots(3).$$

Ex. 1. To find the curves such that the normals all pass through one point.

If we adopt rectangular axes through this point as origin, the condition gives

$$\frac{dy}{dx}=-\frac{x}{y},$$

or
$$x\,dx + y\,dy = 0 \quad\text{.......................}\quad (4),$$
whence
$$x^2 + y^2 = C \quad\text{.......................}\quad (5).$$
The required curves are therefore circles described with the origin as centre. Cf. Art. 56, Ex. 2.

Ex. 2. To find a curve such that the tangents drawn to it from any point are equal.

If we take a fixed tangent as initial line, and its point of contact as origin, then if the two tangents drawn from any point on the initial line be equal, we must have, in the notation of Art. 55,
$$\phi = \theta,$$
and therefore
$$r\frac{d\theta}{dr} = \tan\theta \quad\text{.......................}\quad (6).$$
Hence
$$\frac{dr}{r} = \cot\theta\, d\theta \quad\text{.......................}\quad (7),$$
and
$$\log r = \log\sin\theta + C,$$
or
$$r = a\sin\theta \quad\text{.......................}\quad (8),$$
where a is arbitrary. The circle is therefore the only curve possessing the stated property.

Ex. 3. The equation of rectilinear motion of a particle under an attractive force varying inversely as the square of the distance from a fixed point is
$$v\frac{dv}{dx} = -\frac{\mu}{x^2} \quad\text{.......................}\quad (9).$$

Integrating with respect to x,
$$\tfrac{1}{2}v^2 = \frac{\mu}{x} + C \quad\text{.......................}\quad (10).$$
If v vanish for $x = \infty$, we have $C = 0$. In this case the velocity with which the particle arrives at a distance a from the centre of force is $\sqrt{(2\mu/a)}$, or $\sqrt{(2ga)}$, if $g = \mu/a^2$.

This gives the velocity with which an unresisted particle, falling from rest at a great distance, would reach the Earth, provided a denote the Earth's radius, and g the value of gravity at the surface.

Ex. 4. In a suspension bridge with uniform horizontal load the form of the chain is determined by the condition that any two tangents to the curve intersect on the vertical bisecting the chord of contact.

If the lowest point be taken as origin of rectangular coordinates, and the corresponding tangent as axis of x, the subtangent of any other point must be equal to one-half the abscissa. Hence

$$y \div \frac{dy}{dx} = \tfrac{1}{2}x,$$

or

$$\frac{dy}{y} = 2\,\frac{dx}{x} \quad\text{......................}\quad (11),$$

the integral of which is

$$\log y = 2 \log x + \text{const.},$$

or

$$y = x^2/a \quad\text{..........................}\quad (12),$$

where a is arbitrary. That is, the curve formed by the chain must be a parabola with its axis vertical.

174. Exact Equations.

The case of the preceding Art. comes under the head of 'exact equations.' An equation

$$Mdx + Ndy = 0 \quad\text{....................}\quad (1)$$

is said to be 'exact' when M and N are of the forms $\partial u/\partial x$ and $\partial u/\partial y$ respectively. The form

$$\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = 0 \quad\text{..................}\quad (2)$$

is equivalent to

$$du = 0 \quad\text{..........................}\quad (3),$$

and its integral is

$$u = C \quad\text{..........................}\quad (4),$$

where C is the arbitrary constant*.

It may be shewn that every equation of the type (1) is either exact, or can be rendered exact by a suitable 'integrating factor.' The number of such factors is unlimited; for if we suppose the equation (1) to have been brought to the form (3), it will still be exact when multiplied by $f'(u)$, where $f(u)$ may be *any* function of u. The integral of

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* The rule for ascertaining whether a proposed equation of the first degree is exact will be given in Art. 210.

$$f'(u)\,du = 0 \quad\text{.....................}\quad (5)$$

is

$$f(u) = C \quad\text{.........................}\quad (6),$$

which is obviously equivalent to (4).

Ex. 1. $$(ax + hy + g)\,dx + (hx + by + f)\,dy = 0 \quad\text{.........}\quad (7).$$

This is equivalent to

$$d\,(ax^2 + 2hxy + by^2 + 2gx + 2fy) = 0 \quad\text{..............}\quad (8),$$

whence

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = C \quad\text{..............}\quad (9).$$

Ex. 2. $$xdx + ydy = k\,(xdy - ydx) \quad\text{.............}\quad (10).$$

This may be written

$$d\,(x^2 + y^2) = 2kx^2 d\left(\frac{y}{x}\right) \quad\text{....................}\quad (11),$$

and so becomes exact on division by $x^2 + y^2$, thus

$$\frac{d\,(x^2 + y^2)}{x^2 + y^2} = \frac{2kd\left(\dfrac{y}{x}\right)}{1 + \dfrac{y^2}{x^2}} \quad\text{..................}\quad (12).$$

Hence, integrating,

$$\log\,(x^2 + y^2) = 2k\tan^{-1}\frac{y}{x} + C \quad\text{..............}\quad (13).$$

The equation (10) may also be solved as follows. Its form suggests the substitutions

$$x = r\cos\theta, \quad y = r\sin\theta \quad\text{.................}\quad (14),$$

which give

$$xdx + ydy = rdr, \quad xdy - ydx = r^2 d\theta \quad\text{.........}\quad (15).$$

The equation therefore reduces to

$$\frac{dr}{r} = kd\theta \quad\text{.......................}\quad (16),$$

whence

$$\log r = k\theta + C \quad\text{........................}\quad (17).$$

This is obviously equivalent to (13).

Ex. 3. To find the form of a solid of revolution such that the centre of mass of the volume cut off by any right section shall be at a distance from this section equal to $1/n$th of the length of the axis.

If the axis of x be that of symmetry, and y be the ordinate of the generating curve, we must have (Art. 121)

$$\frac{\int_0^\xi xy^2dx}{\int_0^\xi y^2dx} = \left(1-\frac{1}{n}\right)\xi,$$

or

$$\int_0^\xi xy^2dx = \frac{n-1}{n}\xi\int_0^\xi y^2dx \text{ .... } (18),$$

where ξ is the abscissa of the bounding section. Hence, if η denote the radius of this section, we find, on differentiating with respect to ξ according to the rule of Art. 90,

$$\xi\eta^2 = \frac{n-1}{n}\xi\eta^2 + \frac{n-1}{n}\int_0^\xi y^2dx,$$

or

$$\xi\eta^2 = (n-1)\int_0^\xi y^2dx \text{ .... } (19).$$

A second differentiation gives

$$\frac{d}{d\xi}(\xi\eta^2) = (n-1)\eta^2 \text{ .... } (20),$$

whence

$$\frac{d(\xi\eta^2)}{\xi\eta^2} = (n-1)\frac{d\xi}{\xi} \text{.....................} (21).$$

Integrating, we find $\xi\eta^2 = A\xi^{n-1}$.

The generating curve must therefore be of the type

$$y^2 = Ax^{n-2} \text{ .... } (22).$$

Since we have differentiated twice with respect to ξ, the differential equation actually solved is somewhat more general than the original problem. In fact the same differential equation would have been obtained if, instead of zero, we had had other (and distinct) constants as the lower limits of the two definite integrals in (18). It is therefore necessary to examine *à posteriori* whether the solution finally obtained satisfies the original equation with the actual lower limits. This is easily verified to be the case if $n > 2$.

We note that if $n=3$, the solid is a paraboloid of revolution, and that if $n=4$ it is a cone; cf. Art. 121, Exs. 1, 2.

175. Homogeneous Equation.

Let us suppose that, in the equation

$$M+N\frac{dy}{dx}=0,$$

M, N are homogeneous functions of x and y, of the same degree.

In this case the fraction M/N is a function of y/x only, and we may write

$$\frac{dy}{dx}=f\left(\frac{y}{x}\right)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

If we put $y=xv$ this becomes

$$x\frac{dv}{dx}+v=f(v)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The variables x, v are now separable, viz. we have

$$\frac{dx}{x}=\frac{dv}{f(v)-v}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

whence
$$\log x=\int\frac{dv}{f(v)-v}+C\ldots\ldots\ldots\ldots\ldots(4).$$

After the integration has been effected we must write $v=y/x$.

Ex.
$$(x^2-y^2)\frac{dy}{dx}-2xy=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Here
$$\frac{dy}{dx}=\frac{2\dfrac{y}{x}}{1-\dfrac{y^2}{x^2}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

whence
$$x\frac{dv}{dx}+v=\frac{2v}{1-v^2},\quad\text{or}\quad x\frac{dv}{dx}=\frac{v(1+v^2)}{1-v^2}.$$

Hence
$$\frac{dx}{x}=\frac{1-v^2}{v(1+v^2)}\,dv=\left(\frac{1}{v}-\frac{2v}{1+v^2}\right)dv\ldots\ldots\ldots(7).$$

Integrating, we have

$$\log x=\log v-\log(1+v^2)+\text{const.},$$

which is equivalent to

$$x(1+v^2)=Cv,$$

or

$$x^2+y^2=Cy \qquad\qquad (8).$$

In the geometrical interpretation, the general solution of a homogeneous differential equation must represent a system of similar and similarly situated curves, the origin being a centre of similitude. For the equation (1) shews that where the curves cross any arbitrary straight line ($y/x=m$) through the origin, dy/dx has the same value for each, that is, the tangents are parallel.

Thus, in the above Ex. the solution represents a system of circles touching the axis of x at the origin.

If in (4) we put

$$C=\log c,$$

y/x or v is determined as a function of x/c. In other words, the primitive is homogeneous in respect to x, y, and c, and is therefore of the type

$$\phi\left(\frac{x}{c},\ \frac{y}{c}\right)=0 \qquad\qquad (9).$$

This is in accordance with the geometrical property above stated, since if x, y, and c be altered in any the same ratio, the equation (9) is unaltered. In other words, a change in the value of c merely alters the *scale* of the curve.

EXAMPLES. XLVIII.

1. Integrate $\dfrac{dy}{dx}=\dfrac{y}{x}$. $[y=Cx.]$

2. Integrate $\dfrac{dy}{dx}=\dfrac{y}{x^2-1}$. $\left[y^2=C\dfrac{x-1}{x+1}.\right]$

3. Integrate $\dfrac{dy}{dx}=\cot x\cot y$. $[\sin x\cos y=C.]$

4. Integrate $x^2\dfrac{dy}{dx}+y=1$. $[y=1+Ce^{1/x}.]$

5. Solve $m(y+b)\,dx+n(x+a)\,dy=0.$

[$(x+a)^m(y+b)^n=C.$]

6. Solve $\dfrac{dy}{dx}=\dfrac{1+y^2}{1+x^2}.$ $\left[y=\dfrac{x+C}{1-Cx}.\right]$

7. Solve $(1+y^2)\,dx-xy(1+x^2)\,dy=0.$

[$(1+x^2)(1+y^2)=Cx^2.$]

8. Find the curves in which the angle between the tangent and the radius is one-half the vectorial angle (θ).

[The cardioids $r=a(1-\cos\theta)$.]

9. Find the curves in which the perpendicular from the origin on the tangent is equal to the abscissa of the point of contact. [The circles $r=2a\cos\theta$.]

10. Find the curves such that the portion of the tangent included between the coordinate axes is bisected at the point of contact. [The hyperbolas $xy=C$.]

11. Find the curves in which the subtangent varies as the abscissa. [$y=Cx^m$.]

12. Prove that if the subnormal bears a constant ratio to the abscissa the curve is a conic.

13. Find the curves in which the perpendicular from the foot of the ordinate to the tangent has a constant length a.

[The catenaries $y=a\cosh(x-\alpha)/a$.]

14. Find the curve in which the polar subtangent is constant ($=a$). [$r=a/(\theta-\alpha)$.]

15. Find the curve in which the polar subnormal is constant ($=a$). [$r=a(\theta-\alpha)$.]

16. Find the curves such that the area included between any two ordinates is proportional to the intercepted arc.

[The catenaries $y=a\cosh(x-\alpha)/a$.]

17. Find the curves such that the area included between any ordinate, the axis of x, and the curve may be $1/n$th of the rectangle contained by the ordinate and the corresponding abscissa. [$y=Cx^{n-1}$.]

18. Find the form of a solid of revolution in order that the volume cut off by any right section may be $1/n$th of the product of the area of this section into the length of the axis.

[The equation of the generating curve must be $y^2 = Ax^{n-1}$.]

19. In a suspension-rod of uniform strength the area of the cross-section (S) varies as the total stress across it; prove that if x be measured vertically downwards the relation between S and x must be of the form

$$S = A - B\int_0^x S dx.$$

Hence shew that the form of the rod must be that generated by the revolution of a curve of the type

$$y = be^{-x/a}$$

about the axis of x.

20. Find the form of a curve, symmetrical with respect to the axis of x, such that the centre of mass of the area cut off by any double ordinate may be at a distance from this ordinate equal to $1/n$th of the length of the axis. [$y = Cx^{n-2}$.]

21. Solve $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0.$

22. Solve $xdx + ydy = a^2 \dfrac{xdy - ydx}{x^2 + y^2}.$

$$\left[x^2 + y^2 = 2a^2 \tan^{-1}\frac{y}{x} + C.\right]$$

23. Solve $x\dfrac{dy}{dx} - y = \dfrac{x}{a}\sqrt{(x^2 + y^2)}.$

$$\left[y = x \sinh\frac{x-a}{a}.\right]$$

24. Solve $x\dfrac{dy}{dx} - y = \sqrt{(x^2 + y^2)}.$

Give the geometrical interpretations of the differential equation and of its primitive. [$x^2 = 2Cy + C^2$.]

25. Solve $\dfrac{dx}{x^2 - 2xy} = \dfrac{dy}{y^2 - 2xy}.$ [$xy(x - y) = C$.]

26. Solve $x\dfrac{dy}{dx} + y = xy.$ [$xy = Ce^x$.]

27. Solve $$x\frac{dy}{dx} - y = xy. \qquad [y = Cxe^x.]$$

28. Shew that the equation

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

is rendered homogeneous by the substitutions

$$ax + by + c = \xi, \quad a'x + b'y + c' = \eta.$$

29. Shew that an equation of the type

$$\frac{dy}{dx} = f(ax + by)$$

may be solved by the substitution

$$ax + by = z.$$

30. Shew how to solve any equation of the type

$$\frac{dy}{dx} = f\left(\frac{ax + by + c}{a'x + b'y + c'}\right).$$

176. Linear Equation of the First Order, with Constant Coefficients.

A 'linear' equation is one which involves y and its derivatives only in the first degree. Thus the linear equation of the first order is of the type

$$\frac{dy}{dx} + Py = Q \quad \text{.....................} \quad (1),$$

where P, Q are given functions of x.

We will first take the case where P is a constant, the equation being

$$\frac{dy}{dx} - ay = Q \quad \text{.....................} \quad (2),$$

as this will be of special use to us later.

If $Q = 0$, we have, by separation of the variables,

$$\frac{1}{y}\frac{dy}{dx} - a = 0 \quad \text{.....................} \quad (3),$$

whence $$\log y - ax = A, \quad ye^{-ax} = C,$$

or $$y = Ce^{ax} \qquad (4),$$

where $C, = e^A$, is arbitrary.

It appears from the above process that the factor e^{-ax} renders the left-hand side of (2) an exact differential coefficient. This gives the key to the solution in the general case where $Q \neq 0$. Thus (2) is equivalent to

$$\frac{d}{dx}(e^{-ax}y) = Qe^{-ax} \qquad (5),$$

whence $$e^{-ax}y = \int Qe^{-ax}dx + C,$$

or $$y = e^{ax}\int Qe^{-ax}dx + Ce^{ax} \qquad (6).$$

In accordance with a general usage (see Art. 185), the first term on the right-hand of (6) may be called the 'particular integral,' and the second the 'complementary function.'

The following cases are important:

1°. If $$Q = He^{\alpha x} \qquad (7),$$

we have $$\int Qe^{-ax}dx = H\int e^{(\alpha-a)x}dx = \frac{H}{\alpha - a}e^{(\alpha-a)x},$$

and $$y = \frac{H}{\alpha - a}e^{\alpha x} + Ce^{ax} \qquad (8).$$

That the first term on the right-hand is a particular integral of the proposed equation is verified at once by inspection.

2°. The result (8) needs correction when $\alpha = a$, or

$$Q = He^{ax} \qquad (9).$$

In this case we have

$$\int Qe^{-ax}dx = H\int dx = Hx,$$

and $$y = Hxe^{ax} + Ce^{ax} \qquad (10).$$

3°. If $$Q = Hx^n e^{ax} \qquad (11),$$

we have $$\int Qe^{-ax} = H\int x^n dx = \frac{Hx^{n+1}}{n+1},$$

and
$$y = \frac{Hx^{n+1}}{n+1} e^{ax} + Ce^{ax} \quad\text{...............} (12).$$

Ex. 1. If a particle be subject to a resistance varying as the velocity, and to some other force which is a given function of the time, its equation of motion is of the type

$$\frac{dv}{dt} + kv = f(t) \quad\text{........................}(13).$$

The integral of this is

$$v = Ce^{-kt} + e^{-kt} \int e^{kt} f(t)\, dt \quad\text{................}(14).$$

For example, if
$$f(t) = g,$$

a constant, we have
$$v = Ce^{-kt} + \frac{g}{k} \quad\text{........................}(15).$$

This might have been obtained more simply by writing the differential equation in the form

$$\frac{d}{dt}\left(v - \frac{g}{k}\right) + k\left(v - \frac{g}{k}\right) = 0 \quad\text{................}(16),$$

whence
$$v - \frac{g}{k} = Ce^{-kt} \quad\text{........................}(17).$$

As t increases, v tends asymptotically to the constant value g/k.

Ex. 2. If an electric current of strength x be flowing in a circuit of self-induction L and resistance R, and if E be the extraneous electromotive force in the circuit, we have the equation

$$L\frac{dx}{dt} + Rx = E \quad\text{.....................}(18).$$

If E be a constant, the solution of this is

$$x = \frac{E}{R} + Ce^{-\frac{R}{L}t} \quad\text{.......................}(19),$$

where C is arbitrary. The current therefore tends to the constant value E/R.

If, for example, we suppose that the circuit is completed at time $t = 0$, we have to determine C so that $x = 0$ for $t = 0$; this gives

$$x = \frac{E}{R} - \frac{E}{R} e^{-\frac{R}{L}t}. \quad\text{....................}(20).$$

The second term represents the 'extra current at make.'

Again, if
$$E = E_0 \cos(pt + \epsilon) \quad\text{.....................}(21),$$
we have
$$\frac{d}{dt}\left(xe^{\frac{R}{L}t}\right) = \frac{E_0}{L} e^{\frac{R}{L}t} \cos(pt + \epsilon),$$
whence, integrating, and dividing by $e^{\frac{R}{L}t}$, we find
$$x = Ce^{-\frac{R}{L}t} + \frac{E_0}{L} e^{-\frac{R}{L}t} \int e^{\frac{R}{L}t} \cos(pt + \epsilon)\, dt$$
$$= Ce^{-\frac{R}{L}t} + \frac{E_0}{R^2 + p^2L^2} \{R \cos(pt + \epsilon) + pL \sin(pt + \epsilon)\} \quad\text{...}(22);$$
see Art. 79 (14). Hence as t increases, the current settles down into the steady oscillation
$$x = \frac{E_0}{\surd(R^2 + p^2L^2)} \cos(pt + \epsilon - \epsilon_1) \quad\text{............}(23),$$
where
$$\epsilon_1 = \tan^{-1}\frac{pL}{R} \quad\text{........................}(24).$$

The effect of the self-induction (L) is therefore to diminish the amplitude of the current in the ratio
$$R/\surd(R^2 + p^2L^2),$$
and to retard its phase by ϵ_1.

177. General Linear Equation of the First Order.

We return to the general linear equation of the first order,
$$\frac{dy}{dx} + Py = Q \quad\text{......................}(1).$$
If $Q = 0$, we have
$$\frac{1}{y}\frac{dy}{dx} + P = 0 \quad\text{.......................}(2),$$
whence
$$\log y + \int P dx = A,$$
or
$$ye^{\int P dx} = C \quad\text{..........................}(3).$$
This shews that $e^{\int P dx}$ is an integrating factor of (1), viz. we have
$$e^{\int P dx}\left(\frac{dy}{dx} + Py\right) = \frac{d}{dx}\left(ye^{\int P dx}\right).$$

Hence (1) may be written

$$\frac{d}{dx}(ye^{\int P dx}) = Qe^{\int P dx} \quad\dots\dots\dots\dots\dots\dots (4).$$

Integrating, we find

$$ye^{\int P dx} = \int Qe^{\int P dx}\, dx + C \quad\dots\dots\dots\dots\dots (5).$$

Ex. 1. $$\frac{dy}{dx} + y \cot x = 2 \cos x \quad\dots\dots\dots\dots\dots\dots (6).$$

Here $P = \cot x, \quad \int P dx = \log \sin x, \quad e^{\int P dx} = \sin x.$

Hence, multiplying by $\sin x$,

$$\frac{d}{dx}(y \sin x) = 2 \sin x \cos x \quad\dots\dots\dots\dots\dots\dots (7),$$

$$y \sin x = \sin^2 x + C,$$

$$y = \sin x + \frac{C}{\sin x} \quad\dots\dots\dots\dots\dots\dots (8).$$

Ex. 2. $$(1 - x^2)\frac{dy}{dx} - xy = 1 \quad\dots\dots\dots\dots\dots\dots (9).$$

Dividing by $1 - x^2$, we have

$$\frac{dy}{dx} - \frac{x}{1 - x^2} y = \frac{1}{1 - x^2} \quad\dots\dots\dots\dots\dots\dots (10).$$

Here

$$P = -\frac{x}{1 - x^2}, \quad \int P dx = \tfrac{1}{2}\log(1 - x^2), \quad e^{\int P dx} = \sqrt{(1 - x^2)}.$$

Multiplying (10) by the integrating factor, we get

$$\sqrt{(1 - x^2)}\frac{dy}{dx} - \frac{x}{\sqrt{(1 - x^2)}} y = \frac{1}{\sqrt{(1 - x^2)}},$$

or $$\frac{d}{dx}\{\sqrt{(1 - x^2)}\, y\} = \frac{1}{\sqrt{(1 - x^2)}} \quad\dots\dots\dots\dots (11).$$

Hence, integrating,

$$\sqrt{(1 - x^2)}\, y = \sin^{-1} x + C,$$

or $$y = \frac{\sin^{-1} x}{\sqrt{(1 - x^2)}} + \frac{C}{\sqrt{(1 - x^2)}} \quad\dots\dots\dots\dots\dots (12).$$

The integrating factor will often suggest itself on inspection of the equation, without recourse to the above rule.

Ex. 3.
$$\frac{dy}{dx} + n\frac{y}{x} = x^m \text{ ....} (13).$$

The steps are
$$x^n \frac{dy}{dx} + nx^{n-1}y = x^{m+n},$$
$$x^n y = \frac{x^{m+n+1}}{m+n+1} + C,$$
$$y = \frac{x^{m+1}}{m+n+1} + Cx^{-n} \text{ ....} (14).$$

178. Orthogonal Trajectories.

Suppose that we have a singly-infinite family of curves
$$\phi(x, y, C) = 0 \text{ ....} (1),$$
where C is a variable parameter, and that it is required to determine the curves which cut these everywhere at right angles.

We first form the differential equation of the family, by differentiation of (1) with respect to x, and elimination of C. See Art. 170.

If two curves cut at right angles, and if ψ, ψ' be the angles which the tangents at the intersection make with the axis of x, we have $\psi - \psi' = \pm \frac{1}{2}\pi$, and therefore
$$\tan\psi = -\cot\psi'.$$
Hence the differential equation of one family is obtained from that of the other by writing
$$-1 \Big/ \frac{dy}{dx} \text{ for } \frac{dy}{dx}.$$

Otherwise: if dx, dy be the projections of an element of one of the curves (1) we have
$$\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = 0 \text{ ....} (2).$$

Hence, if dx, dy be the projections of an element of the orthogonal curve through the point (x, y), we have

$$\frac{dx}{\dfrac{\partial\phi}{\partial x}} = \frac{dy}{\dfrac{\partial\phi}{\partial y}} \quad\text{.........................}\quad (3).$$

The differential equation of the trajectories is then obtained by elimination of C between (1) and (3).

Ex. 1. To find the orthogonal trajectories of the rectangular hyperbolas

$$xy = C \quad\text{..............................}\quad (4).$$

Differentiating, we find

$$x\,dy + y\,dx = 0 \quad\text{.........................}\quad (5),$$

and therefore, for the trajectories,

$$x\,dx - y\,dy = 0 \quad\text{.........................}\quad (6),$$

whence
$$x^2 - y^2 = C' \quad\text{.......................}\quad (7).$$

This represents a system of rectangular hyperbolas whose axes coincide in direction with the asymptotes of the former system.

Ex. 2. To find the curves orthogonal to the circles

$$x^2 + y^2 + 2\mu y - k^2 = 0 \quad\text{.....................}\quad (8),$$

where μ is the variable parameter.

Differentiating, we have

$$x\,dx + (y + \mu)\,dy = 0,$$

and therefore, for the trajectory,

$$x\,dy - (y + \mu)\,dx = 0.$$

Eliminating μ between this and (8), we find

$$2xy\frac{dy}{dx} + (x^2 - y^2 - k^2) = 0 \quad\text{.................}\quad (9),$$

or
$$x\frac{d\,(y^2)}{dx} - y^2 = -x^2 + k^2 \quad\text{.................}\quad (10).$$

This is linear, with y^2 as the independent variable. The integrating factor, as found by the rule of Art. 177, or by inspection, is $1/x^2$. Introducing this we have

$$\frac{d}{dx}\left(\frac{y^2}{x}\right) = -1 + \frac{k^2}{x^2},$$

whence
$$\frac{y^2}{x} = -x - \frac{k^2}{x} + 2\lambda,$$

or
$$x^2 + y^2 - 2\lambda x + k^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

λ being arbitrary.

The original equation represents a system of coaxial circles, cutting the axis of x in the points $(\pm k, 0)$. The trajectories (11) consist of a second system of coaxial circles having these points as 'limiting points'; viz. if we put $\lambda = \pm k$ we get the point-circles
$$(x \mp k)^2 + y^2 = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (12);$$
see Fig. 148.

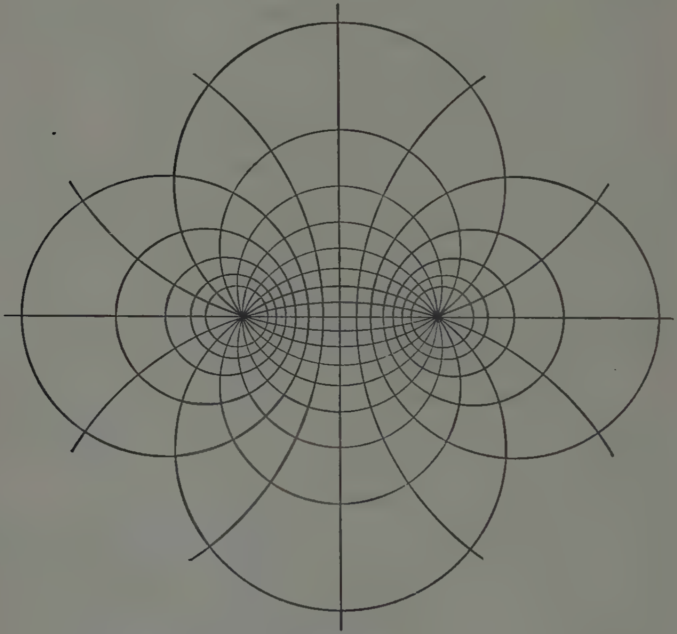

Fig. 148.

If the equation of the given family of curves be in polar coordinates, thus
$$f(r, \theta, C) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13),$$

and if ϕ, ϕ' denote the angles which the tangents to the original curve and to the trajectory make with the radius vector, we have in like manner

$$\tan\phi = -\cot\phi'.$$

Hence the differential equation of one system is obtained from that of the other by writing

$$-\frac{1}{r}\frac{dr}{d\theta} \quad \text{for} \quad \frac{rd\theta}{dr}.$$

Or, differentiating (13) we have

$$\frac{\partial f}{\partial r}dr + \frac{1}{r}\frac{\partial f}{\partial\theta}rd\theta = 0 \quad \ldots\ldots\ldots\ldots\ldots \text{(14)},$$

and therefore, for the trajectory,

$$\frac{\partial f}{\partial r}rd\theta - \frac{1}{r}\frac{\partial f}{\partial\theta}dr = 0 \ldots\ldots\ldots\ldots\ldots \text{(15)}.$$

The elimination of C between (13) and (15) leads to the differential equation of the required system.

Ex. 3. In the circles

$$r = c\cos\theta \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(16)},$$

which pass through the origin, and have their centres on the initial line, we have

$$\frac{dr}{r} = -\tan\theta\,d\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(17)},$$

and therefore, for the trajectory,

$$rd\theta = \tan\theta\,dr, \quad \text{or} \quad \frac{dr}{r} = \cot\theta\,d\theta \ldots\ldots\ldots\ldots \text{(18)}.$$

Integrating, we find $\log r = \log\sin\theta + \text{const.}$,

or

$$r = c'\sin\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(19)},$$

which represents another system of circles, passing through the origin, and touching the initial line.

EXAMPLES. XLIX.

1. Solve $\dfrac{dy}{dx} + y\tan x = \sec x.$ $[y = \sin x + C\cos x.]$

2. Solve $(1-x^2)\dfrac{dy}{dx} + xy = ax.$ $[y = a + C\sqrt{(1-x^2)}.]$

3. Solve $x\dfrac{dy}{dx}+x+y=0.$ $[x^2+2xy=C.]$

4. Solve $\dfrac{dy}{dx}+\dfrac{y}{x}=1.$ $[x^2-2xy=C.]$

5. Solve $\dfrac{dy}{dx}+2xy=1+2x^2.$ $[y=x+Ce^{-x^2}.]$

6. Shew that the equation

$$\frac{dy}{dx}+Py=Qy^n$$

is made linear by the substitution

$$y^{-(n-1)}=z.$$

(Bernoulli's equation.)

7. Solve $x\dfrac{dy}{dx}+y=y^2\log x.$ $\left[\dfrac{1}{y}=1+\log x+Cx.\right]$

8. Solve $\cos x\dfrac{dy}{dx}-y\sin x+y^2=0.$ $\left[\dfrac{1}{y}=\sin x+C\cos x.\right]$

9. If the two plates of a condenser of capacity C are connected by a wire of resistance R (and zero self-induction), the equation connecting the charge (q) with the electromotive force (E) is

$$E=R\frac{dq}{dt}+\frac{q}{C}.$$

Integrate this in the cases $E=0$, $E=\text{const.}$, $E=E_0\cos(pt+\epsilon)$.

10. Find the orthogonal trajectories of the straight lines

$y=Cx.$ [The circles $x^2+y^2=C.$]

11. Find the orthogonal trajectories of the curves

$a^{n-1}y=x^n.$ [The conics $x^2+ny^2=C.$]

12. Find the orthogonal trajectories of the circles

$$x^2+y^2=2cy.$$

[The circles $x^2+y^2=2c'x.$]

13. Find the orthogonal trajectories of the cardioids

$$r=a(1-\cos\theta).$$

[The cardioids $r=b(1+\cos\theta).$]

14. Prove that the orthogonal trajectories of the curves

$$r^m = a^m \cos m\theta$$

are the curves

$$r^m = b^m \sin m\theta.$$

Interpret the cases of $m = 1, -1, 2, -2, \frac{1}{2}, -\frac{1}{2}$, respectively.

15. Prove that the orthogonal trajectories of the curves

$$r^2 = A \cos \theta$$

are the curves

$$r = B \sin^2 \theta.$$

16. Prove that the differential equation of the confocal parabolas

$$y^2 = 4a(x + a),$$

is

$$yp^2 + 2xp - y = 0,$$

where $p = dy/dx$.

Shew that this coincides with the differential equation of the orthogonal curves; and interpret the result.

17. Prove that the differential equation of the confocal conics

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

is

$$xyp^2 + (x^2 - y^2 - a^2 + b^2)\,p - xy = 0.$$

Shew that this coincides with the differential equation of the orthogonal curves, and interpret the result.

18. A system of rectangular hyperbolas pass through the fixed points $(\pm a, 0)$ and have the origin as centre; prove that their orthogonal trajectories are the Cassini's ovals

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2) + C.$$

19. If in bipolar coordinates (Art. 149) the equation of a family of curves be

$$f(r, r') = C,$$

the differential equation of the orthogonal trajectories is

$$r\frac{\partial f}{\partial r}\,d\theta = r'\frac{\partial f}{\partial r'}\,d\theta'.$$

Hence shew that the orthogonal trajectories of the circles

$$r/r' = C,$$

are the circles

$$\theta + \theta' = C.$$

20. Also that the orthogonal trajectories of the Cassini's ovals

$$rr' = C,$$

are the rectangular hyperbolas

$$\theta - \theta' = C.$$

21. Also that the orthogonal trajectories of the equipotential curves

$$\frac{1}{r} - \frac{1}{r'} = C,$$

are the magnetic curves

$$\cos\theta + \cos\theta' = C.$$

179. Equations of Degree higher than the First.

The general type of an equation of the first order and nth degree is

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n-1} p + P_n = 0 \ldots (1),$$

where

$$p = \frac{dy}{dx} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

and P_1, P_2, ... P_n are given functions of x and y. It is usually implied that these functions are algebraic, and rational.

The equation (1), being of the nth degree in p, indicates that n branches of the primitive curves go through any assigned point in the plane xy. Some of these branches may of course be imaginary, and for some ranges of x and y all may be imaginary. There may also be a real locus of points at which two of the values of p coincide; this locus is of special importance in the higher development of the subject.

For example, in the equation of the *second* degree,

$$p^2 + Pp + Q = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$

the values of p will be real and distinct, coincident, or imaginary, according as $P^2 \gtreqless 4Q$. And the locus of points at which the two values of p coincide is the curve $P^2 = 4Q$.

If the left-hand side of (1), considered as a function of p, can be resolved into linear factors, thus

$$(p - p_1)(p - p_2) \ldots (p - p_n) = 0 \ldots\ldots\ldots\ldots (4),$$

where $p_1, p_2, \ldots p_n$ are known functions of x and y, the complete solution will consist of the aggregate of the solutions of the several equations

$$\frac{dy}{dx}=p_1, \quad \frac{dy}{dx}=p_2, \ldots \frac{dy}{dx}=p_n \quad \text{..........} (5).$$

Ex.

$$xyp^2-(x^2-y^2)p-xy=0 \quad \text{..........} (6).$$

This is equivalent to

$$(xp+y)(yp-x)=0 \quad \text{..........} (7);$$

and the solutions of

$$xp+y=0, \quad yp-x=0,$$

are, respectively,

$$xy=C, \quad x^2-y^2=C \quad \text{..........} (8).$$

The product of the two values of p given by (6) is -1. This shews *à priori* that the two branches of the primitive curves which pass through any point (x, y) will be at right angles to one another. Cf. Art. 178, Ex. 1.

180. Clairaut's form.

When the equation (1) of Art. 179 cannot be conveniently resolved into its linear factors, we may in certain cases have recourse to other methods. These are for the most part of somewhat limited utility, and are accordingly passed over here; but an exception may be made in favour of Clairaut's form, which is very simple in theory, and moreover often presents itself in questions where a curve is defined by some property of the tangent.

If we write p for dy/dx, the form in question is

$$y=xp+f(p) \quad \text{..........} (1).$$

It was proved in Art. 53 that the intercepts (α, β) made by the tangent to a curve on the axes of x and y are given by

$$\alpha=(xp-y)/p, \quad \beta=y-xp \quad \text{..........} (2),$$

respectively. Hence any equation of the form (1) expresses a relation between either intercept and the direction of the tangent, or (again) between the two intercepts*. Now it is

---

* Viz. the equation is equivalent to

$$\beta=f(-\beta/\alpha), \text{ or } \phi(\alpha, \beta)=0.$$

evident that this relation is satisfied by any *straight line* whose intercepts have the given relation. Along any such straight line we have

$$p = C \quad\text{.........................(3)},$$

and we thus get the solution

$$y = Cx + f(C) \quad\text{.....................(4)},$$

involving an arbitrary constant C.

But the equation will also be satisfied by the curve which has the family (4) of straight lines as its tangents; in other words, by the *envelope* of this family. This envelope is found by expressing that (4), considered as an equation in C, has a pair of equal roots, *i.e.* by eliminating C between (4) and

$$x + f'(C) = 0 \quad\text{...................(5)};$$

see Art. 156.

Ex. To find the curve whose pedal with respect to the point $(a, 0)$ as pole is the straight line $x = 0$.

The expression of this property is

$$a = p\beta,$$

where β is the intercept on the axis of y, or

$$y = xp + \frac{a}{p} \quad\text{.........................(6)}.$$

This is satisfied by any one of the family of straight lines

$$y = Cx + \frac{a}{C} \quad\text{.........................(7)},$$

and also by their envelope $y^2 = 4ax$ ....(8); see Art. 157, Ex. 2.

The usual method of deducing the above solutions is to differentiate (1) with respect to x; thus

$$p = \frac{dy}{dx} = p + \{x + f'(p)\}\frac{dp}{dx},$$

whence

$$\{x + f'(p)\}\frac{dp}{dx} = 0 \quad\text{.................(9)}.$$

This requires, either that

$$\frac{dp}{dx} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(10)},$$

or that

$$x + f'(p) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(11)}.$$

The former result makes $p = C$, and

$$y = Cx + f(C) \ldots\ldots\ldots\ldots\ldots\ldots \text{(12)}.$$

The alternative result (11), combined with (1), leads, on elimination of p, to a particular relation between x and y. Since the result of eliminating p between (1) and (11) must be the same as that of eliminating C between (4) and (5), we identify this second solution with the envelope aforesaid.

The solution (12), involving an arbitrary constant C, is called the 'complete primitive.' The second, or envelope-solution, is not included in the complete primitive, *i.e.* it cannot be derived from it by giving a particular value to C. It is therefore called a 'singular solution*.'

EXAMPLES. L.

1. Solve $\left(\frac{dy}{dx}\right)^2 - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta = 0.$

[$y = \alpha x + C, \quad y = \beta x + C.$]

2. Solve $\left(\frac{dy}{dx}\right)^2 = \sin^2 x.$ [$y = C \pm \cos x.$]

3. Solve $\left(\frac{dy}{dx}\right)^2 = m^2 y^2.$ [$y = Ce^{\pm mx}.$]

4. Solve $y^2\left(\frac{dy}{dx}\right)^2 = 4a^2.$ [$y^2 = C \pm 4ax.$]

5. Solve $x\left(\frac{dy}{dx}\right)^2 = a.$ [$y = C \pm 2\sqrt{(ax)}.$]

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* The general theory of singular solutions of equations of degree higher than the first must be sought for in books specially devoted to the subject of Differential Equations. It is closely related to, but not altogether co-extensive with, the theory of envelopes.

6. Solve $(1-x^2)\left(\frac{dy}{dx}\right)^2=1.$ $[y=C\pm\sin^{-1}x.]$

7. Solve $\frac{dy}{dx}\left(\frac{dy}{dx}+y\right)=x(x+y).$

$[y=\frac{1}{2}x^2+C, \quad y=1-x+Ce^{-x}.]$

8. Solve $\frac{dy}{dx}\left(\frac{dy}{dx}+x\right)=(x+y)\,y.$

$[y=Ce^{x}, \quad y=1-x+Ce^{-x}.]$

9. Solve $x\left(\frac{dy}{dx}\right)^2-2y\frac{dy}{dx}-x=0.$ $[x^2=2Cy+C^2.]$

10. Solve $y\left(\frac{dy}{dx}\right)^2+2x\frac{dy}{dx}-y=0.$

$[\sqrt{(x^2+y^2)}=C\pm x.]$

11. Find the curve such that the product of the intercepts made by the tangent on the coordinate axes is constant $(=k^2)$.

[The hyperbola $4xy=k^2$.]

12. Find the curve such that the perpendicular from the origin on any tangent is equal to a. [The circle $x^2+y^2=a^2$.]

13. Solve $y=xp+\sqrt{(b^2+a^2p^2)}.$

$\left[\text{Singular solution: } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.\right]$

14. Find the curve such that the product of the perpendiculars from the points $(\pm c, 0)$ on any tangent is equal to b^2.

$\left[\text{The conics } \frac{x^2}{b^2+c^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{c^2-b^2}-\frac{y^2}{b^2}=1.\right]$

15. Find the curve such that the tangent intercepts on the perpendiculars to the axis of x at the points $(\pm a, 0)$ lengths whose product is b^2.

$\left[\text{The conics } \frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1.\right]$

16. Solve $y=xp+ap(1-p).$

[Singular solution: $(x+a)^2=4ay$.]

17. Solve $(x-a)p^2+(x-y)p-y=0.$

[Singular solution: $(x+y)^2=4ay.$]

18. Find the curve such that the sum of the intercepts made by the tangent on the coordinate axes is equal to a.

[The parabola $(x-y)^2-2a(x+y)+a^2=0.$]

19. Shew that any differential equation of the type

$$x+y\frac{dy}{dx}=f\left(\frac{dy}{dx}\right)$$

represents a system of parallel curves.

20. Shew that any differential equation of the type

$$f\left(x, y, p-\frac{1}{p}\right)=0$$

represents two systems of orthogonal curves.

CHAPTER XII.

DIFFERENTIAL EQUATIONS OF THE SECOND ORDER.

181. Equations of the Type $d^2y/dx^2 = f(x)$.

This chapter is devoted principally to differential equations of the second order, and especially to such types as are of most frequent occurrence in the geometrical and physical applications of the Calculus. Occasionally, the methods will admit of extension to equations of higher order.

We begin by the consideration of a few special types, and afterwards proceed to the study of the linear equation, and in particular of the linear equation with constant coefficients.

We take, first, the type

$$\frac{d^2y}{dx^2} = f(x) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

This requires merely two ordinary integrations with respect to x; thus

$$\frac{dy}{dx} = \int f(x)\,dx + A,$$

$$y = \int\{\int f(x)\,dx\}\,dx + Ax + B \quad\ldots\ldots\ldots\ldots(2),$$

where the constants A, B are arbitrary.

Ex. 1. The dynamical equation

$$\frac{d^2x}{dt^2} = f(t) \quad \ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which determines the motion of a particle in a straight line under a force which is a given function of the time, is of the above type, with merely a difference of notation.

In the case of a particle subject to a constant acceleration g we have

$$\frac{d^2x}{dt^2} = g \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

whence

$$\frac{dx}{dt} = gt + A,$$

$$x = \tfrac{1}{2}gt^2 + At + B \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

Again, if

$$\frac{d^2x}{dt^2} = f \sin nt \quad \ldots\ldots\ldots\ldots\ldots\ldots(6),$$

the force varying as a simple-harmonic function of the time, we have

$$\frac{dx}{dt} = -\frac{f}{n}\cos nt + A,$$

$$x = -\frac{f}{n^2}\sin nt + At + B \quad \ldots\ldots\ldots\ldots(7).$$

The constants A, B which occur in these problems may be adjusted so that at any chosen instant the particle shall be in a given position and have a given velocity.

Ex. 2. To solve the equation

$$\mathfrak{B}\frac{d^2y}{dx^2} - W(l - x) = 0 \ldots\ldots\ldots\ldots\ldots\ldots(8),$$

subject to the conditions that $y = 0$ and $dy/dx = 0$ for $x = 0$. This is the problem of determining the flexure of a bar which is clamped in a horizontal position at one end ($x = 0$) and supports a given weight (W) at the other end ($x = l$).

Two successive integrations of (8) give

$$\mathfrak{B}\frac{dy}{dx} = W(lx - \tfrac{1}{2}x^2) + A,$$

$$\mathfrak{B}y = W(\tfrac{1}{2}lx^2 - \tfrac{1}{6}x^3) + Ax + B \quad \ldots\ldots\ldots(9),$$

where A, B are arbitrary. The terminal conditions require that $A=0$, $B=0$, whence

$$y=\tfrac{1}{2}\frac{W}{\mathfrak{B}}x^2\left(l-\tfrac{1}{3}x\right)\;\ldots\ldots\ldots\ldots\ldots\ldots(10).$$

182. Equations of the Type $d^2y/dx^2=f(y)$.

If the equation be of the type

$$\frac{d^2y}{dx^2}=f(y)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

a first integral may be obtained in two ways.

In one of these we multiply both sides by dy/dx, and then integrate with respect to x; thus

$$\frac{dy}{dx}\frac{d^2y}{dx^2}=f(y)\frac{dy}{dx},$$

$$\tfrac{1}{2}\left(\frac{dy}{dx}\right)^2=\int f(y)\frac{dy}{dx}\,dx+A=\int f(y)\,dy+A\;\ldots\ldots\;(2).$$

The second method is to introduce a special symbol (p) for dy/dx. Since this makes

$$\frac{d^2y}{dx^2}=\frac{dp}{dx}=\frac{dp}{dy}\frac{dy}{dx}=p\frac{dp}{dy}\;\ldots\ldots\ldots\ldots\;(3),$$

we have, in place of (1),

$$p\frac{dp}{dy}=f(y)\;\ldots\ldots\ldots\ldots\ldots\ldots\;(4),$$

which may be regarded as an equation of the first order, with p as dependent, and y as independent, variable. Integrating (3) with respect to y, we have

$$\tfrac{1}{2}p^2=\int f(y)\,dy+A\;\ldots\ldots\ldots\ldots\ldots\;(5),$$

which is equivalent to (2).

To complete the solution, we write (2) in the form

$$\frac{dy}{\sqrt{\{2\int f(y)\,dy+2A\}}}=\pm\,dx\ldots\ldots\ldots\ldots\ldots(6).$$

The variables are here separated (Art. 173); but on account mainly of the occurrence of the radical the further integration is often impracticable, even with comparatively simple forms of the function $f(y)$.

A very important case is where $f(y)$ is a linear function of y, so that the equation takes the shape

$$\frac{d^2y}{dx^2} + ay = b \quad\text{.....................}\quad (7).$$

By a change of dependent variable, writing $y_1 + b/a$ for y, and afterwards omitting the suffix, this is reduced to the somewhat simpler form

$$\frac{d^2y}{dx^2} + ay = 0 \quad\text{.....................}\quad (8).$$

The first integral of this is

$$\left(\frac{dy}{dx}\right)^2 + ay^2 = C \quad\text{.......................}\quad (9).$$

If a be positive, we may write

$$m = \sqrt{a}, \quad C = m^2\alpha^2 \quad\text{.................}\quad (10),$$

it being evident that, if we are concerned solely with real quantities, C must be positive. Thus

$$\frac{dy}{\sqrt{(\alpha^2 - y^2)}} = \pm\, m dx \quad\text{..............}\quad (11),$$

whence

$$\cos^{-1}\frac{y}{\alpha} = \pm(mx + \epsilon),$$

or

$$y = \alpha\cos(mx + \epsilon) \quad\text{.................}\quad (12).$$

This is the complete solution of (8), and involves the two arbitrary constants α, ϵ. If we put

$$A = \alpha\cos\epsilon, \quad B = -\alpha\sin\epsilon \quad\text{............}\quad (13),$$

we obtain the equivalent form

$$y = A\cos mx + B\sin mx \quad\text{..............}\quad (14).$$

These results are exceedingly important, and should be remembered.

The case where a is negative, $= -m^2$, say, can be treated in a similar manner, and we should find, as the complete solution

$$y = A\cosh mx + B\sinh mx \text{ .... } (15),$$

where $m = \sqrt{(-a)}$. A simpler method of treating this case will however be given later.

The type (1) is of very great importance in Dynamics. Thus, the equation of rectilinear motion of a particle subject to a force which is a given function of the distance from the origin is of the form

$$\frac{d^2x}{dt^2} = f(x) \text{ .... } (16),$$

which is identical with (1), if regard be had to the difference of notation.

The first method of integration consists in multiplying both sides by dx/dt, thus

$$\frac{dx}{dt}\frac{d^2x}{dt^2} = f(x)\frac{dx}{dt},$$

and integrating both sides with respect to t. In this way we obtain

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = \int f(x)\frac{dx}{dt}\,dt + C = \int f(x)\,dx + C \text{......} (17),$$

which is the 'equation of energy.'

The second method consists in writing v for dx/dt, and therefore $v\,dv/dx$ for d^2x/dt^2; cf. Art. 39. Thus

$$v\frac{dv}{dx} = f(x).$$

Hence, integrating with respect to x, we have

$$\tfrac{1}{2}v^2 = \int f(x)\,dx + C \text{ .... } (18),$$

in agreement with (17).

Ex. 1. If a particle be attracted to the origin with a force varying as the distance, the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu x \text{ .... } (19).$$

This is of the special type (8), and the solution is

$$x = a\cos(\sqrt{\mu}t + \epsilon) \text{ .... } (20).$$

This represents a 'simple-harmonic' motion. The values of x and dx/dt both recur whenever $\sqrt{\mu} t$ increases by 2π; the period of oscillation is therefore $2\pi/\sqrt{\mu}$. The arbitrary constants a and ϵ are in this problem known as the 'amplitude' and the 'epoch,' respectively.

The equation of motion of *any* 'conservative' dynamical system having one degree of freedom, when slightly disturbed from a position of stable equilibrium, is also of the type (19). For example, the accurate equation of motion of a pendulum is

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta \qquad\text{(21)},$$

where g is the acceleration of gravity, and l is a certain length depending on the structure of the pendulum. In the case of a 'simple' pendulum l is the length of the string. If the extreme angular deviation from the equilibrium position be small, we may write θ for $\sin\theta$, thus

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \qquad\text{(22)}.$$

The solution of this equation is

$$\theta = a\cos\left(\sqrt{\frac{g}{l}}\,.\,t + \epsilon\right) \qquad\text{(23)},$$

and the period is therefore $2\pi\sqrt{(l/g)}$.

The accurate equation (21) can be integrated once by the method above explained; we thus find

$$\tfrac{1}{2}l\left(\frac{d\theta}{dt}\right)^2 = g\cos\theta + C \qquad\text{(24)},$$

but the second integration cannot be effected (except in the particular case of $C = g$) without the introduction of elliptic functions.

Ex. 2. If a particle move in a straight line under an attraction varying as the inverse square of the distance from the origin, we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \qquad\text{(25)},$$

whence, as in Art. 173, Ex. 3,

$$\tfrac{1}{2}\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x} + C \qquad\text{(26)}.$$

If the particle start from rest at the distance a, we have $C = -\mu/a$, and

$$\frac{dx}{dt} = -\left(\frac{2\mu}{a}\right)^{\frac{1}{2}} \cdot \left(\frac{a-x}{x}\right)^{\frac{1}{2}} \quad\text{..................(27)},$$

the *minus* sign being taken since the velocity is towards the origin. The second integration is facilitated by the substitution

$$x = a\cos^2\theta \quad\text{........................(28)}.$$

Separating the variables, we find

$$(1+\cos 2\theta)\,d\theta = \left(\frac{2\mu}{a^3}\right)^{\frac{1}{2}} dt \quad\text{...............(29)},$$

$$\theta + \tfrac{1}{2}\sin 2\theta = \left(\frac{2\mu}{a^3}\right)^{\frac{1}{2}} t + C \quad\text{............(30)}.$$

As x diminishes from a to 0, θ increases from 0 to $\frac{1}{2}\pi$. Hence the time (t_1) of falling from rest at the distance a into the centre of force is given by

$$t_1 = \frac{\pi}{2\sqrt{2}}\,\frac{a^{\frac{3}{2}}}{\sqrt{\mu}} \quad\text{.......................(31)}.$$

The time (t_0) in which a particle would describe a circular orbit of radius a about the same centre of force is

$$t_0 = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{\mu}} \quad\text{........................(32)}.$$

Hence

$$\frac{t_1}{t_0} = \frac{1}{4\sqrt{2}} = \cdot 177 \quad\text{.................(33)}.$$

We infer that if the orbital motion of a planet (or of a planet's satellite) were suddenly arrested, the planet (or the satellite) would fall into the sun (or into the primary) in about ·177 of its period of revolution.

183. Equations involving only the First and Second Derivatives.

If the equation be of the type

$$\phi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}\right) = 0 \quad\text{....................(1)},$$

i.e. the variables x, y do not appear (explicitly), then, writing p for dy/dx, we have

$$\phi\left(\frac{dp}{dx}, p\right) = 0 \quad\text{....................(2)},$$

which is an equation of the first order with p as dependent variable.

Ex. 1. To find the curves whose radius of curvature is constant ($=a$, say).

By Art. 152 we have

$$\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\pm\frac{1}{a} \text{ ....(3),}$$

or

$$\frac{dp}{(1+p^2)^{\frac{3}{2}}}=\pm\frac{dx}{a}\text{......................(4).}$$

Integrating this we have (Art. 76 (13))

$$\frac{p}{(1+p^2)^{\frac{1}{2}}}=\pm\frac{x-\alpha}{a} \text{ ....(5),}$$

where α is an arbitrary constant. This gives

$$\frac{dy}{dx}=p=\pm\frac{x-\alpha}{\sqrt{\{a^2-(x-\alpha)^2\}}} \text{ ....(6),}$$

whence

$$y-\beta=\pm\sqrt{\{a^2-(x-\alpha)^2\}}\text{..................(7),}$$

if β be the arbitrary constant introduced by this last integration. The result may be written

$$(x-\alpha)^2+(y-\beta)^2=a^2\text{.....................(8),}$$

and so represents a family of circles of radius a.

This investigation is given merely as an example of the general method; the problem itself can be solved more easily in other ways.

Ex. 2. To determine the rectilinear motion of a particle subject to a force which is a given function of the velocity.

The equation of motion is of the form

$$\frac{d^2x}{dt^2}=f\left(\frac{dx}{dt}\right) \text{ ....(9),}$$

which evidently comes under the type (1). Writing v for dx/dt, we have

$$\frac{dv}{dt}=f(v), \qquad \frac{dv}{f(v)}=dt, \qquad \int\frac{dv}{f(v)}=t+C\text{......(10).}$$

For example, if the particle be subject solely to a resistance varying as the velocity, we have

$$\frac{dv}{dt} = -kv \quad \text{......................(11)},$$

whence, by Art. 176,

$$\frac{dx}{dt} = v = Ae^{-kt}, \qquad x = -\frac{1}{k}Ae^{-kt} + B \quad \text{.........(12)}.$$

Hence, whatever the circumstances of projection, x will approach asymptotically, as t increases, to a constant value (B).

Again, if the resistance vary as the square of the velocity, we have

$$\frac{dv}{dt} = -kv^2, \qquad -\frac{dv}{v^2} = kdt, \qquad \frac{1}{v} = kt + A \text{......... (13)}.$$

Hence $$\frac{dx}{dt} = v = \frac{1}{kt + A}, \text{ and } x = \frac{1}{k}\log(kt + A) + B \text{......(14)}.$$

We see that, although v tends asymptotically to zero, there is now no limit to the space described.

The equation (1) may also be reduced to an equation of the first order, with y as independent variable, by writing as in Art. 182

$$p\frac{dp}{dy} \text{ for } \frac{d^2y}{dx^2};$$

thus $$\phi\left(p\frac{dp}{dy},\ p\right) = 0 \text{.................(15)}.$$

For example, in the dynamical example just given, the equation (9) may be replaced by

$$v\frac{dv}{dx} = f(v) \text{.......................(16)}.$$

Thus, in the case of resistance varying as the velocity, we get

$$\frac{dv}{dx} = -k, \qquad v = -kx + C \quad \text{..............(17)}.$$

Hence $$\frac{dx}{dt} + kx = C \quad \text{.......................(18)},$$

and therefore, by Art. 176,

$$x = \frac{C}{k} + De^{-kt} \text{.......................(19)},$$

where C, D are arbitrary constants. This agrees with (12).

Again, if the resistance vary as the square of the velocity, we have

$$\frac{dv}{dx} = -kv, \quad v = Ce^{-kx} \text{ ....(20).}$$

Hence
$$e^{kx}\frac{dx}{dt} = C, \quad \frac{1}{k}e^{kx} = Ct + D \text{ ....(21)}$$

or
$$kx = \log(kCt + kD) \text{ ....(22),}$$

a form not really distinct from (14), as may be verified by putting

$$A = kD/C, \quad kB = \log C.$$

184. Equations with one Variable absent.

1°. If the *dependent* variable do not appear explicitly, the equation being of the type

$$\phi\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, x\right) = 0 \text{ ....(1),}$$

then, writing p for dy/dx, we have an equation of the first order in p; viz.

$$\phi\left(p, \frac{dp}{dx}, x\right) = 0 \text{ .... (2).}$$

If the solution of this be put in the form

$$p = f(x, A) \text{ .... (3),}$$

where A is the arbitrary constant, a second integration gives

$$y = \int f(x, A)\,dx + B \text{ .... (4).}$$

That one of the arbitrary constants would occur as an addition to y might have been anticipated *à priori*, since the equation (1) is unaltered when we write $y + C$ for y.

Ex. 1.
$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0 \text{ .... (5).}$$

Writing this in the form

$$\frac{1}{p}\frac{dp}{dx} = \frac{x}{1 - x^2},$$

we find
$$\log p = -\tfrac{1}{2}\log(1 - x^2) + \text{const.},$$

or
$$\frac{dy}{dx} = p = \frac{A}{\sqrt{(1 - x^2)}} \text{ ....(6).}$$

Hence
$$y = A\sin^{-1}x + B \text{ ....(7).}$$

Ex. 2. In the theory of the potential we meet with the equation

$$\frac{d^2V}{dr^2}+\frac{2}{r}\frac{dV}{dr}=0 \quad\text{.........................(8).}$$

Regarding dV/dr as the dependent variable, we have

$$\frac{\dfrac{d^2V}{dr^2}}{\dfrac{dV}{dr}}+\frac{2}{r}=0 \quad\text{........................(9),}$$

whence $$\log\frac{dV}{dr}+2\log r=\text{const.},$$

or $$\frac{dV}{dr}=\frac{A}{r^2}\quad\text{..........................(10).}$$

Integrating again, we find

$$V=-\frac{A}{r}+B \quad\text{.......................(11).}$$

2°. If the *independent* variable do not appear explicitly, the type being

$$\phi\left(\frac{d^2y}{dx^2},\ \frac{dy}{dx},\ y\right)=0 \quad\text{.............. (12),}$$

we write as in Art. 182 (3)

$$\frac{dy}{dx}=p,\quad \frac{d^2y}{dx^2}=p\frac{dp}{dy}\quad\text{.................(13),}$$

and obtain $$\phi\left(p\frac{dp}{dy},\ p,\ y\right)=0 \quad\text{.............. (14),}$$

an equation of the first order between p and y.

If the solution of this can be put in the form

$$p=f(y,\ A) \quad\text{..................... (15),}$$

the next integration gives

$$\int\frac{dy}{f(y,\ A)}=x+B\quad\text{.................(16).}$$

Here, again, it might have been anticipated from the form of the given equation (12) that one of the arbitrary constants would consist in an addition to x.

Ex. 3. To find the curves in which the radius of curvature is equal to the normal, but lies on the opposite side of the curve.

Referring to Arts. 53, 152, we see that the expression of the above condition is

$$\frac{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\dfrac{d^2y}{dx^2}} = y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \qquad \text{............(17).}$$

Simplifying, and making the substitutions (13), we find

$$\frac{p}{1+p^2}\frac{dp}{dy} = \frac{1}{y} \qquad \text{.......................(18).}$$

Hence $\qquad \frac{1}{2}\log(1+p^2) = \log y + \text{const.}, \qquad 1+p^2 = \frac{y^2}{c^2}$ ...(19),

where c^2 is written for the arbitrary constant, which must evidently be positive. This gives

$$\frac{dy}{dx} = p = \pm\frac{\sqrt{(y^2-c^2)}}{c} \qquad \text{..................(20).}$$

Separating the variables, we have

$$\frac{dy}{\sqrt{(y^2-c^2)}} = \pm\frac{dx}{c}, \qquad \cosh^{-1}\frac{y}{c} = \pm\frac{(x-a)}{c} \qquad \text{......(21),}$$

where a is the second arbitrary constant. Hence, finally,

$$y = c\cosh\frac{x-a}{c} \qquad \text{.....................(22),}$$

a family of catenaries. Cf. Art. 151, Ex. 1.

185. Linear Equation of the Second Order.

A linear equation of the nth order is one which involves the dependent variable and its first n derivatives in the first degree only, without products. Thus, the general linear equation of the second order may be written

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = V \qquad \text{..................(1),}$$

where P, Q, V are given functions of x.

There are several important properties common to all linear equations. We give the proofs for the equation of the second order, but the generalization will be evident.

1°. The complete solution of (1) may be written

$$y = u + w \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

where w is any function whatever which satisfies (1) as it stands, and u is the general solution of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which differs from (1) by the absence of the right-hand member.

For, assuming that $y = u + w$, where w satisfies (1), and u is to be determined, we find, on substitution in (1),

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu + \frac{d^2w}{dx^2} + P\frac{dw}{dx} + Qw = V,$$

or, since

$$\frac{d^2w}{dx^2} + P\frac{dw}{dx} + Qw = V \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

by hypothesis,

$$\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(5);$$

i.e., the function u must satisfy (3).

The two parts which make up the general solution of (1), viz. w and u, are called the 'particular integral,' and the 'complementary function,' respectively. It is to be observed that the particular integral may be any solution whatever of the original equation; the simpler it is, the better. The complementary function must be the most general solution of (3), and will involve two arbitrary constants.

2°. If u_1, u_2 be any two solutions of (3), the equation will also be satisfied by

$$y = C_1u_1 + C_2u_2 \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

where C_1, C_2 are arbitrary constants. This is easily verified by substitution.

Hence if the functions u_1, u_2 are 'independent,' *i.e.* one is not merely a constant multiple of the other, the formula (6) gives a solution of (3) involving two arbitrary constants.

3°. If a particular integral (v) of the equation (3) be known, the complete solution of (1) is reduced by the substitution

$$y = zv \quad\text{.........................}(7),$$

to the integration of an equation of the first order in dz/dx. For (1) becomes

$$v\frac{d^2z}{dx^2} + \left(2\frac{dv}{dx} + Pv\right)\frac{dz}{dx} + \left(\frac{d^2v}{dx^2} + P\frac{dv}{dx} + Qv\right)z = V,$$

which reduces, in virtue of the hypothesis, to

$$v\frac{d^2z}{dx^2} + \left(2\frac{dv}{dx} + Pv\right)\frac{dz}{dx} = V \quad\text{...........}\ (8).$$

This is linear, of the first order, with dz/dx as the dependent variable.

In particular, if $V = 0$, we have

$$\frac{\dfrac{d^2z}{dx^2}}{\dfrac{dz}{dx}} + \frac{2}{v}\frac{dv}{dx} + P = 0 \quad\text{.................}(9),$$

whence $$\log\frac{dz}{dx} + 2\log v + \int Pdx = \text{const.},$$

or $$\frac{dz}{dx} = \frac{A}{v^2}e^{-\int Pdx} \quad\text{....................}(10).$$

Hence $$z = A\int\frac{e^{-\int Pdx}}{v^2}\,dx + B \quad\text{..............}(11).$$

The complete solution of (3) is therefore

$$y = Av\int\frac{e^{-\int Pdx}}{v^2}\,dx + Bv \quad\text{..............}(12).$$

We add a few examples of the integration of linear equations, by various artifices. The method of integration by series will be noticed in the next Chapter.

Ex. 1. In the theory of Sound, and in other branches of Mathematical Physics, we meet with the equation

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} + k^2\phi = 0 \quad\text{................}(13).$$

If we multiply by r, this is seen to be equivalent to

$$\frac{d^2(r\phi)}{dr^2} + k^2(r\phi) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(14).$$

Hence, by Art. 182,

$$r\phi = A\cos kr + B\sin kr,$$

or

$$\phi = \frac{A\cos kr + B\sin kr}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots(15).$$

Ex. 2. $$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \quad \ldots\ldots\ldots\ldots\ldots(16).$$

A particular solution is obviously $y = x$. We therefore put

$$y = xz \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17),$$

which leads to $$x(1-x^2)\frac{d^2z}{dx^2} + (2-3x^2)\frac{dz}{dx} = 0 \quad \ldots\ldots\ldots\ldots(18).$$

Separating the variables, we have

$$\frac{\dfrac{d^2z}{dx^2}}{\dfrac{dz}{dx}} + \frac{2}{x} - \frac{x}{1-x^2} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

whence $$\frac{dz}{dx} = \frac{A}{x^2\sqrt{(1-x^2)}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(20),$$

$$z = -A\frac{\sqrt{(1-x^2)}}{x} + B \quad \ldots\ldots\ldots\ldots\ldots\ldots(21).$$

The complete solution of (16) is therefore

$$y = A\sqrt{(1-x^2)} + Bx \quad \ldots\ldots\ldots\ldots\ldots\ldots(22).$$

Ex. 3. $$(1+x^2)\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = 0 \ldots\ldots\ldots\ldots\ldots(23).$$

This happens to be an 'exact equation,' *i.e.* the left-hand side is the exact differential coefficient of a function of x, y, and dy/dx, for it may be written

$$\left\{(1+x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}\right\} + \left\{x\frac{dy}{dx} + y\right\} = 0.$$

The integral is $$(1+x^2)\frac{dy}{dx} + xy = A \quad \ldots\ldots\ldots\ldots\ldots\ldots(24).$$

This is linear, of the first order, and the integrating factor is seen to be $1/\sqrt{(1+x^2)}$. We thus find

$$\frac{d}{dx}\{\sqrt{(1+x^2)} \,.\, y\} = \frac{A}{\sqrt{(1+x^2)}},$$

$$\sqrt{(1+x^2)} \,.\, y = A \sinh^{-1} x + B \quad \text{..........} (25).$$

EXAMPLES. LI.

1. $$x\frac{d^2y}{dx^2} = 1. \qquad [y = x \log x + Ax + B.]$$

2. $$\frac{d^2y}{dx^2} = xe^x. \qquad [y = (x-2)e^x + Ax + B.]$$

3. $$x^2\frac{d^2y}{dx^2} = a. \qquad \left[y = a \log \frac{a}{x} + Ax + B.\right]$$

4. The differential equation for the deflection of a horizontal beam subject only to its weight and to the pressures of its supports is

$$\mathfrak{B}\frac{d^4y}{dx^4} = w,$$

where w is the weight per unit length. Integrate this, on the supposition that w is constant, and determine the constants so that $y=0$, $d^2y/dx^2=0$ both for $x=0$ and for $x=l$. (This is the case of a uniform beam of length l supported at its ends.)

$[\mathfrak{B}y = \frac{1}{24}wx(l-x)(l^2+lx-x^2).]$

5. Solve the same equation subject to the conditions that $y=0$, $dy/dx=0$ for $x=0$ and $x=l$. (This is the case of a beam clamped at both ends.) $[\mathfrak{B}y = \frac{1}{24}wx^2(l-x)^2.]$

6. Solve the equation of Ex. 4, subject to the conditions that $y=0$, $dy/dx=0$ for $x=0$, and $d^2y/dx^2=0$, $d^3y/dx^3=0$ for $x=l$. (This is the case of a beam clamped at one end and free at the other.) $[\mathfrak{B}y = \frac{1}{24}wx^2(6l^2-4lx+x^2).]$

7. Solve the equation

$$\frac{d^2x}{dt^2} = -\mu x + f,$$

and interpret the result. $[x = f/\mu + a\cos(\sqrt{\mu}t + \epsilon).]$

8. Shew that the solution of the equation of motion of a particle moving in a straight line under a force of repulsion varying as the distance, viz.

$$\frac{d^2x}{dt^2} = \mu x,$$

is of one or other of the types:

$$x = a \cosh(\sqrt{\mu}t + \epsilon), \quad x = a \sinh(\sqrt{\mu}t + \epsilon), \quad x = ae^{\pm\sqrt{\mu}t+\epsilon};$$

and interpret these results.

9. A particle moves from rest at a distance a towards a centre of force whose accelerative effect is $\mu \times (\text{dist.})^{-3}$. Prove that the time of falling to the centre is $a^2/\sqrt{\mu}$.

10. Obtain a first integral of the general differential equation of central orbits, viz.

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2},$$

where P is a given function of u. $\left[\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\int \frac{P}{h^2u^2}\,du + C.\right]$

11. Solve the equation

$$\frac{d^2r}{dt^2} = \frac{\mu}{r^3},$$

and shew that the solution is equivalent to

$$r^2 = A + 2Bt + Ct^2,$$

where A, B, C are connected by the relation

$$AC - B^2 = \mu.$$

12. $$a\frac{d^2y}{dx^2} = \frac{dy}{dx}. \qquad [y = A + Be^{x/a}.]$$

13. $$2a\frac{dy}{dx}\frac{d^2y}{dx^2} = 1. \qquad [a(y-\beta)^2 = \tfrac{4}{9}(x-\alpha)^3.]$$

14. $$a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0. \qquad \left[y = a \log \frac{x-\alpha}{\beta}.\right]$$

15. $$a\frac{d^2y}{dx^2} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}. \qquad \left[y - \beta = a \cosh \frac{x-\alpha}{a}.\right]$$

16. $$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + 1 = 0. \qquad [y = \beta + \log\cos(x-\alpha).]$$

17. $\dfrac{d^4y}{dx^4} = \dfrac{d^2y}{dx^2}$. $[y = A + Bx + Ce^x + De^{-x}.]$

18. $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 0$. $[y = A + Bx + C\cos x + D\sin x.]$

19. $\dfrac{d^2V}{dr^2} + \dfrac{1}{r}\dfrac{dV}{dr} = 0$. $[V = A\log r + B.]$

20. $x\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx}$. $[y = Ax^2 + B.]$

21. $\dfrac{d^2y}{dx^2} + y\dfrac{dy}{dx} = 0$. $[y = 2\beta\tanh\beta(x - \alpha).]$

22. $(1 + x^2)\dfrac{d^2y}{dx^2} + 1 + \left(\dfrac{dy}{dx}\right)^2 = 0$.

$$\left[y = \beta + \left(1 + \frac{1}{\alpha^2}\right)\log(1 + \alpha x) - \frac{x}{\alpha}.\right]$$

23. $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$. $[y = A + B\sinh^{-1}x.]$

24. $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 0$. $[y = A + B\cosh^{-1}x.]$

25. $(1 + x^2)\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} = 0$. $[y = A\tan^{-1}x + B.]$

26. $(1 - y)\dfrac{d^2y}{dx^2} + 2\left(\dfrac{dy}{dx}\right)^2 = 0$. $\left[y = \dfrac{x + A}{x + B}.\right]$

27. $y\dfrac{d^2y}{dx^2} = 2\left(\dfrac{dy}{dx}\right)^2$. $\left[y = \dfrac{A}{x + B}.\right]$

28. $y\dfrac{d^2y}{dx^2} = 1 - \left(\dfrac{dy}{dx}\right)^2$. $[y^2 = x^2 + Ax + B.]$

29. $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} = 2$. $[y = (\sin^{-1}x)^2 + A\sin^{-1}x + B.]$

30. $\dfrac{d}{d\mu}\left\{(1 - \mu^2)\dfrac{du}{d\mu}\right\} = 0$. $[u = A + B\tanh^{-1}\mu.]$

31. $\dfrac{d}{d\mu}\left\{(1-\mu^2)\dfrac{du}{d\mu}\right\}+2u=0.$ $[u=A\mu+B(1-\mu\tanh^{-1}\mu).]$

32. Find the curves in which the radius of curvature is equal to the normal, on the same side of the curve.

[The circles $(x-a)^2+y^2=\beta^2$.]

33. Find the curves in which the radius of curvature is double the normal, on the opposite side of the curve.

[The parabolas $(x-a)^2=4\beta(y-\beta)$.]

34. Find the curves in which the projection of the radius of curvature on the axis of y has a given constant value (a).

$$\left[y=\beta\log\sec\frac{x-a}{a}.\right]$$

35. Find the curves in which the radius of curvature varies as the cube of the normal.

[Conics having the axis of x as an axis of symmetry.]

36. $$(1+x)\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2=\frac{dy}{dx}.$$

$$\left[y=\beta+\log(x^2-a^2)-\frac{1}{a}\log\frac{x+a}{x-a}.\right]$$

37. $$(1-x^2)\frac{d^2y}{dx^2}-\frac{1}{x}\frac{dy}{dx}+x^2=0.$$

$$[y=A+B\sqrt{(1-x^2)}+\tfrac{1}{2}x^2.]$$

38. $\dfrac{d^2y}{dx^2}+P\dfrac{dy}{dx}+\dfrac{dP}{dx}y=0.$ $[y=e^{-\int P dx}(A\int e^{\int P dx}dx+B).]$

39. ✓ $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}-y=0.$ $\left[y=\dfrac{Ae^x+Be^{-x}}{x}.\right]$

40. $(1+x^2)\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0.$ $[y=Ax+B(1-x^2).]$

41. $(1+x^2)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}=y.$ $[y=Ax+B\sqrt{(1+x^2)}.]$

42. $$\frac{d^2y}{dx^2}+2n\cot nx\frac{dy}{dx}+(m^2-n^2)y=0.$$

$$\left[y=\frac{A\cos mx+B\sin mx}{\sin nx}.\right]$$

43. $$(1-x^2)\frac{d^2y}{dx^2}-3x\frac{dy}{dx}-y=0. \qquad \left[y=\frac{A\sin^{-1}x+B}{\sqrt{(1-x^2)}}.\right]$$

44. $$x^2y\frac{d^2y}{dx^2}+\left(y-x\frac{dy}{dx}\right)^2=0.$$

$[y^2=Ax+Bx^2.]$

186. Linear Equation of the Second Order with Constant Coefficients. Complementary Function.

A special variety is the linear equation with constant coefficients. The treatment of this is much facilitated by the use of a few simple properties of the operator D, or d/dx, where x is the independent variable.

It has been shewn in Art. 36 that the operation indicated by D is 'distributive,' viz. u, v being any functions of x, we have

$$D(u+v)=Du+Dv \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Again, if a be a constant, we have

$$(D+a)u=\frac{du}{dx}+au=au+\frac{du}{dx}=(a+D)u \;\ldots\; (2),$$

and $$D(au)=a\frac{du}{dx}=aDu \quad\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

so that the operator D, in conjunction with constant multipliers, obeys the 'commutative' laws.

Further, the symbol D is subject to the 'index-law,' viz.

$$D^mD^nu=D^{m+n}u \quad\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

Hence the operator D, both by itself, and in conjunction with constant multipliers, is subject to the fundamental laws of ordinary Algebra. We can therefore assume at once, so far as they have a meaning in the present application, all the results which in Algebra follow from these laws.

For example, if λ_1, λ_2 be any constants, we have

$$\begin{aligned}(D-\lambda_1)(D-\lambda_2)u &= (D-\lambda_1)\left(\frac{du}{dx}-\lambda_2 u\right)\\ &= \frac{d}{dx}\left(\frac{du}{dx}-\lambda_2 u\right)-\lambda_1\left(\frac{du}{dx}-\lambda_2 u\right)\\ &= \frac{d^2u}{dx^2}-(\lambda_1+\lambda_2)\frac{du}{dx}+\lambda_1\lambda_2 u\\ &= \{D^2-(\lambda_1+\lambda_2)D+\lambda_1\lambda_2\}u \quad\text{......... (5).}\end{aligned}$$

We proceed to the equation of the second order, which occurs very frequently in dynamical problems. To find the complementary function, we have to solve an equation of the form

$$\frac{d^2y}{dx^2}+a\frac{dy}{dx}+by=0 \quad\text{.................. (6),}$$

or

$$(D^2+aD+b)y=0 \quad\text{.................. (7).}$$

If $\frac{1}{4}a^2>b$, this is equivalent to

$$(D-\lambda_1)(D-\lambda_2)y=0 \quad\text{.............. (8),}$$

where λ_1, λ_2 are the roots of

$$\lambda^2+a\lambda+b=0 \quad\text{...................... (9),}$$

viz.

$$\left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\}=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{4}a^2-b)} \quad\text{.............. (10).}$$

If we write

$$(D-\lambda_2)y=z \quad\text{.................... (11),}$$

the equation (8) becomes

$$(D-\lambda_1)z=0 \quad\text{.................... (12),}$$

a linear equation of the first order. The solution of this is, by Art. 176,

$$z=Ae^{\lambda_1 x} \quad\text{...................... (13).}$$

Substituting in (11), we have

$$(D-\lambda_2)y=Ae^{\lambda_1 x} \quad\text{.................. (14),}$$

whence, by Art. 176, 1°,

$$y=C_1e^{\lambda_1 x}+C_2e^{\lambda_2 x} \quad\text{.................. (15),}$$

if $C_1 = A/(\lambda_1 - \lambda_2)$. Since A is arbitrary, the constants C_1, C_2 are both arbitrary; and the process shews that (15) is the most general solution of the proposed equation (6).

If $\frac{1}{4}a^2 = b$, the roots of the equation (9) in λ are coincident, and the equation (14) becomes

$$(D - \lambda_1)\, y = Ae^{\lambda_1 x} \quad\text{..................(16).}$$

The general solution of this, as found in Art. 176, 2°, is

$$y = (Ax + B)\, e^{\lambda_1 x} \quad\text{..................(17).}$$

If $\frac{1}{4}a^2 < b$, the values of λ_1, λ_2 are imaginary, but we can still obtain, by the foregoing process, a symbolical solution involving $\sqrt{(-1)}$. Into the question as to what meaning can be attributed to such a result it is not necessary to enter here, as the difficulty can be evaded in the following manner.

If we write

$$y = e^{-\frac{1}{2}ax} z \quad\text{..................... (18),}$$

we find

$$Dy = e^{-\frac{1}{2}ax}(D - \tfrac{1}{2}a)\, z, \quad D^2 y = e^{-\frac{1}{2}ax}(D^2 - aD + \tfrac{1}{4}a^2)\, z,$$

so that the equation (6) becomes

$$\frac{d^2 z}{dx^2} + (b - \tfrac{1}{4}a^2)\, z = 0 \quad\text{............... (19).}$$

The solution of this, when $b > \frac{1}{4}a^2$, has been shewn in Art. 182 to be

$$z = A \cos \beta x + B \sin \beta x \quad\text{...............(20),}$$

where

$$\beta = \sqrt{(b - \tfrac{1}{4}a^2)} \quad\text{.................. (21).}$$

Hence the solution of (6), in the present case, is

$$y = e^{-\frac{1}{2}ax}(A \cos \beta x + B \sin \beta x) \quad\text{......... (22).}$$

This is also equivalent to

$$y = Ce^{-\frac{1}{2}ax} \cos(\beta x + \epsilon) \quad\text{...............(23),}$$

where the constants C, ϵ are arbitrary.

To summarize our results:

(a) If $\frac{1}{4}a^2 > b$, the solution of (6) is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x},$$

where λ_1, λ_2 are the roots of

$$\lambda^2 + a\lambda + b = 0;$$

(*b*) If $\frac{1}{4}a^2 = b$, the solution is

$$y = (Ax + B)\, e^{-\frac{1}{2}ax};$$

(*c*) If $\frac{1}{4}a^2 < b$, the solution is

$$y = e^{-\frac{1}{2}ax}\,(A \cos \beta x + B \sin \beta x),$$

if

$$\beta^2 = b - \tfrac{1}{4}a^2.$$

Ex. 1.

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0 \quad \text{.....................(24).}$$

The equation in λ is

$$\lambda^2 + \lambda - 6 = 0, \text{ whence } \lambda = 2, \text{ or } -3.$$

Hence

$$y = Ae^{2x} + Be^{-3x} \quad \text{.....................(25).}$$

Ex. 2.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

The equation in λ, viz. $(\lambda + 2)^2 = 0$,
has the double root -2. Hence

$$y = (Ax + B)\, e^{-2x} \quad \text{.....................(26).}$$

Ex. 3. The free oscillations of a pendulum in a medium whose resistance varies as the velocity are determined by an equation of the form

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = 0 \quad \text{..................(27),}$$

where k is a coefficient of friction. The same equation also serves to represent the motion of a galvanometer-needle as affected by the viscosity of the air and by the electro-magnetic action of the currents induced by its motion in neighbouring masses of metal.

When regard is had to the difference of notation, the solution of (27), when the friction falls below a certain limit, is

$$x = Ce^{-\frac{1}{2}kt} \cos(n_1 t + \epsilon) \quad \text{..................(28),}$$

where

$$n_1 = \sqrt{(\mu - \tfrac{1}{4}k^2)} \quad \text{.....................(29).}$$

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The motion represented by (28) may be described as a simple-harmonic vibration of period $2\pi/n_1$, whose amplitude diminishes asymptotically to zero according to the law $e^{-\frac{1}{2}kt}$.

The solution (28) assumes that $k^2 < 4\mu$. When $k^2 > 4\mu$, the proper form is

$$x = Ae^{\lambda_1 t} + Be^{\lambda_2 t} \quad \text{..................(30)},$$

where λ_1, λ_2 are the roots of

$$\lambda^2 + k\lambda + \mu = 0 \quad \text{..................(31)}.$$

By hypothesis, these roots are real. Since their product (μ) is positive they have the same sign; and since their sum ($-k$) is negative, the sign is *minus*. Hence the displacement x sinks asymptotically to zero after passing once (at most) through this value. This case is realized in a 'dead-beat' galvanometer, or in a pendulum swinging in a very viscous fluid.

In the critical case of $k^2 = 4\mu$, we have

$$x = (A + Bt)\, e^{-\frac{1}{2}kt} \quad \text{..................(32)}.$$

The first factor increases (in absolute value) indefinitely with t, whilst the second diminishes. The decrease of the second factor prevails however over the increase of the first, and the limiting value of the product, for $t = \infty$, is zero. Cf. Art. 28, Ex. 3.

187. Determination of Particular Integrals.

We have next to consider the problem of finding a particular integral of the linear equation of the second order with constant coefficients, when this equation has a right-hand member, thus:

$$(D^2 + aD + b)\, y = V \quad \text{..................(1)},$$

where V is a given function of x. As already remarked, *any* particular integral, however obtained, will serve the purpose. Thus, we may omit from the particular integral any terms which occur in the complementary function, since these will contribute nothing to the left-hand side of (1). Conversely, if for any purpose it is convenient to do so, we may add to the particular integral any groups of terms taken from the complementary function.

Again, if V be composed of a series of terms, the problem consists in finding values of y which, when substituted on the left-hand side of (1) will reproduce the several terms, and adding the results.

It will be sufficient here to notice the most useful cases.

1°. If V contains a term

$$He^{\alpha x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

the corresponding term of the particular integral is

$$y = \frac{H}{\alpha^2 + a\alpha + b} e^{\alpha x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

For if we perform the operation $D^2 + aD + b$ on the right-hand side of this, we reproduce (2).

This rule fails if $\alpha^2 + a\alpha + b = 0$, *i.e.* if $e^{\alpha x}$ be one of the terms which occur in the complementary function. Using the notation of the preceding Art., we will suppose that $\alpha = \lambda_1$, so that the equation to be solved is

$$(D - \lambda_1)(D - \lambda_2) y = He^{\lambda_1 x} \quad\ldots\ldots\ldots\ldots (4).$$

If we write, for a moment,

$$(D - \lambda_2) y = z \quad\ldots\ldots\ldots\ldots\ldots\ldots (5),$$

this takes the form

$$(D - \lambda_1) z = He^{\lambda_1 x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (6).$$

It was found in Art. 176, 2°, that a particular solution of (6) is

$$z = Hxe^{\lambda_1 x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

It remains only to solve

$$(D - \lambda_2) y = Hxe^{\lambda_1 x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (8).$$

The integrating factor is $e^{-\lambda_2 x}$; thus

$$D(ye^{-\lambda_2 x}) = Hxe^{(\lambda_1 - \lambda_2) x} \quad\ldots\ldots\ldots\ldots\ldots (9).$$

Integrating the right-hand member by parts, and omitting a term already included in the complementary function, we find

$$ye^{-\lambda_2 x} = \frac{H}{\lambda_1 - \lambda_2} xe^{(\lambda_1 - \lambda_2) x},$$

or

$$y = \frac{H}{\lambda_1 - \lambda_2} xe^{\lambda_1 x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

A further modification is necessary if α be a *double* root of the equation $D^2 + aD + b = 0$. The equation to be solved has now the form

$$(D - \lambda)^2 y = He^{\lambda x} \quad\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

The first step is as before, but in place of (8) we have

$$(D-\lambda)\,y = Hxe^{\lambda x} \quad\text{..................(12).}$$

We found in Art. 176, 3°, that a particular integral of this is

$$y = \tfrac{1}{2}Hx^2e^{\lambda x} \quad\text{..................... (13).}$$

The *forms* of these results being once established, the student will probably find it the easiest and safest course to assume

$$y = Ce^{\alpha x}, \quad y = Cxe^{\alpha x}, \quad \text{or } y = Cx^2e^{\alpha x} \quad\text{.........(14),}$$

as the case may be, and to determine the value of C by actual substitution in the equation

$$(D^2 + aD + b)\,y = He^{\alpha x} \quad\text{..................(15).}$$

The work is facilitated by formulæ to be given in Art. 188.

Ex. 1.
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = e^{3x} + e^{-3x} \quad\text{...............(16).}$$

As in Art. 186, Ex. 1, the complementary function is

$$y = Ae^{2x} + Be^{-3x}.$$

If we assume $y = Ce^{3x}$, we find on substitution that the first term on the right-hand side of (16) is reproduced provided $C = \frac{1}{6}$. The second term comes under one of the exceptional cases above discussed, since -3 is a root of $\lambda^2 + \lambda - 6 = 0$. If we assume $y = Cxe^{-3x}$, we find that the term in question is reproduced, provided $C = -\frac{1}{5}$.

The complete solution of (16) is therefore

$$y = Ae^{2x} + Be^{-3x} + \tfrac{1}{6}e^{3x} - \tfrac{1}{5}xe^{-3x} \quad\text{...........(17).}$$

Ex. 2.
$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{2x} + e^{-2x} \quad\text{...............(18).}$$

The complementary function has been found in Art. 186, Ex. 2, to be

$$y = (Ax + B)\,e^{-2x}.$$

To reproduce the first term on the right hand, we assume $y = Ce^{2x}$, and find $C = \frac{1}{16}$. The second term corresponds to a double root in the equation for λ; assuming, therefore, $y = Cx^2e^{-2x}$, we find $C = \frac{1}{2}$.

Hence the complete solution of (18) is

$$y = (Ax + B)\,e^{-2x} + \tfrac{1}{16}e^{2x} + \tfrac{1}{2}x^2e^{-2x} \quad\text{...........(19).}$$

2°. If V contains terms of the form

$$H \cos \alpha x + K \sin \alpha x \quad \ldots\ldots\ldots\ldots (20),$$

we may assume

$$y = A \cos \alpha x + B \sin \alpha x \quad \ldots\ldots\ldots\ldots (21).$$

Substituting in (1), we obtain, on the left-hand side,

$$(-\alpha^2 A + a\alpha B + bA) \cos \alpha x + (-\alpha^2 B - a\alpha A + bB) \sin \alpha x.$$

Hence the terms (20) are reproduced, provided

$$(-\alpha^2 + b) A + a\alpha B = H, \quad -a\alpha A + (-\alpha^2 + b) B = K \ldots (22).$$

Except in the particular case where $a = 0$, $\alpha^2 = b$, which will be considered presently, these equations determine A and B; thus

$$A = \frac{(-\alpha^2 + b) H - a\alpha K}{(\alpha^2 - b)^2 + a^2\alpha^2}, \quad B = \frac{a\alpha H + (-\alpha^2 + b) K}{(\alpha^2 - b)^2 + a^2\alpha^2} \ldots (23).$$

Ex. 3. To find a particular integral of

$$\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = f \cos (pt + \epsilon) \quad \ldots\ldots\ldots\ldots (24).$$

This is the equation of motion of a pendulum subject to a resistance varying as the velocity, and acted on by a force which is a simple-harmonic function of the time.

We assume $\quad x = A \cos (pt + \epsilon) + B \sin (pt + \epsilon) \quad \ldots\ldots\ldots (25)$,

and find, on substitution,

$$\left.\begin{aligned} -p^2 A + kpB + \mu A &= f, \\ -p^2 B - kpA + \mu B &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (26),$$

whence $\quad A = \dfrac{\mu - p^2}{(\mu - p^2)^2 + k^2p^2} f, \quad B = \dfrac{kp}{(\mu - p^2)^2 + k^2p^2} f, \ \ldots (27).$

If we put $\quad A = R \cos \epsilon_1, \quad B = R \sin \epsilon_1 \quad \ldots\ldots\ldots\ldots (28)$,

the solution (25) takes the form

$$x = R \cos (pt + \epsilon - \epsilon_1) \quad \ldots\ldots\ldots\ldots (29),$$

where $\quad R = \dfrac{f}{\sqrt{\{(\mu - p^2)^2 + k^2p^2\}}}, \quad \epsilon_1 = \tan^{-1} \dfrac{kp}{\mu - p^2} \ldots\ldots (30).$

We have thus determined the 'forced oscillations' due to the given periodic force. The 'free oscillations,' which are in general

$\epsilon \quad \epsilon' + \frac{\pi}{2}$ $\qquad r \cos(pt + \epsilon' - \frac{\pi}{2})$

superposed on these, are given by the complementary function (Art. 186, Ex. 3). Unless $k=0$ they gradually die out as t increases.

The foregoing results simplify when the coefficient a in the differential equation is zero; a particular integral of

$$\frac{d^2y}{dx^2}+by=H\cos \alpha x+K\sin \alpha x \quad\text{.........}\quad (31)$$

being obviously

$$y=\frac{H}{b-\alpha^2}\cos \alpha x+\frac{K}{b-\alpha^2}\sin \alpha x \quad\text{.........}\quad (32).$$

A singularity arises, however, when $\alpha^2=b$. To find the proper form in this case we may assume

$$y=u\cos \alpha x+v\sin \alpha x \quad\text{...............}\quad (33).$$

This makes

$$\frac{d^2y}{dx^2}+\alpha^2y=(2\alpha Dv+D^2u)\cos \alpha x+(-2\alpha Du+D^2v)\sin \alpha x \quad\text{......}\quad (34).$$

Hence (31) will be satisfied in this case, provided

$$Du=-K/2\alpha,\quad Dv=H/2\alpha,$$

or

$$u=-\frac{Kx}{2\alpha},\quad v=\frac{Hx}{2\alpha} \quad\text{..............,}\quad (35).$$

A particular integral is, therefore,

$$y=\frac{H}{2\alpha}x\sin \alpha x-\frac{K}{2\alpha}x\cos \alpha x \quad\text{...........}\quad (36).$$

Ex. 4. For the forced oscillations of an unresisted pendulum we have

$$\frac{d^2x}{dt^2}+n^2x=f\cos(pt+\epsilon) \quad\text{..................}\quad (37).$$

A particular integral is

$$x=\frac{f}{n^2-p^2}\cos(pt+\epsilon) \quad\text{..................}\quad (38).$$

This fails when $p=n$. Assuming that in this case

$$x=Ct\sin(nt+\epsilon) \quad\text{.....................}\quad (39),$$

we find on substitution that (37) is satisfied provided

$$2nC = f, \text{ or } C = f/2n \text{ ....(40).}$$

The interpretation of (39) is that, if an unresisted pendulum be acted on by a periodic force whose period coincides with that natural to the pendulum, the amplitude of the forced oscillations will at first* increase proportionally to the time.

188. Properties of the Operator D.

The methods of Arts. 186, 187 admit of extension to the general linear equation with constant coefficients

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + A_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = V \text{...(1),}$$

or, as we may write it for shortness,

$$f(D)\, y = V \text{ ....(2),}$$

$f(D)$ denoting a rational integral function of D. We shall however content ourselves with indicating how a solution of (1), involving n distinct arbitrary constants, can be obtained when $V = 0$, and how a particular integral can be found for certain forms of V. The proof that the solution thus arrived at is the most general which the equation admits of is omitted; but from the point of view of practical applications it is sufficient to have at our disposal the proper number of arbitrary constants required to satisfy the remaining conditions of the problem.

The following properties of the operator D will be useful.

1°. If $\psi(D)$ be any rational integral function of D, say

$$\psi(D) \equiv A_0 D^n + A_1 D^{n-1} + A_2 D^{n-2} + \dots + A_{n-1} D + A_n \text{...(3),}$$

then

$$\psi(D)\, e^{\lambda x} = \psi(\lambda)\, e^{\lambda x} \text{ ....(4).}$$

For

$$D^r e^{\lambda x} = \lambda^r e^{\lambda x},$$

and thus the several terms of $\psi(D)$ give rise to the several terms of $\psi(\lambda)$ as factors of $e^{\lambda x}$.

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* Usually, in the physical applications, the equation (37) is approximate only, being obtained by the neglect of powers of x higher than the first. Hence when the amplitude increases beyond a certain limit, the equation ceases to apply, even approximately, to the subsequent motion.

2°. With the same meaning of $\psi(D)$, if u be any function of x, then

$$\psi(D) . e^{\lambda x} u = e^{\lambda x} . \psi(D+\lambda) u \text{ ....(5).}$$

For we find in succession

$$\begin{aligned} D \;. e^{\lambda x} u &= e^{\lambda x} (D+\lambda) u, \\ D^2 . e^{\lambda x} u &= D\{e^{\lambda x}(D+\lambda) u\} = e^{\lambda x}(D+\lambda)(D+\lambda) u \\ &= e^{\lambda x}(D+\lambda)^2 u, \end{aligned}$$

and so on; the general result being

$$D^r . e^{\lambda x} u = e^{\lambda x}(D+\lambda)^r u.$$

Hence the several terms of the operator $\psi(D)$ give rise to the several terms of the result given in (5).

3°. If $\psi(D)$ contain only even powers of D, it may be denoted by $\phi(D^2)$. It appears from Art. 63 that if

$$u = A \cos ax + B \sin ax \text{ .... (6),}$$

then

$$D^2 u = -a^2 u,$$

and therefore

$$\phi(D^2) u = \phi(-a^2) u \text{ ....(7).}$$

189. General Linear Equation with Constant Coefficients. Complementary Function.

To obtain solutions of the equation

$$f(D) y = 0 \text{ ....(1),}$$

we remark that if $f(D)$ be resolved into any two rational integral factors, thus

$$f(D) \equiv \phi(D) . \psi(D) \text{ ....(2),}$$

the equation (1) is obviously satisfied by any solution of

$$\psi(D) y = 0 \text{ ....(3).}$$

And since the factors are commutative (Art. 186), it is also satisfied by any solution of

$$\phi(D) y = 0 \text{ ....(4).}$$

Hence (1) is satisfied by the sum of any solutions of (3) and (4). Continuing the resolution we see that if

$$f(D) \equiv \phi_1(D) . \phi_2(D) . \phi_3(D) \dots \text{ ....(5),}$$

the equation (1) will be satisfied by the sum of the several solutions of

$$\phi_1(D)\,y = 0, \quad \phi_2(D)\,y = 0, \quad \phi_3(D)\,y = 0, \ldots \quad \ldots(6).$$

By a theorem of Algebra, already referred to in Art. 84, the function $f(D)$ can be resolved (if its coefficients be real) into real factors of the first and second degrees, the sum of the degrees of the several factors being equal to the degree (n, say) of the function. Moreover the factors of the first degree are of the forms

$$D - \lambda_1, \quad D - \lambda_2, \; D - \lambda_3, \ldots$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots$ are the real roots of

$$f(\lambda) = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (7).$$

If λ be a simple root of (7), one of the equations (6) is of the form

$$(D - \lambda)\,y = 0 \;\ldots\ldots\ldots\ldots\ldots\ldots\; (8),$$

the integral of which is known to be

$$y = Ce^{\lambda x} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (9).$$

And if the roots of (7) be all real, say they are $\lambda_1, \lambda_2, \ldots \lambda_n$, a solution of (1) involving n arbitrary constants is

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \ldots + C_n e^{\lambda_n x} \;\ldots\ldots\ldots (10);$$

cf. Art. 186 (15).

If the equation (7) has a multiple root, two or more of the terms on the right hand of (10) coalesce, and the number of distinct solutions thus obtained is less than n. To supply the deficiency, we remark that, if λ be an r-fold root of (7), $f(D)$ contains a factor $(D - \lambda)^r$. To solve

$$(D - \lambda)^r\,y = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11),$$

we assume

$$y = e^{\lambda x}\,z \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12),$$

which makes

$$(D - \lambda)^r\,y = (D - \lambda)^r \,.\, e^{\lambda x}\,z = e^{\lambda x}\,D^r\,z,$$

by Art. 188, 2°., and the solution of $D^r z = 0$ is obviously

$$z = B_0 + B_1 x + B_2 x^2 + \ldots + B_{r-1} x^{r-1},$$

whence

$$y = (B_0 + B_1 + B_2 x^2 + \ldots + B_{r-1} x^{r-1})\,e^{\lambda x} \;\ldots\ldots (13).$$

We have here r arbitrary constants, corresponding to the r-fold factor of $f(D)$. Cf. Art. 186 (17).

If $f(D)$ contains, once only, an irreducible quadratic factor D^2+aD+b, where $\frac{1}{4}a^2<b$, then part of the solution of (1) consists of the solution of

$$(D^2+aD+b)\,y=0 \quad \text{..............} \quad (14).$$

If we put $\quad y=e^{-\frac{1}{2}ax}z, \quad \beta^2=b-\frac{1}{4}a^2 \quad \text{..........} \quad (15).$

we have, by Art. 188 (5),

$$\begin{aligned}(D^2+aD+b)\,y&=\{(D+\tfrac{1}{2}a)^2+\beta^2\}\,e^{-\frac{1}{2}ax}z\\&=e^{-\frac{1}{2}ax}(D^2+\beta^2)\,z.\end{aligned}$$

And the solution of $\quad (D^2+\beta^2)\,z=0,$

is $\quad z=E\cos\beta x+F\sin\beta x.$

We thus find $\quad y=e^{-\frac{1}{2}ax}(E\cos\beta x+F\sin\beta x) \quad \text{........} \quad (16),$

as in Art. 186 (22). Hence for every distinct quadratic factor of $f(D)$ we obtain a solution involving two arbitrary constants.

Finally, if $f(D)$ contains an irreducible quadratic factor which occurs r times, we have to solve

$$(D^2+aD+b)^r y=0 \quad \text{..............} \quad (17),$$

or, making the substitutions (15), as before,

$$(D^2+\beta^2)^r z=0 \quad \text{................} \quad (18).$$

To solve this, we assume

$$z=u\cos\beta x+v\sin\beta x \quad \text{..............} \quad (19).$$

Now, by actual differentiation, we find

$$\begin{aligned}(D^2+\beta^2)\,.\,u\cos\beta x&=2\beta\,.\,Du\,.\cos(\beta x+\tfrac{1}{2}\pi)+\ldots,\\(D^2+\beta^2)^2\,.\,u\cos\beta x&=(2\beta)^2\,.\,D^2u\,.\cos(\beta x+\pi)+\ldots,\end{aligned}$$

and, generally,

$$(D^2+\beta^2)^r\,.\,u\cos\beta x=(2\beta)^r\,.\,D^ru\,.\cos(\beta x+\tfrac{1}{2}r\pi)+\ldots \quad \text{...} \quad (20),$$

where only the terms of lowest order in the derivatives of u are expressed. Similarly

$$(D^2+\beta^2)^r\,.\,v\sin\beta x=(2\beta)^r\,.\,D^rv\,.\sin(\beta x+\tfrac{1}{2}r\pi)+\ldots \quad \text{...} \quad (21).$$

Hence the equation (18) is satisfied, provided

$$D^r u = 0, \quad D^r v = 0 \quad \text{..................} (22),$$

i.e.

$$\left.\begin{aligned} u &= E_0 + E_1 x + E_2 x^2 + \ldots + E_{r-1} x^{r-1} \\ v &= F_0 + F_1 x + F_2 x^2 + \ldots + F_{r-1} x^{r-1} \end{aligned}\right\} \quad \text{......} (23).$$

The complete solution of (17) is therefore

$$y = (E_0 + E_1 x + E_2 x^2 + \ldots + E_{r-1} x^{r-1}) e^{-\frac{1}{2}ax} \cos \beta x$$
$$+ (F_0 + F_1 x + F_2 x^2 + \ldots + F_{r-1} x^{r-1}) e^{-\frac{1}{2}ax} \sin \beta x \ldots (24),$$

involving $2r$ arbitrary constants.

Ex. 1.
$$\frac{d^3 y}{dx^3} = \frac{d^2 y}{dx^2} \quad \text{.......................} (25).$$

This may be written $\quad D^2 (D - 1) y = 0,$

and the complete solution is accordingly made up of the solutions of $D^2 y = 0$, $(D - 1) y = 0$, thus

$$y = A + Bx + Ce^x \quad \text{....................} (26).$$

Ex. 2.
$$\frac{d^4 y}{dx^4} = m^4 y \quad \text{.......................} (27).$$

This may be written

$$(D - m)(D + m)(D^2 + m^2) y = 0.$$

Adding together the solutions of

$$(D - m) y = 0, \quad (D + m) y = 0, \quad (D^2 + m^2) y = 0,$$

we obtain $\quad y = Ae^{mx} + Be^{-mx} + E \cos mx + F \sin mx \quad$ ......(28).

Ex. 3.
$$\frac{d^4 y}{dx^4} + \frac{d^2 y}{dx^2} + y = 0 \quad \text{....................} (29).$$

This is equivalent to

$$(D^2 + D + 1)(D^2 - D + 1) y = 0.$$

Hence

$$y = e^{-\frac{1}{2}x} \left(A \cos \frac{\sqrt{3}}{2} x + B \sin \frac{\sqrt{3}}{2} x \right) + e^{\frac{1}{2}x} \left(A' \cos \frac{\sqrt{3}}{2} x + B' \sin \frac{\sqrt{3}}{2} x \right)$$
$$\text{............} (30).$$

Ex. 4.
$$\frac{d^4 y}{dx^4} + 2m^2 \frac{d^2 y}{dx^2} + m^4 y = 0 \quad \text{..................} (31).$$

Writing this in the form

$$(D^2 + m^2)^2 y = 0,$$

we find $\quad y = (E_0 + E_1 x) \cos mx + (F_0 + F_1 x) \sin mx \quad$ ......(32).

190. Particular Integrals.

We proceed to the determination of a particular integral of the equation

$$f(D)y = V \quad\text{.........................}\quad (1),$$

in the two most important cases.

1°. If V contain a term

$$He^{\alpha x} \quad\text{...........................}\quad (2),$$

the corresponding term in the particular integral is

$$y = \frac{H}{f(\alpha)} e^{\alpha x} \quad\text{.....................}\quad (3),$$

for this makes $\quad f(D)y = \dfrac{H}{f(\alpha)} f(D) e^{\alpha x} = He^{\alpha x},$

by Art. 188 (4).

The rule fails when α is a root of

$$f(D) = 0 \quad\text{.........................}\quad (4).$$

If it be a simple root, we may write

$$f(D) \equiv \phi(D)(D - \alpha) \quad\text{.................}\quad (5),$$

where $\phi(D)$ does not contain the factor $D - \alpha$. The equation

$$\phi(D)(D - \alpha)y = He^{\alpha x} \quad\text{...............}\quad (6),$$

is satisfied provided $\quad (D - \alpha)y = \dfrac{H}{\phi(\alpha)} e^{\alpha x},$

and we have seen, in Art. 176, 2°, that a particular integral of this is

$$y = \frac{H}{\phi(\alpha)} xe^{\alpha x} \quad\text{..................}\quad (7).$$

If α be an r-fold root of (4), we may write

$$f(D) \equiv \phi(D)(D - \alpha)^r \quad\text{...............}\quad (8),$$

where $\phi(D)$ does not contain $D - \alpha$ as a factor.

The equation $\quad \phi(D)(D - \alpha)^r y = He^{\alpha x} \quad\text{.......... ...}\quad (9),$

is satisfied, provided

$$(D - \alpha)^r y = \frac{H}{\phi(\alpha)} e^{\alpha x}.$$

Now if we put $\quad y = e^{\alpha x} z,$

we have, by Art. 188 (5),

$$(D-\alpha)^r y = e^{\alpha x} D^r z,$$

whence
$$D^r z = \frac{H}{\phi(\alpha)}.$$

This is evidently satisfied by

$$z = \frac{H}{r!\,\phi(\alpha)} x^r.$$

A particular integral of (9) is therefore

$$y = \frac{H}{r!\,\phi(\alpha)} x^r e^{\alpha x} \quad\text{..................}\quad (10).$$

2°. Let V contain terms of the type

$$H \cos \alpha x + K \sin \alpha x \quad\text{...............}\quad (11).$$

Since the operation $f(D)$, performed on

$$y = A \cos \alpha x + B \sin \alpha x \quad\text{...............}\quad (12),$$

must result in an expression of the same form with altered coefficients, a particular integral can in general be found by substituting this value of y in the equation

$$f(D)\, y = H \cos \alpha x + K \sin \alpha x \quad\text{............}\quad (13),$$

and determining A and B by equating coefficients of $\cos \alpha x$ and $\sin \alpha x$.

In one very frequent case, the values of A and B can be written down at once, viz. when the equation is of the type

$$\phi(D^2)\, y = H \cos \alpha x + K \sin \alpha x \quad\text{.........}\quad (14),$$

i.e. $f(D)$ contains only even powers of D. We have, then, by Art. 188, 3°

$$y = \frac{H}{\phi(-\alpha^2)} \cos \alpha x + \frac{K}{\phi(-\alpha^2)} \sin \alpha x \quad\text{.........}\quad (15).$$

This result fails if $\phi(-\alpha^2) = 0$, *i.e.* if $\phi(D^2)$ contain $D^2 + \alpha^2$ as a factor, in which case terms of the type (11) occur in the complementary function. If the factor $D^2 + \alpha^2$ occur once only, we write

$$\phi(D^2) \equiv \chi(D^2)(D^2 + \alpha^2) \quad\text{............}\quad (16).$$

Now the equation

$$\chi(D^2)(D^2 + \alpha^2)\, y = H \cos \alpha x + K \sin \alpha x \quad\text{......}\quad (17),$$

will be satisfied if

$$(D^2+\alpha^2)\,y=\frac{H}{\chi(-\alpha^2)}\cos\alpha x+\frac{K}{\chi(-\alpha^2)}\sin\alpha x \ldots (18).$$

The problem is thus reduced to one already solved under Art. 187, 2°, viz. we have the particular integral

$$y=\frac{H}{2\alpha\chi(-\alpha^2)}\,x\sin\alpha x-\frac{K}{2\alpha\chi(-\alpha^2)}\,x\cos\alpha x \ldots\ldots (19).$$

If the factor $D^2+\alpha^2$ occur r times in $\phi(D^2)$, we write

$$\phi(D^2)\equiv\chi(D^2)\,(D^2+\alpha^2)^r \ldots\ldots\ldots\ldots (20),$$

and the problem is reduced to finding a particular integral of

$$(D^2+\alpha^2)^r\,y=\frac{H}{\chi(-\alpha^2)}\cos\alpha x+\frac{K}{\chi(-\alpha^2)}\sin\alpha x \ldots (21).$$

If we assume

$$y=u\cos(\alpha x-\tfrac{1}{2}r\pi)+v\sin(\alpha x-\tfrac{1}{2}r\pi) \ldots\ldots (22)^*,$$

we find

$$(D^2+\alpha^2)^r y=(2\alpha)^r.D^r u.\cos\alpha x+(2\alpha)^r.D^r v.\sin\alpha x+\ldots$$

by Art. 189 (20), (21). Hence (21) is satisfied provided

$$D^r u=\frac{H}{(2\alpha)^r\chi(-\alpha^2)},\quad D^r v=\frac{K}{(2\alpha)^r\chi(-\alpha^2)} \ldots (23),$$

or

$$u=\frac{Hx^r}{r\,!\,(2\alpha)^r\chi(-\alpha^2)},\quad v=\frac{Kx^r}{r\,!\,(2\alpha)^r\chi(-\alpha^2)} \ldots (24).$$

A particular integral is therefore

$$y=\frac{x^r}{r\,!\,(2\alpha)^r\chi(-\alpha^2)}\{H\cos(\alpha x-\tfrac{1}{2}r\pi)+K\sin(\alpha x-\tfrac{1}{2}r\pi)\} \ldots\ldots\ldots\ldots (25).$$

In the general case, where $f(D)$ contains both odd and even powers of D, the assumption (12) fails in like manner if, and only if, $f(D)$ contains the factor $D^2+\alpha^2$. Writing

$$f(D)=\chi(D)\,(D^2+\alpha^2)^r \ldots\ldots\ldots\ldots\ldots (26),$$

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[*] The assumption $y=u\cos\alpha x+v\sin\alpha x$ would serve equally well, but the form in the text enables us to write the final result in a more compact manner.

where $\chi(D)$ does not contain the factor $D^2 + \alpha^2$, we first obtain a particular integral of the equation

$$\chi(D)\, y = H \cos \alpha x + K \sin \alpha x \quad\text{.........}\ (27),$$

in the form

$$y = H_1 \cos \alpha x + K_1 \sin \alpha x \quad\text{.........}\ (28).$$

It then remains only to solve

$$(D^2 + \alpha^2)^r\, y = H_1 \cos \alpha x + K_1 \sin \alpha x \text{.........}(29).$$

This has been treated above.

Another case where a particular integral can be obtained is that of

$$f(D)\, y = X \text{........................}(30),$$

where X is a rational integral function, of (say) degree r. We put $y = x^m v$, where m is the lowest index of D which occurs in $f(D)$, and v is a rational integral function of x, of degree r. The coefficients in v are then determined by substitution.

191. Homogeneous Linear Equation.

An equation of the type

$$x^n \frac{d^n y}{dx^n} + A_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + A_r x^r \frac{d^r y}{dx^r} + \ldots$$
$$+ A_{n-1} x \frac{dy}{dx} + A_n y = V \text{.........}(1),$$

is sometimes called a 'homogeneous linear equation.' The complementary function in this case consists in general of a series of terms of the form Cx^m, where C is arbitrary, and the values of m are to be found by substitution on the left-hand side. Moreover, if V contains a term Hx^p, the corresponding term in the particular integral will in general be Bx^p, provided B be properly determined.

To see the truth of these statements we may take the homogeneous linear equation of the second order. To solve

$$x^2 \frac{d^2 y}{dx^2} + ax \frac{dy}{dx} + by = 0 \text{................}(2),$$

we assume

$$y = Cx^m \text{ ....} (3).$$

This will satisfy the equation provided

$$\{m(m-1) + am + b\}\, Cx^m = 0 \text{..............}(4).$$

Equating to zero the expression in { }, we have a quadratic in m. If m_1, m_2 be the roots of this, the solution is

$$y = C_1 x^{m_1} + C_2 x^{m_2} \qquad \text{.................. (5).}$$

Again, a particular integral of the equation

$$x^2 \frac{d^2y}{dx^2} + ax \frac{dy}{dx} + by = Hx^p \qquad \text{..............(6),}$$

will be

$$y = Bx^p \qquad \text{..........................(7),}$$

provided

$$\{p(p-1) + ap + b\} B = H \qquad \text{..............(8).}$$

Ex. 1.

$$\frac{d^2V}{dr^2} + \frac{2}{r}\frac{dV}{dr} = 0 \qquad \text{.......................(9).}$$

If we multiply by r^2 this comes under the form (1); thus

$$r^2 \frac{d^2V}{dr^2} + 2r \frac{dV}{dr} = 0 \qquad \text{..................(10).}$$

Assuming

$$V = Cr^m,$$

we find

$$m(m-1) + 2m = 0, \text{ or } m(m+1) = 0.$$

The admissible values of m are 0 and -1; and the solution is therefore

$$V = A + \frac{B}{r} \qquad \text{......................(11).}$$

Cf. Art. 184, Ex. 2.

Ex. 2.

$$x^2 \frac{d^2y}{dx^2} + 2x \frac{dy}{dx} - 2y = x^2 \qquad \text{................(12).}$$

To find the complementary function, we assume $y = Cx^m$, and obtain

$$m(m-1) + 2m - 2 = 0, \text{ or } (m-1)(m+2) = 0,$$

whence $m = 1$ or -2. Again, a particular integral is $y = Cx^2$, provided

$$(2-1)(2+2)C = 1, \text{ or } C = \tfrac{1}{4}.$$

Hence

$$y = Ax + \frac{B}{x^2} + \tfrac{1}{4}x^2 \qquad \text{.....................(13).}$$

Complications arise when the equation in m has imaginary roots, or when it has coincident roots; there are, further, difficulties in connection with the particular integral when V contains terms of the type x^p, where p is a root of the equation in m. To avoid special investigation, we shew how the equation (1) can always, by a change of independent variable, be transformed into a linear equation with constant coefficients.

If we put
$$x = e^{\theta} \qquad\qquad (14),$$
then, u being any function whatever of x, we have
$$\frac{du}{dx} = \frac{du}{d\theta} \div \frac{dx}{d\theta} = e^{-\theta}\frac{du}{d\theta},$$
or
$$x\frac{du}{dx} = \frac{du}{d\theta} \qquad\qquad (15).$$

We will denote the operator $d/d\theta$, which is seen to be equivalent to xd/dx, by ϑ. Then, D standing as usual for d/dx, we have
$$xD(x^m D^m u) = x^{m+1} D^{m+1} u + mx^m D^m u,$$
or
$$x^{m+1} D^{m+1} u = (xD - m)(x^m D^m u) = (\vartheta - m)(x^m D^m u) \qquad\qquad (16).$$

Putting $m = 0, 1, 2, \ldots$ in this formula we can express
$$xDu, \quad x^2D^2u, \quad x^3D^3u, \ldots$$
in succession in terms of $\vartheta u, \vartheta^2 u, \vartheta^3 u, \ldots$. Thus, since the operator $\vartheta, = d/d\theta$, is commutative,
$$xDu = \vartheta u,$$
$$x^2D^2u = \vartheta(\vartheta - 1)u,$$
$$x^3D^3u = \vartheta(\vartheta - 1)(\vartheta - 2)u,$$
and so on, the general formula being
$$x^rD^ru = \vartheta(\vartheta - 1)(\vartheta - 2)\ldots(\vartheta - r + 1)u \qquad (17).$$

If we substitute for the several terms of (1) their values as given by (17), we get a linear equation with constant coefficients, of the form
$$f(\vartheta)y = V, \quad \text{or} \quad f\left(\frac{d}{d\theta}\right)y = V \qquad\qquad (18).$$

Ex. 3.
$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x \qquad\qquad (19).$$
This becomes
$$\{\vartheta(\vartheta - 1) - \vartheta + 1\}y = e^{\theta},$$
or
$$(\vartheta - 1)^2 y = e^{\theta}.$$

Allowing for the difference of notation, the solution of this is, by Art. 186,

$$y=(A+B\theta)\,e^{\theta}+\tfrac{1}{2}\theta^2e^{\theta},$$

or, in terms of x,

$$y=(A+B\log x)\,x+\tfrac{1}{2}x\,(\log x)^2 \quad\text{...........}(20).$$

Ex. 4.

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^3 \quad\text{.................}(21).$$

This gives

$$(\vartheta^2+1)\,y=e^{3\theta},$$

$$\begin{aligned}y&=A\cos\theta+B\sin\theta+\tfrac{1}{10}e^{3\theta}\\&=A\cos(\log x)+B\sin(\log x)+\tfrac{1}{10}x^3 \quad\text{.........}(22).\end{aligned}$$

192. Simultaneous Differential Equations.

In dynamical and other problems we often meet with systems of simultaneous differential equations, involving two or more functions of a single independent variable, and their derivatives, the number of equations being always equal to that of the dependent variables. We may denote the dependent variables by the letters $x, y, \ldots$, and the independent variable by t.

Without entering into questions of general theory, it will be sufficient here to give a few examples exhibiting the methods which are most generally useful.

In the first place, it may happen that each of the given equations involves only one of the dependent variables, and so can be treated separately.

Ex. 1. In the case of a projectile under gravity, if the axes of x and y be horizontal and vertical, we have

$$\frac{d^2x}{dt^2}=0,\quad \frac{d^2y}{dt^2}=-g \quad\text{....................}(1).$$

Hence

$$x=A+A't,\quad y=B+B't-\tfrac{1}{2}gt^2 \text{..............}(2).$$

The arbitrary constants A, A', B, B' enable us to satisfy the initial conditions as to position and velocity.

Ex. 2. In the case of a particle attracted to a fixed centre (the origin) with a force varying as the distance, we have

$$\frac{d^2x}{dt^2}=-\mu x,\quad \frac{d^2y}{dt^2}=-\mu y \quad\text{.................}(3),$$

whence

$$x = A_1 \cos \sqrt{\mu}t + A_2 \sin \sqrt{\mu}t, \quad y = B_1 \cos \sqrt{\mu}t + B_2 \sin \sqrt{\mu}t.$$

If we eliminate t, we find

$$(B_1x - A_1y)^2 + (B_2x - A_2y)^2 = (A_1B_2 - A_2B_1)^2 \quad\text{.........(4)},$$

which shews that the path is an ellipse.

If the given equations, which we will suppose to be n in number, are not of this simple type, then by differentiations, and algebraical combinations, we may eliminate all the dependent variables $x, y, z \ldots$, save one (say x). If we substitute the general value of x, hence derived, in the original system, we shall find that this now reduces to $n-1$ equations involving the $n-1$ dependent variables $y, z, \ldots$. The process can be repeated until each dependent variable in turn has been determined as a function of t and of arbitrary constants.

In particular cases various modifications of the above method may suggest themselves. We shall content ourselves with a few illustrations taken from physical problems.

Ex. 3. If x, y be the coordinates of any point in a rigid plane which is rotating with angular velocity n about the origin, we have the equations

$$\frac{dx}{dt} = -ny, \quad \frac{dy}{dt} = nx \quad\text{.....................(5)}.$$

Eliminating y, we have

$$\frac{d^2x}{dt^2} = -n\frac{dy}{dt} = -n^2x,$$

whence

$$x = a\cos(nt + \epsilon) \quad\text{......................(6)},$$

the constants a and ϵ being arbitrary. Substituting in the first of the equations (5), we find

$$y = a\sin(nt + \epsilon) \quad\text{......................(7)}.$$

The results (6) and (7) shew that each point describes a circle about the origin with angular velocity n.

Ex. 4. In the theory of electro-magnetic induction we meet with the equations

$$\left.\begin{aligned} L\frac{dx}{dt} + M\frac{dy}{dt} + Rx = E, \\ M\frac{dx}{dt} + N\frac{dy}{dt} + Sy = F \end{aligned}\right\} \quad\text{..................(8)}.$$

Here x and y denote the currents in two circuits subject to mutual influence; R and S are the resistances of the circuits, L and N the coefficients of self-induction, M that of mutual induction, and E, F are the extraneous electro-motive forces.

Let us first suppose that $E = 0$, $F = 0$. The equations are then satisfied by

$$x = Ae^{\lambda t}, \quad y = Be^{\lambda t} \quad \text{.....................(9)},$$

provided

$$\left.\begin{array}{l}(L\lambda + R)A + M\lambda B = 0, \\ M\lambda A + (N\lambda + S)B = 0\end{array}\right\} \quad \text{.............. (10)}.$$

Eliminating the ratio $A : B$, we have

$$(L\lambda + R)(N\lambda + S) - M^2\lambda^2 = 0,$$

or

$$(LN - M^2)\lambda^2 + (LS + NR)\lambda + RS = 0 \text{.........(11)}.$$

Since

$$(LS + NR)^2 - 4RS(LN - M^2)$$
$$= (LS - NR)^2 + 4M^2RS,$$

a positive quantity, it appears that the roots of the quadratic (11) are always real. Again, for physical reasons, LN is necessarily greater than M^2. Hence (11) shews that the two values of λ must have the same sign (since their product is positive), and further that this sign must be negative (since the sum is negative). Hence, denoting the roots by $-\lambda_1$, $-\lambda_2$, we have the solutions

$$\left.\begin{array}{l}x = A_1e^{-\lambda_1 t}, \quad y = B_1e^{-\lambda_1 t}, \\ x = A_2e^{-\lambda_2 t}, \quad y = B_2e^{-\lambda_2 t}\end{array}\right\} \quad \text{...............(12)},$$

and

where the relation between A_1 and B_1, or A_2 and B_2, is given by either of equations (10) with $-\lambda_1$ or $-\lambda_2$ written for λ. On account of their linear character, the equations (8), with $E = F = 0$, are satisfied by the sums of the above values of x and y, respectively.

If E and F are not zero, but given constants, a particular integral of (8) is evidently

$$x = \frac{E}{R}, \quad y = \frac{F}{S};$$

and the complete solution is

$$\left.\begin{array}{l}x = \dfrac{E}{R} + A_1e^{-\lambda_1 t} + A_2e^{-\lambda_2 t}, \\ y = \dfrac{F}{S} + B_1e^{-\lambda_1 t} + B_2e^{-\lambda_2 t}\end{array}\right\} \quad \text{...............(13)},$$

where the relations between A_1 and B_1, and between A_2 and B_2 are as above indicated.

The first terms in these values of x and y represent the steady currents due to the given electro-motive forces; the remaining terms represent the effects of induction. Since we have, virtually, two arbitrary constants, these can be determined so as to make the actual currents have any given initial values.

Another important case is where E is a simple-harmonic function of the time, and F is zero. Putting, then,

$$E = E_0 \cos pt, \quad F = 0 \quad \text{..................(14)},$$

a particular integral of the equations (8) may be found by assuming

$$\left.\begin{aligned} x &= A \cos pt + A' \sin pt, \\ y &= B \cos pt + B' \sin pt \end{aligned}\right\} \quad \text{...............(15)}.$$

On substitution we find, equating separately the coefficients of $\cos pt$ and $\sin pt$,

$$\left.\begin{aligned} pLA' + pMB' + RA &= E_0, \\ -pLA - pMB + RA' &= 0, \\ pMA' + pNB' + SB &= 0, \\ -pMA - pNB + SB' &= 0 \end{aligned}\right\} \text{..............(16)}.$$

These formulæ give A, A', B, B', and so determine the electrical oscillations in the two circuits due to the given periodic electro-motive force. The free currents are given by terms of the same form as in (12) Their values depend on the initial circumstances, and in any case they die out as t increases.

Ex. 5. As a final example, we take the equations

$$\left.\begin{aligned} A\frac{d^2x}{dt^2} + H\frac{d^2y}{dt^2} + ax + hy &= X, \\ H\frac{d^2x}{dt^2} + B\frac{d^2y}{dt^2} + hx + by &= Y \end{aligned}\right\} \quad \text{............(17)},$$

which determine the motion of any 'conservative' dynamical system, having two degrees of freedom, in the neighbourhood of a position of equilibrium.

To find the free motion, we put $X = 0$, $Y = 0$, and assume

$$x = Fe^{\lambda t}, \quad y = Ge^{\lambda t} \text{.....................(18)}.$$

We thus get
$$\left.\begin{aligned}(A\lambda^2+a)F+(H\lambda^2+h)G=0,\\(H\lambda^2+h)F+(B\lambda^2+b)G=0\end{aligned}\right\}\ \ldots\ldots\ldots\ldots(19).$$
Eliminating the ratio $F : G$, we have
$$(A\lambda^2+a)(B\lambda^2+b)-(H\lambda^2+h)^2=0\ \ \ldots\ldots\ldots(20),$$
or
$$(AB-H^2)\lambda^4+(Ab+Ba-2Hh)\lambda^2+(ab-h^2)=0\ \ldots(21).$$
This is a quadratic in λ^2.

The expressions
$$\tfrac{1}{2}\left\{A\left(\frac{dx}{dt}\right)^2+2H\frac{dx}{dt}\frac{dy}{dt}+B\left(\frac{dy}{dt}\right)^2\right\}\ \ldots\ldots\ldots(22),$$
and
$$\tfrac{1}{2}(ax^2+2hxy+by^2)\ \ \ldots\ldots\ldots\ldots\ldots\ldots(23),$$
represent the kinetic and potential energies of the system. The former is essentially positive; hence A, B are positive and $AB > H^2$. It follows that the left-hand side of (20) or (21) will be positive both for $\lambda^2 = +\infty$ and for $\lambda^2 = -\infty$, whilst for $\lambda^2 = 0$ the sign is that of $ab - h^2$. Also, from (20), it appears that the left-hand member is negative for $\lambda^2 = -a/A$ and for $\lambda^2 = -b/B$.

Hence if the expression (23) be essentially negative, so that a, b are negative, whilst $ab - h^2$ is positive, the equation (21) is satisfied by two positive values of λ^2, one of which is greater, and the other less, than either of the quantities $-a/A$, $-B/b$. Denoting these roots by λ_1^2, λ_2^2, we have the solutions
$$\left.\begin{aligned}x=F_1e^{\lambda_1t}+F_1'e^{-\lambda_1t}+F_2e^{\lambda_2t}+F_2'e^{-\lambda_2t},\\y=G_1e^{\lambda_1t}+G_1'e^{-\lambda_1t}+G_2e^{\lambda_2t}+G_2'e^{-\lambda_2t}\end{aligned}\right\}\ \ldots\ldots\ldots(24).$$
Of the eight coefficients, only four are arbitrary. The ratio $F_1 : G_1$, which is the same as $F_1' : G_1'$, is determined by (19), with λ_1^2 written for λ^2. Similarly, the ratio $F_2 : G_2$ or $F_2' : G_2'$ is determined by the same equations, with λ_2^2 written for λ^2. The four arbitrary constants which remain may be utilized to give any prescribed initial values to x, y, dx/dt, dy/dt. It appears that x and y will increase indefinitely with t, unless the initial circumstances be specially adjusted to make F_1 and F_2 vanish. Hence if the potential energy in the equilibrium position be greater than in any neighbouring position, an arbitrarily started disturbance will in general increase indefinitely; so that the equilibrium position is unstable.

If, on the other hand, the expression (23) be essentially positive, so that a, b, $ab - h^2$ are positive, the roots of the quadratic in λ^2 will both be negative, viz. one will lie between

0 and the numerically smaller of the two quantities $-a/A$, $-b/B$, and the other will lie between the numerically greater of these quantities and $-\infty$. This indicates that in place of (18) the proper assumption now is

$$x = F\cos pt + F'\sin pt, \quad y = G\cos pt + G'\sin pt \ldots(25).$$

This leads to equations of the forms (19) and (21), with $-p^2$ written for λ^2. It follows that the two roots of the quadratic in p^2 will be real and positive; denoting them by p_1^2 and p_2^2, we get the solutions

$$\left.\begin{aligned} x &= F_1\cos p_1t + F_1'\sin p_1t + F_2\cos p_2t + F_2'\sin p_2t, \\ y &= G_1\cos p_1t + G_2'\sin p_1t + G_2\cos p_2t + G_2'\cos p_2t \end{aligned}\right\}\ldots(26),$$

where the ratios F_1/G_1, F_1'/G_1', F_2/G_2, F_2'/G_2' are determined in the same manner as before. Since $F_1'/G_1' = F_1/G_1$, and $F_2'/G_2' = F_2/G_2$, the results may also be written

$$\left.\begin{aligned} x &= F_1\cos(p_1t + \epsilon_1) + F_2\cos(p_2t + \epsilon_2), \\ y &= G_1\cos(p_1t + \epsilon_1) + G_2\cos(p_2t + \epsilon_2) \end{aligned}\right\}\ldots\ldots\ldots(27),$$

where F_1/G_1 and F_2/G_2 are determinate. Hence when the potential energy in the equilibrium position is less than in any neighbouring position, a slight disturbance will merely cause the system to oscillate about the equilibrium position, which is therefore stable.

The two roots of the quadratic in λ^2 (or in p^2) have been assumed to be distinct. It may be proved that they cannot be equal unless $a/A = b/B = h/H$; and that if these conditions be fulfilled the solution is of one or other of the two types:

$$x = Fe^{\lambda t} + F'e^{-\lambda t}, \quad y = Ge^{\lambda t} + G'e^{-\lambda t} \quad \ldots\ldots\ldots(28),$$

$$x = F\cos pt + F'\sin pt, \quad y = G\cos pt + G'\cos pt \ldots(29),$$

where the four constants are in each case independent.

Finally, we have the case where the expression (23) for the potential energy may be sometimes positive and sometimes negative. In this case $ab - h^2$ is negative, and one root of the quadratic in λ^2 is positive, the other negative. The complete solution is now of the type

$$\left.\begin{aligned} x &= Fe^{\lambda t} + F'e^{-\lambda t} + F''\cos pt + F'''\sin pt, \\ y &= Ge^{\lambda t} + G'e^{-\lambda t} + G''\cos pt + G'''\sin pt \end{aligned}\right\}\ \ldots\ldots(30).$$

It is clear that an arbitrary disturbance will in general increase indefinitely, so that the equilibrium position must be reckoned as unstable.

A slightly different method of treating the question is to assume

$$y = \mu x \text{ ....(31).}$$

The equations to be solved now take the form

$$\left.\begin{aligned}(A+\mu H)\frac{d^2x}{dt^2}+(a+\mu h)\,x=0,\\ (H+\mu B)\frac{d^2x}{dt^2}+(h+\mu b)\,x=0\end{aligned}\right\} \text{ ....(32).}$$

These are both satisfied by

$$x = Fe^{\lambda t} \text{ ....(33),}$$

provided

$$\frac{A+\mu H}{a+\mu h}=\frac{H+\mu B}{h+\mu b}=-\frac{1}{\lambda^2} \text{ .... (34).}$$

Hence μ is determined by the quadratic

$$(Hb-Bh)\,\mu^2+(Ab-Ba)\,\mu+(Ah-Ha)=0 \text{ ......(35).}$$

If μ_1, μ_2 be the roots of this, the corresponding values of λ^2 are given by (34). In this way we obtain two solutions, which, on account of the linearity of the differential equations, we may superpose.

It may be shewn that if we eliminate μ from (34), we get the same quadratic to determine λ^2 as before. Hence the condition for the reality of the roots of (35) must be the same as in the case of the quadratic (21). This is easily verified.

If λ^2 be negative, the solutions are of the types

$$\left.\begin{aligned}x=F_1\cos(p_1t+\epsilon_1),\quad y=\mu_1F_1\cos(p_1t+\epsilon_1),\\ x=F_2\cos(p_2t+\epsilon_2),\quad y=\mu_2F_2\cos(p_2t+\epsilon_2)\end{aligned}\right\} \text{......(36),}$$

where F_1, F_2, ϵ_1, ϵ_2 are arbitrary. Either of these by itself represents what is called a 'normal mode' of vibration of the system.

To find the *forced* oscillations when the extraneous forces X, Y are of the type

$$X=\alpha\cos(nt+\epsilon),\quad Y=\beta\sin(nt+\epsilon) \text{ ....(37),}$$

we may assume

$$x=F\cos(nt+\epsilon),\quad Y=G\sin(nt+\epsilon) \text{ ....(38),}$$

and determine the coefficients F, G by substitution in (17). A case of failure may arise, when the expression (23) is essentially positive, owing to n^2 coinciding with one of the roots of the quadratic in p^2.

EXAMPLES. LII.

1. $$\frac{d^2y}{dx^2}+\frac{dy}{dx}=0. \qquad [y=A+Be^{-x}.]$$

2. $$\frac{d^2y}{dx^2}-3\frac{dy}{dx}-10y=0. \qquad [y=Ae^{5x}+Be^{-2x}.]$$

3. $$2\frac{d^2y}{dx^2}-3\frac{dy}{dx}+y=0. \qquad [y=Ae^{x}+Be^{\frac{1}{2}x}.]$$

4. $$\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0. \qquad [y=Ae^{x}+Be^{2x}+Ce^{3x}.]$$

5. $$\frac{d^4y}{dx^4}=m^4y. \quad [y=Ae^{mx}+Be^{-mx}+C\cos mx+D\sin mx.]$$

6. $$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}+\frac{dy}{dx}-y=0. \quad [y=Ae^{x}+B\cos x+C\sin x.]$$

7. $$\frac{d^2y}{dx^2}-2m\frac{dy}{dx}+(m^2+n^2)\,y=0.$$
$$[y=e^{mx}(A\cos nx+B\sin nx).]$$

8. $$\frac{d^2y}{dx^2}-4\frac{dy}{dx}+13y=0. \quad [y=e^{2x}(A\cos 3x+B\sin 3x).]$$

9. $$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+5y=0. \quad [y=e^{-x}(A\cos 2x+B\sin 2x).]$$

10. $$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+13y=0. \quad [y=e^{3x}(A\cos 2x+B\sin 2x).]$$

11. $$\frac{d^4y}{dx^4}+m^4y=0.$$
$$\left[y=\left(A\cosh\frac{mx}{\sqrt{2}}+B\sinh\frac{mx}{\sqrt{2}}\right)\cos\frac{mx}{\sqrt{2}}\right.$$
$$\left.+\left(A'\cosh\frac{mx}{\sqrt{2}}+B'\sinh\frac{mx}{\sqrt{2}}\right)\sin\frac{mx}{\sqrt{2}}.\right]$$

12. $$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}-\frac{dy}{dx}-y=0. \quad [y=Ae^{x}+(B+Cx)\,e^{-x}.]$$

13. $$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=0. \qquad [y=(A+Bx)\,e^x+Ce^{-x}.]$$

14. $$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+4y=0. \qquad [y=(A+Bx)\,e^{2x}+Ce^{-x}.]$$

15. $$\frac{d^3y}{dx^3}-3\frac{dy}{dx}+2y=0. \qquad [y=(A+Bx)\,e^x+Ce^{-2x}.]$$

16. $$\frac{d^3y}{dx^3}-3\frac{d^2y}{dx^2}+3\frac{dy}{dx}-y=0. \qquad [y=(A+Bx+Cx^2)\,e^x.]$$

17. $$\frac{d^4y}{dx^4}+(m^2+n^2)\frac{d^2y}{dx^2}+m^2n^2y=0.$$

$$[y=A\cos(mx+\alpha)+B\cos(nx+\beta).]$$

18. Shew that the solution of the equation

$$\frac{d^2x}{dt^2}+k\frac{dx}{dt}-\mu x=0$$

is of the form $x=Ae^{-\alpha t}+Be^{\beta t}$,

where α, β are both positive (if k and μ are positive) and $\alpha>\beta$.

19. $$\frac{d^2y}{dx^2}-2m\frac{dy}{dx}+m^2y=\sin nx.$$

$$\left[y=\frac{(m^2-n^2)\sin nx+2mn\cos nx}{(m^2+n^2)^2}+\&c.\right]$$

20. $$\frac{d^3y}{dx^3}-\frac{d^2y}{dx^2}-\frac{dy}{dx}+y=\cosh x. \qquad [y=\tfrac{1}{8}x^2e^x+\tfrac{1}{8}xe^{-x}+\&c.]$$

21. $$\frac{d^2y}{dx^2}+\frac{dy}{dx}=1+\cosh x. \qquad [y=x+\tfrac{1}{4}e^x-\tfrac{1}{2}xe^{-x}+\&c.]$$

22. $$\frac{d^4y}{dx^4}-m^4y=\cos kx+\cosh kx.$$

$$\left[y=\frac{1}{k^4-m^4}(\cos kx+\cosh kx)+\&c.\right]$$

23. $$x^2\frac{d^3y}{dx^3}+x\frac{d^2y}{dx^2}-4\frac{dy}{dx}=0. \qquad \left[y=A+\frac{B}{x}+Cx^3.\right]$$

24. Solve $$\frac{d^2V}{dr^2}+\frac{1}{r}\frac{dV}{dr}=0$$

as a homogeneous linear equation. $[V=A\log r+B.]$

25. $$x^2\frac{d^3y}{dx^3}-2\frac{dy}{dx}=0. \qquad [y=A+B\log x+Cx^3.]$$

26. $$x^2\frac{d^2y}{dx^2}-2y=x. \qquad \left[y=Ax^2+\frac{B}{x}-\tfrac{1}{2}x.\right]$$

27. $$x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=\frac{1}{x}. \qquad \left[y=\frac{A}{x}+\frac{B}{x^2}+\frac{\log x}{x}.\right]$$

28. $$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=x^3. \qquad \left[y=Ax+\frac{B}{x}+\tfrac{1}{8}x^3.\right]$$

29. $$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+4y=x^3. \qquad [y=(A+B\log x)\,x^2+x^3.]$$

30. $$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)^2 f(r)=0. \qquad \left[f(r)=\frac{A}{r}+Br+Cr^2+Dr^4.\right]$$

31. Prove that
$$f\left(x\frac{d}{dx}\right)x^m u=x^m f\left(x\frac{d}{dx}+m\right)u.$$

32. Prove that
$$f\left(x\frac{d}{dx}\right)x^m\log x=x^m\{f(m)\log x+f'(m)\}.$$

33. $$\frac{dx}{dt}+7x-y=0, \qquad \frac{dy}{dt}+2x+5y=0.$$
$$[x=\tfrac{1}{2}\{(A+B)\sin t+(A-B)\cos t\}\,e^{-6t},$$
$$y=(A\cos t+B\sin t)\,e^{-6t}.]$$

34. $$\frac{dx}{dt}=3x-y, \qquad \frac{dy}{dt}=x+y.$$
$$[x=(A+Bt)\,e^{2t}, \quad y=(A-B+Bt)\,e^{2t}.]$$

35. $$\frac{dx}{dt}+5x+y=e^t, \qquad \frac{dy}{dt}+3y-x=e^{2t}.$$
$$[x=(A+Bt)\,e^{-4t}+\tfrac{4}{25}e^t-\tfrac{1}{36}e^{2t},$$
$$y=-(A+B+Bt)\,e^{-4t}+\tfrac{1}{25}e^t+\tfrac{7}{36}e^{2t}.]$$

36. $$\frac{d^2x}{dt^2}+3x-4y+3=0, \qquad \frac{d^2y}{dt^2}+x-y+5=0.$$
$$[x=(A+Bt)\cos t+(A'+B't)\sin t-17,$$
$$y=\tfrac{1}{2}(A+B'+Bt)\cos t+\tfrac{1}{2}(A'-B+B't)\sin t-12.]$$

37. Determine the constants in the solution of the simultaneous equations

$$\frac{d^2x}{dt^2}=\mu x, \quad \frac{d^2y}{dt^2}=\mu y$$

so that, for $t=0$,

$$x=a, \quad y=0, \quad \frac{dx}{dt}=0, \quad \frac{dy}{dt}=V.$$

$[x=a\cosh\sqrt{\mu}t, \quad y=V/\sqrt{\mu}\,.\sinh\sqrt{\mu}t.]$

38. The equations of motion of electricity in a circuit of self-induction L, and resistance R, which is interrupted by a condenser of capacity C, are

$$L\frac{dx}{dt}+Rx=-\frac{q}{C}, \quad \frac{dq}{dt}=x,$$

where x is the current, and q the charge of the condenser. Find the condition that the discharge should be oscillatory. $[L>\frac{1}{4}R^2C.]$

39. Solve $$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=-\frac{\mu}{y^3},$$

and shew that the solution represents a conic symmetrical with respect to the axis of x.

40. Solve $$\frac{d^2x}{dt^2}=0, \quad \frac{d^2y}{dt^2}=\mu y^3,$$

and prove that the curves represented by the solution include a family of hyperbolas.

41. Solve $$\frac{d^2x}{dt^2}=-n\frac{dy}{dt}+f, \quad \frac{d^2y}{dt^2}=n\frac{dx}{dt}.$$

$$\left[x=\alpha+a\cos(nt+\epsilon), \quad y=\beta+\frac{f}{n}t+a\sin(nt+\epsilon).\right]$$

42. Solve $$\frac{d^2x}{dt^2}-2n\frac{dy}{dt}+m^2x=0,$$
$$\frac{d^2y}{dt^2}+2n\frac{dx}{dt}+m^2y=0.$$

$$\left[\begin{aligned} x&=A\cos(pt+\epsilon)+A'\cos(p't+\epsilon'),\\ y&=-A\sin(pt+\epsilon)+A'\sin(p't+\epsilon'),\end{aligned}\right.$$

$$\left.\text{where } \begin{matrix}p\\p'\end{matrix}\right\}=\sqrt{(m^2+n^2)}\pm n\Big].$$

43. Solve $\dfrac{d^2x}{dt^2} + m^2y = 0, \quad \dfrac{d^2y}{dt^2} - m^2x = 0.$

$$\left[x = Ae^{\frac{mt}{\sqrt{2}}} \cos\left(\frac{mt}{\sqrt{2}} + a\right) + Be^{-\frac{mt}{\sqrt{2}}} \cos\left(\frac{mt}{\sqrt{2}} + \beta\right),\right.$$

$$\left. y = Ae^{\frac{mt}{\sqrt{2}}} \sin\left(\frac{mt}{\sqrt{2}} + a\right) - Be^{-\frac{mt}{\sqrt{2}}} \sin\left(\frac{mt}{\sqrt{2}} + \beta\right).\right]$$

44. Shew that the integrals of

$$\frac{dx}{dt} = ax + hy, \quad \frac{dy}{dt} = hx + by,$$

are $\quad x = A_1e^{\lambda_1 t} + A_2e^{\lambda_2 t}, \quad y = \mu_1 A_1 e^{\lambda_1 t} + \mu_2 A_2 e^{\lambda_2 t},$

where μ_1, μ_2 are the roots of

$$\mu^2 + \frac{a-b}{h}\mu - 1 = 0,$$

and $\quad \lambda_1 = a + h\mu_1, \quad \lambda_2 = a + h\mu_2.$

45. Solve $\dfrac{d^2x}{dt^2} = ax + hy, \quad \dfrac{d^2y}{dt^2} = hx + by.$

46. Solve $\dfrac{d^2x}{dt^2} + m^2x - n^2y = 0, \quad \dfrac{d^2y}{dt^2} + m^2y - n^2x = 0.$

$$[x + y = A \cos \sqrt{(m^2 - n^2)}\, t + A' \sin \sqrt{(m^2 - n^2)}\, t,$$

$$x - y = B \cos \sqrt{(m^2 + n^2)}\, t + B' \sin \sqrt{(m^2 + n^2)}\, t.]$$

47. Solve

$$\left.\begin{aligned} \frac{d^2x}{dt^2} + (\mu + 1)\frac{g}{a}x - \mu\frac{g}{b}y = 0, \\ \frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{g}{b}y = 0 \end{aligned}\right\}.$$

(The equations of motion of a double pendulum, the lengths of the upper and lower strings being a and b, and μ being the ratio of the mass of the lower to that of the upper particle.)

Prove that the periods of the normal vibrations are $2\pi/p_1$, $2\pi/p_2$, if p_1^2, p_2^2 be the roots of

$$p^4 - (1 + \mu)\, g\left(\frac{1}{a} + \frac{1}{b}\right)p^2 + (1 + \mu)\frac{g^2}{ab} = 0.$$

Shew that the roots of this quadratic in p^2 are real, positive, and distinct.

CHAPTER XIII.

DIFFERENTIATION AND INTEGRATION OF INFINITE SERIES.

193. Statement of the Question.

The main object of this Chapter is to justify, under proper conditions, the application of the processes of differentiation and integration to functions expressed by 'power-series,' *i.e.* by series of the type

$$A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n + \ldots \ldots \quad (1),$$

where the coefficients are constants. Thus, if $S(x)$ denote the sum of this series, assumed to be convergent for all values of x extending over a certain range, we have to examine under what conditions it can be asserted that $S(x)$ is a continuous and differentiable function of x, and that, moreover,

$$S'(x) = A_1 + 2A_2 x + 3A_3 x^2 + \ldots + nA_n x^{n-1} + \ldots \ldots \quad (2),$$

and

$$\int_0^x S(x)\,dx = A_0 x + \tfrac{1}{2} A_1 x^2 + \tfrac{1}{3} A_2 x^3 + \ldots + \frac{1}{n+1} A_n x^{n+1} + \ldots \ldots \quad (3),$$

respectively.

If the number of terms in (1) were *finite*, the statements in question would need no proof, beyond what has already been indicated in the course of this work (see Arts. 36, 73), but it must be remembered that the word 'sum' as applied to an infinite series bears an artificial sense, and that we are not entitled to assume without examination that statements which are true when the word has one meaning remain true when it is used in another.

We shall find it convenient, throughout this Chapter, to denote the sum of the first n terms of the series (1) by $S_n(x)$; thus

$$S_n(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_{n-1} x^{n-1} \quad \ldots\ldots (4),$$

a rational integral function of x, of degree $n-1$. We shall further write

$$S(x) = S_n(x) + R_n(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The quantity $R_n(x)$ is called the 'remainder after n terms'; it is of course the sum of the series

$$A_n x^n + A_{n+1} x^{n+1} + \ldots \quad \ldots\ldots\ldots\ldots (6).$$

By hypothesis, the sequence

$$S_1(x), \quad S_2(x), \quad S_3(x), \ldots \quad \ldots\ldots\ldots\ldots (7)$$

has (Art. 6) the limiting value $S(x)$. It follows that the sequence

$$R_1(x), \quad R_2(x), \quad R_3(x), \ldots \quad \ldots\ldots\ldots\ldots (8)$$

must have the limiting value zero.

Ex. In the case of the geometric series

$$1 + x + x^2 + \ldots + x^n + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

we have

$$S_n(x) = \frac{1-x^n}{1-x}, \quad S(x) = \frac{1}{1-x}, \quad R_n(x) = \frac{x^n}{1-x} \quad \ldots\ldots (10),$$

provided $|x| < 1$.

It is to be noted that in each of the questions above propounded we have to deal with what is known as a 'double limit.' Thus, to establish the continuity of $S(x)$ for $x = a$ we have to shew that

$$\lim_{n=\infty} \lim_{x=a} S_n(x) = \lim_{x=a} \lim_{n=\infty} S_n(x) \quad \ldots\ldots\ldots\ldots (11).$$

Again, the formulæ (2) and (3) may be written

$$\frac{d}{dx} \lim_{n=\infty} S_n(x) = \lim_{n=\infty} \frac{d}{dx} S_n(x) \quad \ldots\ldots\ldots (12),$$

and

$$\int_0^x \{\lim_{n=\infty} S_n(x)\}\, dx = \lim_{n=\infty} \int_0^x S_n(x)\, dx \quad \ldots\ldots (13),$$

respectively. Since a derived function is the limit of a quotient, and an integral is the limit of a sum, these forms also come under the description of a double limit.*

194. Uniform Convergence of Power-Series.

We first give a simple, and frequently sufficient, test of the convergency of the series

$$A_0 + A_1x + A_2x^2 + \dots + A_nx^n + \dots \quad \dots\dots(1).$$

If a quantity α can be found, such that the quantities

$$A_0,\ A_1\alpha,\ A_2\alpha^2,\ \dots\dots\ A_n\alpha^n,\ \dots \quad \dots\dots\dots\dots(2)$$

do not increase indefinitely (in absolute magnitude), the series (1) will be 'essentially' convergent (see Art. 6) for all values of x such that

$$|x| < |\alpha|.$$

For, writing the series in the form

$$A_0 + A_1\alpha\left(\frac{x}{\alpha}\right) + A_2\alpha^2\left(\frac{x}{\alpha}\right)^2 + \dots + A_n\alpha^n\left(\frac{x}{\alpha}\right)^n + \dots \quad \dots(3),$$

and denoting by M the upper limit of the absolute values of the quantities (2), we see that the several terms of (3) are in absolute value less than the corresponding terms of the convergent geometrical series

$$M(1 + t + t^2 + \dots + t^n + \dots) \quad \dots\dots\dots\dots (4),$$

where $$t = |x/\alpha|.$$

Hence the series (3) is itself essentially convergent.

It follows that if the series (1) be convergent for any one value (α) of x, other than 0, it will be convergent for values of x ranging from $x = -k$ to $x = k$, inclusively, where k may be any positive quantity less than $|\alpha|$. For if the series be convergent for $x = \alpha$, the limiting value of $A_n\alpha^n$, as n is increased indefinitely, must be 0.

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* **For further examples see Art. 114 (11) and Art. 210.**

Ex. The coefficients of the series

$$1+\tfrac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\dots \qquad\dots\dots\dots\dots(5)$$

are all less than unity. The series is therefore convergent for all values of x *between* ± 1.

The argument does not entitle us to assert that the series (5) is convergent for the extreme values $x=\pm 1$; it is in fact divergent for $x=1$.

Again, we have

$$R_n(x)=A_n\alpha^n\left(\frac{x}{\alpha}\right)^n+A_{n+1}\alpha^{n+1}\left(\frac{x}{\alpha}\right)^{n+1}+\dots \qquad\dots(6),$$

and therefore, considering absolute values,

$$|R_n(x)|<M(t^n+t^{n+1}+\dots) \quad\text{or}\quad \frac{Mt^n}{1-t}\dots\dots\dots(7).$$

Hence if x do not exceed the limits $\pm k$, where $k<|\alpha|$, we have, *a fortiori*,

$$|R_n(x)|<\frac{Mt_1{}^n}{1-t_1}\dots\dots\dots\dots\dots\dots\dots\dots(8),$$

where $\qquad t_1=|k/\alpha|$.

Hence, σ being any assigned magnitude, however small, we can always find a finite number n such that the quantities

$$R_n(x),\quad R_{n+1}(x),\quad R_{n+2}(x),\dots \qquad\dots\dots\dots\dots(9),$$

shall remain *constantly* less in absolute value than σ as x ranges from $-k$ to k, inclusively.

The word italicized is important. If a series

$$S(x)=u_0(x)+u_1(x)+u_2(x)+\dots+u_n(x)+\dots \qquad\dots(10),$$

whose several terms are any given functions of x, be convergent for all values of x extending over a certain range, it is implied that for each such value of x a value of n can be found such that every member of the sequence

$$R_n(x),\ R_{n+1}(x),\ R_{n+2}(x),\dots \qquad\dots\dots(11),$$

where $R_n(x)$ denotes the remainder after n terms, shall be less* than σ, where σ may be any assigned quantity however small. If we suppose σ to be fixed, the lowest value of n fulfilling this condition will in general vary with x; in other words, the number of terms of the series (10) which must be taken in order to obtain a prescribed degree of approximation to the sum $S(x)$ will vary with x. But if, for each value of σ, a value of n can be assigned such that the above condition is fulfilled for all values of x belonging to the range in question, the series (10) is said to be 'uniformly' convergent over that range.

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* In absolute value.

The preceding investigation leads to the result that a power-series is uniformly convergent over any range which is included *within* the range of convergency*. More generally, if it can be shewn that for all values of x included in a certain range ($a \leqq x \leqq b$, say)† each term in the series (10) is less in absolute value than the corresponding term in a series of constant positive quantities

$$\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_n + \ldots \quad \ldots\ldots\ldots\ldots\ldots(12),$$

which is known to be convergent, the series (10) will be uniformly convergent throughout that range. For if ρ_n denote the remainder after n terms in (12), we have

$$|R_n(x)| < \rho_n, \quad |R_{n+1}(x)| < \rho_{n+1} < \rho_n, \quad |R_{n+2}(x)| < \rho_{n+2} < \rho_n, \quad \ldots(13).$$

for $a \leqq x \leqq b$, and we can determine n so that ρ_n shall be less than any prescribed quantity σ, however small.

It is not difficult to construct infinite series which shall be convergent, but not uniformly convergent, over a certain range. We have only to choose a suitable form for $S_n(x)$; the corresponding series is then obtained by writing

for $m > 1$, and

$$\left.\begin{aligned} u_m(x) &= S_{m+1}(x) - S_m(x), \\ u_0(x) &= S_1(x) \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots(14).$$

Ex. 1. If

$$S_n(x) = \frac{2nx}{1 + n^2x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

we have

$$S(x) = \lim_{n=\infty} S_n(x) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(16),$$

for all values of x without exception. The series is however not *uniformly* convergent over any range which includes $x = 0$. For

$$R_n(x) = S(x) - S_n(x) = -\frac{2nx}{1 + n^2x^2} \quad \ldots\ldots\ldots\ldots (17),$$

and for any given value of n this attains the values ∓ 1 for $x = \pm 1/n$.

---

* Thus a power-series which is convergent for $x = a$, where (for definiteness) we suppose a positive, but not for $x > a$, has been proved to be uniformly convergent for values of x ranging from 0 to any fixed value short of a. As a matter of fact it will be uniformly convergent up to $x = a$, *inclusively*, but this cannot be established by the above method.

† By this notation it is meant that x may range from a to b *inclusively*.

The question may be illustrated graphically* by drawing the curve $y = S(x)$, and also the 'approximation-curves' $y = S_n(x)$, for $n = 1, 2, 3, \ldots$†. If the series be uniformly convergent, then, however small the value of σ, from some finite value of n onwards the approximation-curves will all lie within the strip of the plane xy bounded by the curves

$$y = S(x) \pm \sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(18).$$

In the above Ex. the approximation-curve $y = S_n(x)$ is derived from Fig. 17, p. 31, by contracting all the abscissae in the ratio $1/n$. The curves tend ultimately to lie between the limits $y = \pm\sigma$ for the greater part of their course, but (if $\sigma < 1$) they will always transgress these boundaries in the neighbourhood of the origin. See Fig. 149.

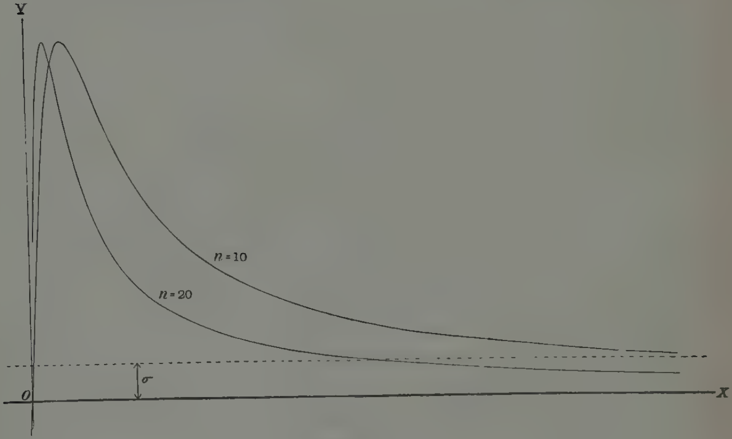

Fig. 149.

Ex. 2. If $$S_n(x) = \tanh nx \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19),$$

we have $$S(x) = \lim_{n=\infty} \tanh nx = -1,\ 0,\ +1 \quad \ldots\ldots\ldots\ldots(20),$$

according as x is negative, zero, or positive; see Fig. 22, p. 43. The annexed Fig. 150 shews that no approximation-curve lies wholly within the required limits, in the immediate neighbourhood of the origin.

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* The author is indebted, here and in Arts. 196, 198, to a paper by Prof. W. F. Osgood, "A Geometrical Method for the Treatment of Uniform Convergence and Certain Double Limits," *Bull. Amer. Math. Soc.*, 2nd ser., t. 3 (1896).

† See for example Fig. 152, Art. 203.

We have here an instance of a series whose sum is a discontinuous function of x, although the individual terms are themselves continuous.

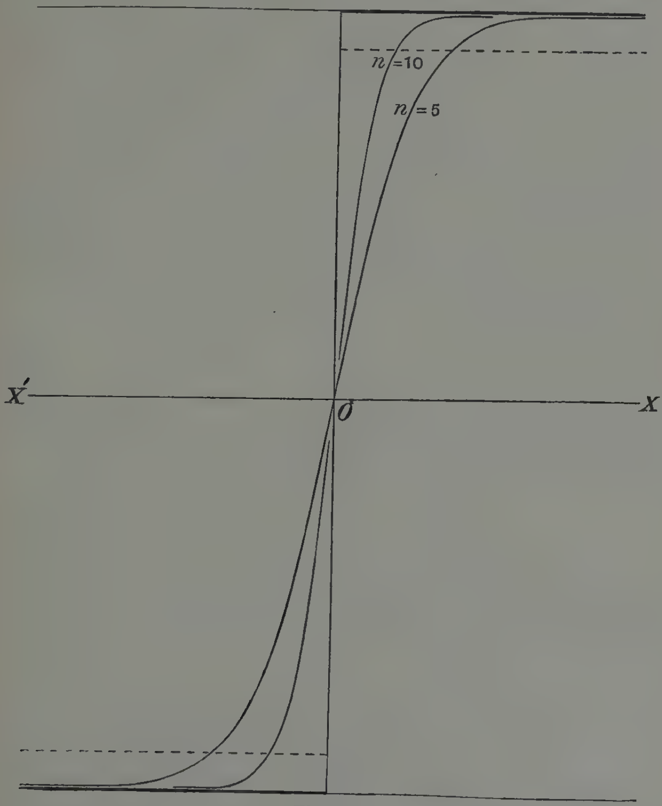

Fig. 150.

195. **Continuity of the Sum of a Power-Series.**

We can now shew that the sum of a convergent power-series is a continuous function of x for all values of x *within* the range of convergence.

If x, x' denote any two values of the variable within this range, we have, with the notation of Art. 193,

$$S(x') - S(x) = S_n(x') - S_n(x) + R_n(x') - R_n(x) \ldots (1),$$

and the condition of continuity is that, σ being any assigned magnitude, however small, a value of δx shall exist such that

$$|S(x') - S(x)| < \sigma \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

for values of x' ranging from x to $x + \delta x$, inclusively.

It appears from Art. 194 that a finite value of n can be found such that

$$|R_n(x)| < \tfrac{1}{3}\sigma, \text{ and } |R_n(x')| < \tfrac{1}{3}\sigma,$$

for *all* values of x and x' within the range of convergence, and therefore such that

$$|R_n(x') - R_n(x)| < \tfrac{2}{3}\sigma \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Again, n being chosen so as to satisfy this condition, $S_n(x)$ is a rational integral function of finite degree n, and is therefore continuous for all values of x, by Art. 14. It follows that a quantity δx exists such that

$$|S_n(x') - S_n(x)| < \tfrac{1}{3}\sigma \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

for values of x' ranging from x to $x + \delta x$.

It follows then from (1), and from the inequalities (3) and (4), that the condition (2) is satisfied.

The reader will notice that the only assumptions which have been made with respect to the series are that its terms are continuous functions of x, and that it is 'uniformly' convergent. Hence we can assert that if a series whose terms are continuous functions of x be uniformly convergent over a certain range of x, the sum will be continuous over the same range.

On the other hand, if the series be convergent, but not uniformly convergent, the sum may or may not be continuous. This is illustrated by the Exx. given in Art. 194.

196. Integration of a Power-Series.

With the same notation as before we can shew further that, if x lie within the range of convergence of the series (1) of Art. 193, the integral

$$\int_0^x S(x)\,dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

will be determinate and equal to the sum of the series

$$A_0x + \tfrac{1}{2}A_1x + \tfrac{1}{3}A_2x^3 + \ldots + \frac{1}{n+1}A_nx^{n+1} + \ldots \;\ldots (2),$$

obtained by integrating the series $S(x)$ term by term. That this latter series is convergent follows from the fact that its terms are respectively less in absolute value than those of the series

$$x(A_0 + A_1x + A_2x^2 + \ldots + A_nx^n + \ldots).$$

We write as usual

$$S(x) = S_n(x) + R_n(x) \;\ldots\ldots\ldots\ldots\ldots (3).$$

Since $S_n(x)$ is the sum of a finite number of integrable functions, we have

$$\int_0^\xi S_n(x)\,dx = \int_0^\xi (A_0 + A_1x + \ldots + A_{n-1}x^{n-1})\,dx$$
$$= A_0\xi + \tfrac{1}{2}A_1\xi^2 + \tfrac{1}{3}A_2\xi^3 + \ldots + \frac{1}{n}A_{n-1}\xi^n \;\ldots\ldots (4).$$

Now let ξ be any quantity within the range of convergence of the original series, and let σ be the upper limit to the absolute values of the function $R_n(x)$ over the range from 0 to ξ. If we refer to the fundamental definition of Art. 86, it will appear that the process there described, when applied to the function on the right-hand of (3), leads to a result which is so far determinate that it must be intermediate in value to the two quantities

$$\int_0^\xi S_n(x)\,dx \pm \sigma\xi \;\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

This conclusion holds, however great the value of n. It has been shewn in Art. 194 that as n is indefinitely increased, σ tends to the limiting value 0; and the limiting value of the first term in (5) is the series

$$A_0\xi + \tfrac{1}{2}A_1\xi^2 + \tfrac{1}{3}A_2\xi^3 + \ldots + \frac{1}{n+1}A_n\xi^{n+1} + \ldots \quad \ldots(6).$$

It follows that the integral

$$\int_0^\xi S(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

is completely determinate, and that it is equal to the sum of the series (6).

If we now write x for ξ, we have the result as stated at the head of this Art.

By a slight generalization of the above method it may be shewn that if a series whose terms are continuous functions of x be *uniformly* convergent over a given range, the series derived from this by integrating it, term by term, between limits belonging to the aforesaid range, will be convergent, and will have for its sum the integral of the sum of the former series. In symbols, if $S(x)$ denote the sum of the infinite series

$$u_0(x) + u_1(x) + u_2(x) + \ldots \quad \ldots\ldots\ldots\ldots\ldots(8),$$

where $u_0(x)$, $u_1(x)$, $u_2(x)$, ... are continuous functions of x, and the series be uniformly* convergent over a given range which includes the values a, b, then the integral

$$\int_a^b S(x)\,dx \quad \ldots\ldots\ldots\ldots\ldots\ldots(9)$$

will be determinate, and the series

$$\int_a^b u_0(x)\,dx + \int_a^b u_1(x)\,dx + \int_a^b u_2(x)\,dx + \ldots \quad \ldots\ldots(10)$$

will be convergent, and the sum of the series will be equal to the value of the integral.

---

* This condition is essential to the above proof. If it be not fulfilled the statement is not necessarily true. The correct theory of the matter is due to Weierstrass.

Ex. 1. It was shewn in Art. 109 that the perimeter of an ellipse whose semi-major axis is a and eccentricity e is

$$4a\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\phi)^{\frac{1}{2}}\,d\phi \;\ldots\ldots\ldots\ldots\ldots\ldots(11).$$

The radical can be expanded by the Binomial Theorem (see Art. 200) in the series

$$1-\frac{1}{2}e^2\sin^2\phi-\frac{1\,.\,1}{2\,.\,4}e^4\sin^4\phi-\frac{1\,.\,1\,.\,3}{2\,.\,4\,.\,6}e^6\sin^6\phi-\ldots\ldots(12),$$

which is uniformly convergent from $\phi=0$ to $\phi=\frac{1}{2}\pi$, since $e^2<1$. Integrating this term by term, and making use of the formula (8) of Art. 95, we obtain, for the value of the perimeter, the expression

$$2\pi a\left(1-\frac{1}{2^2}e^2-\frac{1^2\,.\,3}{2^2\,.\,4^2}e^4-\frac{1^2\,.\,3^2\,.\,5}{2^2\,.\,4^2\,.\,6^2}e^6-\ldots\right)\ldots\ldots(13).$$

Ex. 2. As an example of a series which is not integrable term by term, let

$$S_n(x)=nxe^{-nx^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14).$$

If $x\neq 0$, we have

$$S(x)=\lim_{n=\infty} nxe^{-nx^2}=\frac{1}{x}\lim_{z=\infty} ze^{-z}=0,$$

by Art. 28, Ex. 3, whilst it is obvious that $S_n(0)=0$ and therefore $S(0)=0$. Hence, ω being any quantity,

$$\int_0^\omega S(x)\,dx=0 \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15).$$

On the other hand the sum of the integrals of the several terms is

$$\lim_{n=\infty}\int_0^\omega S_n(x)\,dx=\lim_{n=\infty}\left[-\tfrac{1}{2}e^{-nx}\right]_0^\omega$$

$$=\lim_{n=\infty}\tfrac{1}{2}(1-e^{-n\omega^2})=\tfrac{1}{2}\;\ldots\ldots\ldots\ldots\ldots\ldots(16).$$

The reason why the previous argument does not apply in this case is that the original series is not uniformly convergent over any range which includes the origin. It is easily ascertained that the approximation-curve $y=S_n(x)$ has maximum and minimum ordinates $\pm\sqrt{(n/2e)}$ for $x=\pm 1/\sqrt{(2n)}$, respectively. These ordinates increase indefinitely, and approach the axis of y indefinitely, as n is increased, but the area included between the curve and (say) the positive part of the axis of x remains finite and > 0.

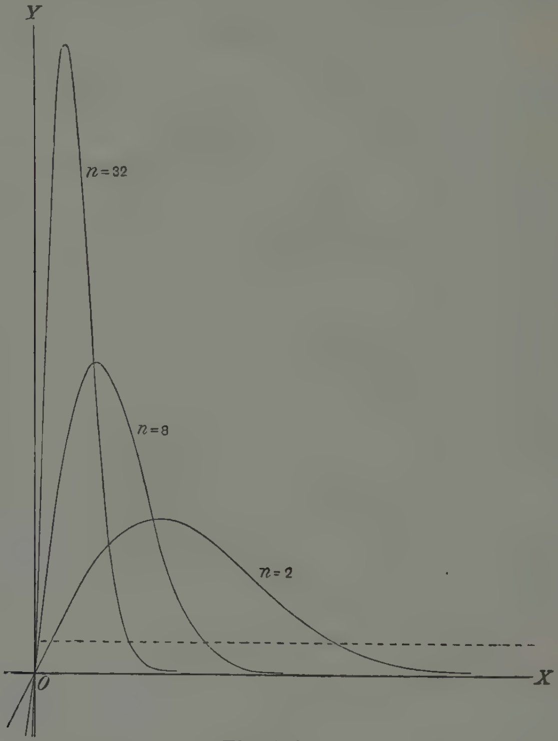

Fig. 151.

197. Derivation of the Logarithmic Series, and of Gregory's Series.

The following are important applications of the theorem just proved:

1°. If $|x| < 1$, we have

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots \qquad \text{...........(1)}.$$

Hence, integrating between the limits 0 and x,

$$\log(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots \quad \dots\dots\dots(2),$$

which is the 'logarithmic series*.'

The proof applies only for $|x| < 1$; and we cannot assert without examination that the result is valid for $x = \pm 1$.

For $x=1$, the terms on the right-hand of (2) are alternately positive and negative, and diminish continually, and without limit, in absolute value. Hence, by a known theorem (Art. 6), the sum is finite, and the remainder after n terms is in absolute value less than $1/(n+1)$. Hence this remainder can be made less than any assigned magnitude by taking n sufficiently great. The same result holds *à fortiori* when x is positive and < 1. Hence the series on the right-hand of (2) is uniformly convergent up to the point $x=1$ inclusive; it follows by the method of Art. 195 that the sum is continuous for values of x ranging up to unity, inclusive. The two sides of (2) are therefore finite and continuous up to $x=1$, and since they have been proved equal for values of x which may be taken as nearly equal to unity as we please, it follows that they are rigorously equal for $x=1$. Hence

$$\log 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots \quad \dots\dots\dots\dots(3).$$

This formula, though exact, is not suited for actual calculation, on account of the very slow convergence of the series. It may be shewn that about 10^n terms would have to be included in order to obtain a result accurate to n places of decimals.

If we subtract from (2) the corresponding series obtained by reversing the sign of x, we obtain

$$\log \frac{1+x}{1-x} = 2\left(x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \dots\right) \dots\dots\dots(4),$$

which might also have been obtained directly by integration from the identity

$$\frac{1}{1+x} + \frac{1}{1-x} = \frac{2}{1-x^2} = 2\left(1 + x^2 + x^4 + \dots\right)\dots\dots(5).$$

---

* Apparently first obtained by N. Mercator in 1668.

If in (4) we put $x = 1/(2m+1)$, we obtain

$$\log(m+1) - \log m = \log\frac{m+1}{m}$$
$$= 2\left\{\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \dots\right\} \quad\dots\dots\dots(6).$$

This series is very convergent, even for $m = 1$. Putting $m = 1$, 2, 3, ..., we obtain the values of

$$\log 2,\ \log 3 - \log 2,\ \log 4 - \log 3, \dots$$

in succession, and thence the values of the logarithms (to base e) of the natural numbers 2, 3, ... When log 10 has been found, its reciprocal gives the modulus μ by which logarithms to base e must be multiplied in order to convert them into logarithms to base 10*.

2°. If $x^2 < 1$ we have

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots \quad\dots\dots\dots\dots(7).$$

Hence, by integration between the limits 0 and x,

$$\tan^{-1} x = x - \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 + \dots \quad\dots\dots\dots(8).$$

This is known as 'Gregory's Series†.'

The function $\tan^{-1} x$ here appears as the equivalent of

$$\int_0^x \frac{dx}{1+x^2},$$

and must therefore be taken to denote that value of the 'multiform' function $\tan^{-1} x$ which lies between $\pm\frac{1}{4}\pi$.

For reasons similar to those given at length in connection with the formula (2), the result (8) holds up to both the limits ± 1 of x, inclusively. Putting $x = 1$, we find

$$\tfrac{1}{4}\pi = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \dots \quad\dots\dots\dots\dots\dots(9).$$

---

* The most rapid way of determining μ is by means of the identity
$$\log 10 = 3\log 2 + \log\tfrac{5}{4}.$$
The two logarithms on the right hand are found by putting $m = 1$, $m = 4$, in (6).

† After the discoverer James Gregory (1671).

The series (9) converges very slowly, and has been superseded for the calculation of π by others. Euler used the identity

$$\tfrac{1}{4}\pi = \tan^{-1}\tfrac{1}{2} + \tan^{-1}\tfrac{1}{3} \quad \ldots\ldots\ldots\ldots\ldots\ldots (10),$$

which gives

$$\tfrac{1}{4}\pi = \left(\frac{1}{2} - \frac{1}{3\,.\,2^3} + \frac{1}{5\,.\,2^5} - \ldots\right) + \left(\frac{1}{3} - \frac{1}{3\,.\,3^3} + \frac{1}{5\,.\,3^5} - \ldots\right) \ldots (11).$$

Machin had previously employed the formula

$$\tfrac{1}{4}\pi = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} \quad \ldots\ldots\ldots\ldots\ldots (12),$$

which, like (10), is proved in most elementary books on Trigonometry. This leads to

$$\tfrac{1}{4}\pi = 4\left(\frac{1}{5} - \frac{1}{3\,.\,5^3} + \frac{1}{5\,.\,5^5} - \ldots\right) - \left(\frac{1}{239} - \frac{1}{3\,.\,239^3} + \frac{1}{5\,.\,239^5} - \ldots\right)$$
$$\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

On account of the importance of the matter, it is worth while to give the details of the calculation of π from Machin's formula. To calculate $\tan^{-1}\frac{1}{5}$, we first draw up the following table:

n	$\frac{1}{5^n}$	$\pm\frac{1}{n\,.\,5^n}$
1	·200 000 000 0	+ ·200 000 000 0
3	8 000 000 0	− 2 666 666 7
5	320 000 0	+ 64 000 0
7	12 800 0	− 1 828 6
9	512 0	+ 56 9
11	20 5	− 1 9
13	8	+ 1

The sum of the positive terms in the last column is

+ ·200 064 057 0,

and that of the negative terms is

− ·002 668 497 2.

Hence $\tan^{-1}\frac{1}{5}$ = ·197 395 559 8.

Again to calculate $\tan^{-1}\frac{1}{239}$ we have the table:

n	$\dfrac{1}{239^n}$	$\pm\dfrac{1}{n\,.\,239^n}$
1	·004 184 100 4	+ ·004 184 100 4
3	73 2	− 24 4

This makes $\tan^{-1}\frac{1}{239} = \cdot 004\ 184\ 076\ 0.$

Hence
$$\begin{aligned}\tfrac{1}{4}\pi &= 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239}\\ &= +\cdot 789\ 582\ 239\ 2\\ &\quad -\cdot 004\ 184\ 076\ 0\\ &= \cdot 785\ 398\ 163\ 2,\\ \pi &= 3\cdot 141\ 592\ 652\ 8.\end{aligned}$$

The last figures are of course liable to error. To estimate the possible error of the final result, we remark that in the calculation of $\tan^{-1}\frac{1}{5}$ there were five errors, each not exceeding half a unit in the last place, and that there are two such errors in the computation of $\tan^{-1}\frac{1}{239}$. The errors, therefore, in the inferred value of π, even if cumulative, cannot exceed

$$4 \times (4 \times 5 \times \tfrac{1}{2} + 2 \times \tfrac{1}{2}),\ = 44,$$

times the unit of the last place. The first seven decimal places cannot therefore be affected, and we can assert that the last three must lie within the limits 484 and 572. As a matter of fact the errors are not all in the same direction, and the correct value of π to ten places is

$$\pi = 3\cdot 141\ 592\ 653\ 6.$$

3°. If $|x| < 1$, we have, by the Binomial Theorem (see Art. 200),

$$\frac{1}{\sqrt{(1-x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \ldots\ldots(14).$$

Hence, integrating term by term between the limits 0 and x,

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{x^5}{5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{x^7}{7} + \ldots\ldots(15);$$

a series due to Newton.

If we put $x=\frac{1}{2}$, we get

$$\pi=6\left\{\frac{1}{2}+\frac{1}{2\,.\,3\,.\,2^3}+\frac{1\,.\,3}{2\,.\,4\,.\,5\,.\,2^5}+\ldots\right\}\ldots\ldots(16),$$

from which π can be calculated without much trouble.

198. Differentiation of a Power-Series.

If the series

$$A_0+A_1x+A_2x^2+\ldots+A_nx^n+\ldots\ldots\ldots\ldots(1)$$

be convergent for any one value (α) of x, the series

$$A_1+2A_2x+3A_3x^2+\ldots+nA_nx^{n-1}+\ldots\ldots\ldots\ldots(2),$$

composed of the derivatives of the successive terms of (1), will be essentially convergent for all values of x such that $|x|<|\alpha|$.

For the hypothesis requires that $|A_n\alpha^n|$ shall tend with increasing n to the limiting value 0. Hence if M denote the greatest of the quantities

$$|A_1\alpha|,\quad |A_2\alpha^2|,\ldots|A_n\alpha^n|,\ldots\ldots\ldots\ldots(3),$$

the terms of (2) will be in absolute value respectively less than the corresponding terms of the series

$$\frac{M}{\alpha}(1+2t+3t^2+\ldots+nt^{n-1}+\ldots)\ \ldots\ldots\ldots\ (4),$$

where $$t=|x/\alpha|.$$

The ratio of successive terms in (4) is of the form

$$\frac{n+1}{n}t\quad\text{or}\quad\left(1+\frac{1}{n}\right)t,$$

and the limiting value of this, for increasing n, is t, which is by hypothesis <1. Hence after a finite number of terms the series (4) will converge more rapidly than a geometric progression whose common ratio is t_1, where t_1 may be any quantity between t and 1. The series (4), and *à fortiori* the series (2), is therefore essentially convergent.

We can now shew that if $S(x)$ denote the sum of the series (1), and $f(x)$ that of the series (2), then

$$f(x) = S'(x) \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

i.e. the sum of (2) is the derivative of the sum of (1), for all values of x *within* the range of convergence of (1). Since

$$f(x) = A_1 + 2A_2x + 3A_3x^2 + \ldots + nA_nx^{n-1} + \ldots \ldots (6),$$

we have, by the theorem of Art. 196,

$$\int_0^\xi f(x) = A_1\xi + A_2\xi^2 + A_3\xi^3 + \ldots + A_n\xi^n + \ldots$$
$$= S(\xi) - A_0 \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

provided ξ lie within the range of convergence of (6). Hence, differentiating both sides with respect to ξ, and afterwards writing x for ξ, we obtain the result (5).

We have thus ascertained that, for values of x within the range of convergence, the sum of a power-series is a differentiable, as well as a continuous, function of x, and that its derivative is obtained by differentiating the series term by term*.

A more general theorem is that if the series

$$S(x) = u_0(x) + u_1(x) + u_2(x) + \ldots + u_n(x) + \ldots \quad \ldots (8)$$

be convergent for values of x extending over a certain range, and if the series

$$u_0'(x) + u_1'(x) + u_2'(x) + \ldots + u_n'(x) + \ldots \quad \ldots\ldots (9),$$

composed of the derived functions of the several terms of (8), be *uniformly* convergent over this range, the sum of this latter series will be $S'(x)$.

Ex. 1. If $|x| < 1$, we know that

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \quad \ldots\ldots\ldots\ldots(10).$$

---

* The plan of deducing this theorem as a corollary from that of Art. 196 is attributed to Darboux. It has considerable advantage in point of simplicity over the inverse order more usually followed.

Differentiating both sides, we obtain

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \dots \quad \dots\dots\dots(11).$$

A second differentiation leads to

$$\frac{1}{(1-x)^3} = \tfrac{1}{2}(1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \dots) \quad \dots(12).$$

Ex. 2. On the other hand, consider the series (considered already in Art. 194) in which

$$S_n(x) = \frac{2nx}{1+n^2x^2} \quad \dots\dots\dots\dots\dots\dots\dots(13).$$

This makes $S(x) = 0$, and therefore $S'(x) = 0$ for all values of x; but, if we differentiate term by term, the sum of the resulting series is

$$\lim_{n=\infty} S_n'(x) = \lim_{n=\infty} \frac{2n(1-n^2x^2)}{(1+n^2x^2)^2} \quad \dots\dots\dots\dots(14).$$

This vanishes if $x \neq 0$, but it becomes infinite for $x = 0$; the series of derivatives is in fact divergent for $x = 0$.

199. Integration of Differential Equations by Series.

Given a linear differential equation, with coefficients which are rational integral functions of the independent variable (x), it is often possible to obtain a solution in the form of an ascending power-series, thus

$$y = A_0 + A_1x + A_2x^2 + \dots + A_nx^n + \dots \quad \dots\dots(1).$$

If we assume, provisionally, that this series is convergent for a certain range of x, it can be differentiated once, twice, ... with respect to x, by the theorem of Art. 198. Substituting in the differential equation, we find that this can be satisfied if certain relations between the coefficients A_0, A_1, A_2, ... are fulfilled. In this way we obtain a series involving one or more arbitrary constants; and if this series proves to be in fact convergent, we have obtained a solution of the proposed differential equation. Whether it is the *complete* solution, or how far it may require to be supplemented, are of course distinct questions, which remain to be discussed independently.

One or two simple examples will make the method clear.

1°. Let the equation be

$$\frac{dy}{dx} = y \quad \text{..........................(2).}$$

The solution of this is of course known otherwise; but if we were ignorant of it we might obtain a series, as follows. Assuming the form (1), and substituting in (2), we get

$$(A_1 - A_0) + (2A_2 - A_1)x + (3A_3 - A_2)x^2 + \dots \\ + (nA_n - A_{n-1})x^{n-1} + \dots = 0 \quad \text{......(3),}$$

which will be satisfied identically provided

$$A_1 = A_0, \quad A_2 = \tfrac{1}{2}A_1 = \tfrac{1}{2}A_0, \quad A_3 = \tfrac{1}{3}A_2 = \frac{1}{2 \, . \, 3}A_0,$$

and, generally,

$$A_n = \frac{1}{n}A_{n-1} = \dots = \frac{1}{n!}A_0, \quad \text{........... (4).}$$

We thus obtain $y = A_0 E(x)$..........................(5),

where, as in Art. 17,

$$E(x) = 1 + x + \frac{x^2}{1 \, . \, 2} + \dots + \frac{x^n}{n!} + \dots \quad \text{.........(6),}$$

and A_0 is arbitrary.

If we were previously ignorant of the existence and properties of the exponential function, it would present itself naturally in Mathematics as involved in the solution of the problem: To determine a function whose rate of increase shall be proportional to the function itself.

That (5) constitutes the *complete* solution of (2) may easily be shewn. For, writing

$$y = vE(x) \quad \text{.......................(7),}$$

where v is to be determined, and substituting in (2), we find

$$\frac{dv}{dx}E(x) + v\{E'(x) - E(x)\} = 0 \quad \text{............(8),}$$

or, since $E(x)$ satisfies (2),

$$\frac{dv}{dx}=0, \text{ or } v=C \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9).$$

2°. Let the equation be

$$\frac{d^2y}{dx^2}+y=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10).$$

Assuming the form (1), and substituting, we find

$$(1.2A_2+A_0)+(2.3A_3+A_1)x+(3.4A_4+A_2)x^2+\ldots$$
$$+\{(n-1)nA_n+A_{n-2}\}x^{n-2}+\ldots=0 \ldots (11),$$

which is satisfied identically, provided

$$A_2=-\frac{1}{1.2}A_0, \qquad A_3=-\frac{1}{2.3}A_1,$$

$$A_4=-\frac{1}{3.4}A_2=\frac{1}{4!}A_0, \qquad A_5=-\frac{1}{4.5}A_3=\frac{1}{5!}A_1,$$

and, generally,

$$\left.\begin{aligned} A_{2n} &= -\frac{1}{(2n-1)\,2n}A_{2n-2}=(-)^n\frac{1}{2n!}A_0, \\ A_{2n+1} &= -\frac{1}{2n\,(2n+1)}=(-)^n\frac{1}{(2n+1)!}A_1, \end{aligned}\right\}\ldots (12).$$

We thus obtain the solution

$$y=A_0\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)+A_1\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)\ldots (13).$$

The series in brackets are easily seen to be convergent, and their sums therefore continuous, for all values of x.

It has been shewn in Art. 182 that the complete solution of (10) is

$$y=A\cos x+B\sin x \quad\ldots\ldots\ldots\ldots\ldots (14).$$

Hence, given A_0 and A_1, it must be possible to determine A and B so that the expressions (13) and (14) shall be identical.

For example, putting $A_0 = 1$, $A_1 = 0$, we must have

$$1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots = A\cos x + B\sin x,$$

and, changing the sign of x,

$$1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots = A\cos x - B\sin x.$$

Hence we must have $B = 0$, and putting $x = 0$, we find $A = 1$. We thus obtain the formula

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots \quad \ldots\ldots\ldots\ldots(15).$$

In the same way, if we put $A_0 = 0$, $A_1 = 1$, we find $A = 0$, $B = 1$, and therefore

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots \quad \ldots\ldots\ldots\ldots(16).$$

The foregoing method is, for various reasons, not always practicable. It may also lead to a solution which is *incomplete*; thus in the case of the linear equation of the second order, the method may yield only *one* series, with one arbitrary constant. This occurs not infrequently in the physical applications of the subject. The solution may, in this case, be completed, at least symbolically, by the method of Art. 185, 3°.

200. Expansions by means of Differential Equations.

The method of the preceding Art. may sometimes be utilized to obtain the expansion of a given function in a power-series, provided we can form a linear differential equation, with rational integral coefficients, which the function satisfies*.

For example, let $\quad y = (1 + x)^m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$

where m may be integral or fractional, positive or negative. Taking logarithms of both sides, and then differentiating, we have

$$\frac{1}{y}\frac{dy}{dx} = \frac{m}{1 + x},$$

---

* This method was first employed by Newton, to whom the series for $\cos x$ and $\sin x$ are also due. The manner of obtaining these series was, however, different.

or
$$(1+x)\frac{dy}{dx}-my=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Assuming
$$y=A_0+A_1x+A_2x^2+\ldots+A_nx^n+\ldots \quad\ldots\ldots(3),$$
and substituting, we have
$$(1+x)(A_1+2A_2x+\ldots+nA_nx^{n-1}+\ldots)$$
$$-m(A_0+A_1x+A_2x^2+\ldots+A_nx^n+\ldots)=0,$$
or
$$(A_1-mA_0)+\{2A_2-(m-1)A_1\}x+\{3A_3-(m-2)A_2\}x^2+\ldots$$
$$+\{nA_n-(m-n+1)A_{n-1}\}x^{n-1}+\ldots=0 \quad\ldots\ldots(4),$$
which is satisfied identically provided
$$A_1=\frac{m}{1}A_0,$$
$$A_2=\frac{m-1}{2}A_1=\frac{m(m-1)}{1.2}A_0,$$
$$A_3=\frac{m-2}{3}A_2=\frac{m(m-1)(m-2)}{1.2.3}A_0,$$
and, generally,
$$A_n=\frac{m-n+1}{n}A_{n-1}=\frac{m(m-1)\ldots(m-n+1)}{1.2.3\ldots n}A_0, \ldots(5).$$

We thus obtain
$$y=A_0\left\{1+\frac{m}{1}x+\frac{m(m-1)}{2!}x^2+\ldots\right.$$
$$\left.+\frac{m(m-1)\ldots(m-n+1)}{n!}x^n+\ldots\right\} \ldots(6),$$
as a solution of (2); and it is easily verified that the series is convergent so long as $|x|<1$.

Now if we retrace the steps by which the differential equation (2) was formed, we see that its complete solution is
$$y=C(1+x)^m \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where C is arbitrary. Hence (6) must be equivalent to (7), and putting $x=0$ in both, we see that $C=A_0$. Hence

$$(1+x)^m = 1 + \frac{m}{1}x + \frac{m(m-1)}{2!}x^2 + \dots$$
$$+ \frac{m(m-1)\dots(m-n+1)}{n!}x^n + \dots \quad \dots\dots(8),$$

for all values of x such that $|x|<1$. This is the well-known 'Binomial Expansion*.'

As a further example, we take the function

$$y = \frac{\sin^{-1} x}{\surd(1-x^2)} \quad \dots\dots\dots\dots\dots\dots(9).$$

Multiplying up by $\surd(1-x^2)$, and then differentiating, we find

$$\surd(1-x^2)\frac{dy}{dx} - \frac{x}{\surd(1-x^2)}y = \frac{1}{\surd(1-x^2)},$$

or
$$(1-x^2)\frac{dy}{dx} - xy = 1 \quad \dots\dots\dots\dots\dots\dots (10).$$

Assuming
$$y = A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n + \dots \quad \dots\dots(11),$$

we find

$$(1-x^2)(A_1 + 2A_2 x + 3A_3 x^2 + \dots + nA_n x^{n-1} + \dots)$$
$$- x(A_0 + A_1 x + A_2 x^2 + \dots + A_n x^n + \dots) = 1 \dots(12),$$

or
$$(A_1 - 1) + (2A_2 - A_0)x + (3A_3 - 2A_1)x^2 + \dots$$
$$+ \{nA_n - (n-1)A_{n-2}\}x^{n-1} + \dots = 0 \dots(13),$$

which is satisfied identically, provided

$$\left.\begin{array}{ll} A_1 = 1, & A_2 = \frac{1}{2}A_0, \\ A_3 = \frac{2}{3}A_1 = \frac{2}{3}, & A_4 = \frac{3}{4}A_2 = \frac{1.3}{2.4}A_0, \\ A_5 = \frac{4}{5}A_3 = \frac{2.4}{3.5}, & A_6 = \frac{5}{6}A_4 = \frac{1.3.5}{2.4.6}A_0, \\ \dots\dots\dots\dots & \dots\dots\dots\dots \end{array}\right\} \dots\dots(14).$$

---

* Newton (1676). The cases of $x = \pm 1$ would require special investigation.

We thus obtain the solution

$$y = x + \frac{2}{3}x^3 + \frac{2\,.\,4}{3\,.\,5}x^5 + \frac{2\,.\,4\,.\,6}{3\,.\,5\,.\,7}x^7 + \dots$$
$$+ A_0\left(1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}x^6 + \dots\right) \dots(15).$$

Now if we retrace the steps by which the linear equation (10) was formed, we see that its general solution is

$$\sqrt{(1 - x^2)}\,.\,y = \sin^{-1}x + A \dots\dots\dots\dots\dots\dots(16)^*,$$

or

$$y = \frac{\sin^{-1}x}{\sqrt{(1 - x^2)}} + \frac{A}{\sqrt{(1 - x^2)}} \dots\dots\dots\dots\dots (17),$$

and, by the nature of the case, (15) must be included in this form. If we put $x = 0$ in (15) and (17), we see that $A = A_0$. The identity of the two expressions for y then requires that

$$\frac{\sin^{-1}x}{\sqrt{(1 - x^2)}} = x + \frac{2}{3}x^3 + \frac{2\,.\,4}{3\,.\,5}x^5 + \dots \dots\dots\dots(18),$$

and

$$\frac{1}{\sqrt{(1 - x^2)}} = 1 + \frac{1}{2}x^2 + \frac{1\,.\,3}{2\,.\,4}x^4 + \dots \dots\dots\dots(19).$$

These series are both of them convergent for $|x| < 1$.

The result (19) is a mere reproduction of the binomial expansion of $(1 - x^2)^{-\frac{1}{2}}$.

If we put $x = \sin\theta$, the former series may be written

$$\theta = \sin\theta\cos\theta\left(1 + \frac{2}{3}\sin^2\theta + \frac{2\,.\,4}{3\,.\,5}\sin^4\theta + \dots\right) \dots (20).$$

Again, if we put $\tan\theta = z$, we obtain the form

$$\tan^{-1}z = \frac{z}{1 + z^2}\left\{1 + \frac{2}{3}\frac{z^2}{1 + z^2} + \frac{2\,.\,4}{3\,.\,5}\left(\frac{z^2}{1 + z^2}\right)^2 + \dots\right\}\dots(21).$$

This series has been made the basis of several ingenious methods of calculating π. It may be shewn, for example, that

$$\tfrac{1}{4}\pi = 5\tan^{-1}\tfrac{1}{7} + 2\tan^{-1}\tfrac{3}{79},$$

whence

$$\pi = \frac{28}{10}\left\{1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\,.\,4}{3\,.\,5}\left(\frac{2}{100}\right)^2 + \dots\right\}$$
$$+ \frac{30336}{100000}\left\{1 + \frac{2}{3}\left(\frac{144}{100000}\right) + \frac{2\,.\,4}{3\,.\,5}\left(\frac{144}{100000}\right)^2 + \dots\right\}\dots(22).$$

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* See also Art. 177, Ex. 2.

These series are rapidly convergent, and are otherwise very convenient for computation, owing to the powers of 10 in the denominators*.

Another remarkable series follows by integration from (18), viz.

$$\tfrac{1}{2}(\sin^{-1}x)^2 = \frac{x^2}{2} + \frac{2}{3}\frac{x^4}{4} + \frac{2 \cdot 4}{3 \cdot 5}\frac{x^6}{6} + \ldots \ldots\ldots\ldots \quad (23).$$

EXAMPLES. LIII.

1. Prove by repeated differentiation of the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots,$$

where $|x| < 1$, that, if m be a positive integer

$$(1-x)^{-m} = 1 + mx + \frac{m(m+1)}{1 \cdot 2}x^2 + \frac{m(m+1)(m+2)}{1 \cdot 2 \cdot 3}x^3 + \ldots.$$

2. If $|x| < 1$, prove that

$$\tanh^{-1}x = x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \ldots.$$

3. If $|x| < 1$, prove that

$$\frac{x^2}{1 \cdot 2} - \frac{x^3}{2 \cdot 3} + \frac{x^4}{3 \cdot 4} - \ldots = (1+x)\log(1+x) - x.$$

Does the result hold for $x = \pm 1$?

4. If $|x| < 1$, prove that

$$\frac{x^2}{1 \cdot 2} - \frac{x^4}{3 \cdot 4} + \frac{x^6}{5 \cdot 6} - \ldots = x\tan^{-1}x - \tfrac{1}{2}\log(1+x^2).$$

Hence shew that

$$1 - \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} - \ldots = \cdot 43882\ldots.$$

5. Prove that

$$\int_0^x \frac{\sin x}{x}dx = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \ldots,$$

$$\int_0^x \frac{\sinh x}{x}dx = x + \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} + \ldots.$$

---

* For the history of these series, see Glaisher, *Mess. of Math.*, t. ii., p. 119 (1873).

6. Prove that

$$\int_a^b \frac{e^x}{x}\,dx = \log\frac{b}{a} + (b-a) + \frac{b^2-a^2}{2\,.\,2\,!} + \frac{b^3-a^3}{3\,.\,3\,!} + \dots.$$

7. Obtain the following results by calculation from the series (6) of Art. 197:

$$\begin{array}{ll}
\log 2 = \cdot693\ 147\ 181 & \log 7 = 1\cdot945\ 910\ 149 \\
\log 3 = 1\cdot098\ 612\ 289 & \log 8 = 2\cdot079\ 441\ 542 \\
\log 4 = 1\cdot386\ 294\ 361 & \log 9 = 2\cdot197\ 224\ 577 \\
\log 5 = 1\cdot609\ 437\ 912 & \log 10 = 2\cdot302\ 585\ 093 \\
\log 6 = 1\cdot791\ 759\ 469 & \mu = \cdot434\ 294\ 482.
\end{array}$$

8. Prove that

$$\begin{aligned}
\log 2 &= 7a - 2b + 3c, \\
\log 3 &= 11a - 3b + 5c, \\
\log 5 &= 16a - 4b + 7c,
\end{aligned}$$

and thence

$$\log 10 = 23a - 6b + 10c,$$

where

$$\begin{aligned}
a &= \frac{1}{10} + \frac{1}{2\,.\,10^2} + \frac{1}{3\,.\,10^3} + \dots = \cdot1053605157, \\
b &= \frac{4}{100} + \frac{4^2}{2\,.\,100^2} + \frac{4^3}{3\,.\,100^3} + \dots = \cdot0408219945, \\
c &= \frac{1}{80} - \frac{1}{2\,.\,80^2} + \frac{1}{3\,.\,80^3} - \dots = \cdot0124225200.
\end{aligned}$$

Apply this to find log 10. (Adams.)

9. Prove that

$$\begin{aligned}
\log 2 &= 7P + 5Q + 3R, \\
\log 3 &= 11P + 8Q + 5R, \\
\log 5 &= 16P + 12Q + 7R,
\end{aligned}$$

and thence

$$\log 10 = 23P + 17Q + 10R,$$

where

$$\begin{aligned}
P &= 2\left(\frac{1}{31} + \frac{1}{3\,.\,31^3} + \frac{1}{5\,.\,31^5} + \dots\right) = \cdot0645385211, \\
Q &= 2\left(\frac{1}{49} + \frac{1}{3\,.\,49^3} + \frac{1}{5\,.\,49^5} + \dots\right) = \cdot0408219945, \\
R &= 2\left(\frac{1}{161} + \frac{1}{3\,.\,161^3} + \frac{1}{5\,.\,161^5} + \dots\right) = \cdot0124225200.
\end{aligned}$$

Apply this to find log 10. (Glaisher.)

10. Shew graphically that the series in which

$$u_n(x) = (1-x)x^n$$

is not uniformly convergent in the neighbourhood of $x = 1$.

11. In the series for which

$$u_n(x) = \frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}$$

we have $S(x) = x$ for values of x ranging from 0 to 1 inclusively, whilst the series of derivatives of the several terms has the sum 1 for $1 > x > 0$, but the sum 0 for $x = 1$. Account for this.

12. Examine the character of the convergency of the series for which $S_n(x)$ has the forms

$$\frac{1}{1+x^{2n}},\quad x^{\frac{1}{2n-1}},\quad nx^2e^{-nx^2},\quad \sin^n x,\quad \frac{\sin nx}{nx},$$

respectively.

13. If
$$S_n(x) = \frac{1}{n}\log\cosh nx,$$

prove that $S(x) = |x|$ for all values of x.

14. If the series

$$a_0 + a_1 + a_2 + \dots + a_n + \dots$$

be essentially convergent, the series

$$a_0 + a_1\cos x + a_2\cos 2x + \dots + a_n\cos nx + \dots$$

will be uniformly convergent for all values of x.

15. If a_0, a_1, a_2, ... a_n, ... be a descending sequence of positive qualities whose lower limit is 0, prove that the series

$$a_0 + a_1\cos x + a_2\cos 2x + \dots + a_n\cos nx + \dots$$

will be uniformly convergent over any range of x which does not include any of the points $x = 2s\pi$, where $s = 0, \pm 1, \pm 2, \pm 3, \dots$

16. Prove that

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = 1 + \frac{1}{2}\cdot\frac{1}{5} + \frac{1.3}{2.4}\cdot\frac{1}{9} + \frac{1.3.5}{2.4.6}\cdot\frac{1}{13} + \dots.$$

17. From the formula

$$\frac{ds}{d\psi} = a\frac{1-e^2}{(1-e^2\sin^2\psi)^{\frac{3}{2}}}$$

for the radius of curvature of an ellipse, shew that the perimeter is equal to

$$2\pi a(1-e^2)\left(1+\frac{3}{2^2}e^2+\frac{3^2\,.\,5}{2^2\,.\,4^2}e^4+\frac{3^2\,.\,5^2\,.\,7}{2^2\,.\,4^2\,.\,6^2}e^6+\ldots\right),$$

and verify that this is equivalent to the result of Art. 196 (13).

18. The time of a complete oscillation of a simple pendulum of length l, oscillating through an angle $\alpha\ (<\pi)$ on each side of the vertical, is

$$4\sqrt{\frac{l}{g}}\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)}};$$

prove that this is equal to

$$2\pi\sqrt{\frac{l}{g}}\left(1+\frac{1^2}{2^2}\sin^2\tfrac{1}{2}\alpha+\frac{1^2\,.\,3^2}{2^2\,.\,4^2}\sin^4\tfrac{1}{2}\alpha+\frac{1^2\,.\,3^2\,.\,5^2}{2^2\,.\,4^2\,.\,6^2}\sin^6\tfrac{1}{2}\alpha+\ldots\right).$$

19. The perimeter of an ellipse of small eccentricity e exceeds that of a circle of the same area in the ratio $1+\frac{3}{64}e^4$, approximately.

20. The surface of an ellipsoid of revolution (prolate or oblate) of small eccentricity e exceeds that of a sphere of equal volume by the fraction $\frac{2}{45}e^4$ of itself.

21. Prove that

$$\pi = 2\sqrt{3}\left(1-\frac{1}{3\,.\,3}+\frac{1}{5\,.\,3^2}-\frac{1}{7\,.\,3^3}+\ldots\right).$$

22. Assuming the series for $\sin x$, prove Huyghens' rule for calculating approximately the length of a circular arc, viz.: From eight times the chord of half the arc subtract the chord of the whole arc, and divide the result by three.

Prove that in an arc of 45° the proportional error is less than 1 in 20000.

23. Obtain a particular solution of the equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+my=0$$

in the form

$$y=A\left(1-\frac{mx}{1^2}+\frac{m^2x^2}{1^2.2^2}-\frac{m^3x^3}{1^2.2^2.3^2}+\dots\right).$$

24. Obtain a particular solution of the equation

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+k^2\phi=0$$

in the form $$\phi=A\left(1-\frac{k^2r^2}{2^2}+\frac{k^4r^4}{2^2.4^2}-\dots\right).$$

25. Integrate $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}=0,$

by series, and deduce the expansion of $\sin^{-1}x$ (Art. 197 (15)).

26. Prove that $y=\sinh^{-1}x$

satisfies the differential equation

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0.$$

Hence shew that, for $|x|<1$,

$$\log\{x+\sqrt{(1+x^2)}\}=x-\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}-\dots.$$

27. Obtain a solution of the equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{du}{d\mu}\right\}+n(n+1)u=0$$

in the form

$$u=A\left(1-\frac{n(n+1)}{2!}\mu^2+\frac{(n-2)n(n+1)(n+3)}{4!}\mu^4-\dots\right)$$

$$+B\left(\mu-\frac{(n-1)(n+2)}{3!}\mu^3+\frac{(n-3)(n-1)(n+2)(n+4)}{5!}\mu^5-\dots\right).$$

28. Obtain a solution of the equation

$$x(1-x)\frac{d^2y}{dx^2}+\{\gamma-(\alpha+\beta+1)x\}\frac{dy}{dx}-\alpha\beta y=0$$

in the form

$$y=A\left(1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2\right.$$
$$\left.+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\ldots\right).$$

29. Obtain a solution of the equation

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\left(k^2-\frac{n^2}{r^2}\right)\phi=0$$

in the form

$$\phi=Ar^n\left(1-\frac{k^2r^2}{2(2n+2)}+\frac{k^4r^4}{2.4(2n+2)(2n+4)}-\ldots\right).$$

30. Obtain the solution of the equation

$$\frac{d^2R}{dr^2}+\frac{2(n+1)}{r}\frac{dR}{dr}+k^2R=0$$

in the form

$$R=A\left(1-\frac{k^2r^2}{2(2n+3)}+\frac{k^4r^4}{2.4(2n+3)(2n+5)}-\ldots\right)$$
$$+Br^{-2n-1}\left(1-\frac{k^2r^2}{2(1-2n)}+\frac{k^4r^4}{2.4(1-2n)(3-2n)}-\ldots\right).$$

31. If $$y=\sin(m\sin^{-1}x),$$

prove that $$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

Hence shew that

$$\frac{\sin m\theta}{m\sin\theta}=1-\frac{m^2-1}{3!}\sin^2\theta+\frac{(m^2-1)(m^2-3^2)}{5!}\sin^4\theta-\ldots,$$

$$\cos m\theta=1-\frac{m^2}{2!}\sin^2\theta+\frac{m^2(m^2-2^2)}{4!}\sin^4\theta-\ldots.$$

32. Obtain a solution, by a series, of

$$(1-x^2)\frac{d^2y}{dx^2}+n(n-1)y=0,$$

and give the symbolical expression of the complete solution.

CHAPTER XIV.

TAYLOR'S THEOREM.

201. Form of the Expansion.

Let $f(x)$ be any function of x which admits of expansion in a convergent power-series for all values of x within certain limits $\pm\alpha$. It has been proved, in Art. 198, that the derived function $f'(x)$ will be given by a similar series, obtained by differentiating the original series term by term, for all values of x between $\pm\alpha$. By a second application of the theorem cited, the value of $f''(x)$ will be obtained, for values of x between the above limits, by differentiating the series for $f'(x)$ term by term. And so on. Hence, writing

$$f(x) = A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n + \ldots \ldots (1),$$

we have

$$\left.\begin{array}{ll} f'(x) = & A_1 + 2A_2 x + \ldots + nA_n x^{n-1} + \ldots, \\ f''(x) = & 2.1A_2 + \ldots + n(n-1)A_n x^{n-2} + \ldots, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f^{(n)}(x) = & n(n-1)\ldots 2.1A_n + \ldots, \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\}\ldots(2).$$

Putting $x=0$ in these equations, we find

$$A_0 = f(0),\ A_1 = f'(0),\ A_2 = \frac{1}{2!}f''(0), \ldots A_n = \frac{1}{n!}f^{(n)}(0) \ldots (3),$$

where the symbols $f(0), f'(0), f''(0), \ldots$ are used to express that x is put $=0$ *after* the differentiations have been performed.

The original expansion may now be written

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2\,!} f''(0) + \ldots + \frac{x^n}{n\,!} f^{(n)}(0) + \ldots \qquad \ldots(4).$$

This investigation was given by Maclaurin*.

It will be noticed that the proof depends entirely on the initial assumption that the function $f(x)$ admits of being expanded in a convergent power-series. The question as to when, and under what limitations, such an expansion is possible will be discussed later (Arts. 203, 204).

If we write $$\phi(a+x) = f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots(5),$$
we can deduce the form of the expansion of $\phi(a+x)$ in a power-series, when such expansion is possible. For if we write, for a moment,

$$u = a + x,$$

we have

$$f(x) = \phi(u),$$

$$f'(x) = \frac{d}{dx}\phi(u) = \frac{d}{du}\phi(u)\,.\,\frac{du}{dx} = \phi'(u),$$

$$f''(x) = \frac{d}{dx}\phi'(u) = \frac{d}{du}\phi'(u)\,.\,\frac{du}{dx} = \phi''(u),$$

and so on (Art. 39, 1°). Hence, putting $x = 0$, $u = a$, we find

$$f(0) = \phi(a),\ f'(0) = \phi'(a),\ f''(0) = \phi''(a),\ \ldots f^{(n)}(0) = \phi^{(n)}(a) \quad \ldots\ldots\ldots\ldots\ldots(6);$$

so that (4) takes the form

$$\phi(a+x) = \phi(a) + x\phi'(a) + \frac{x^2}{1\,.\,2}\phi''(a) + \ldots + \frac{x^n}{n\,!}\phi^{(n)}(a) + \ldots \qquad \ldots\ldots\ldots(7).$$

This is known as Taylor's Theorem†. We have deduced it from Maclaurin's Theorem, but the two theorems are only slightly different expressions of the same result. Thus assuming (7), we deduce Maclaurin's expansion if we put $a = 0$§.

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* *Treatise on Fluxions* (1742). The theorem had been previously noticed by Stirling.

† Given (under a slightly different form) as a corollary from a theorem in Finite Differences, *Methodus Incrementorum* (1716).

§ The virtual identity of (4) with Taylor's Theorem was clearly recognized by Maclaurin.

202. Particular Cases.

Before proceeding to a more fundamental treatment of the problem suggested in the preceding Art., the student will do well to make himself familiar with the mode of formation of the series. In the following examples the possibility of the expansion is assumed to begin with; and the results obtained are therefore not to be considered as established, at all events by this method.

1°. If $$\phi(a) = a^m \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have

$$\phi'(a) = ma^{m-1},\ \phi''(a) = m(m-1)a^{m-2}, \ldots$$
$$\phi^{(n)}(a) = m(m-1)\ldots(m-n+1)a^{m-n}, \ldots \quad \ldots(2).$$

Taylor's formula then gives

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{1.2}a^{m-2}x^2 + \ldots$$
$$+ \frac{m(m-1)\ldots(m-n+1)}{1.2\ldots m}a^{m-n}x^n + \ldots \quad \ldots(3),$$

which is the well-known Binomial Expansion.

That Taylor's Theorem cannot hold in all cases, without qualification, is shewn by the fact that the series on the right-hand is divergent if $|x| > |a|$. For $|x| < |a|$ the series is convergent, but it is not legitimate to affirm on the basis of the investigation of Art. 201 that its sum is then equal to $(a+x)^m$*. A valid proof of the equality has been given in Art. 200.

2°. If $$f(x) = e^x \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

we have $$f^{(n)}(x) = e^x,$$

and therefore $$f(0) = 1,\quad f^{(n)}(0) = 1.$$

Maclaurin's expansion is therefore

$$e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots + \frac{x^n}{n!} + \ldots \quad \ldots\ldots(5).$$

This is in any case a mere verification, as the series on the right-hand was adopted in Art. 17 as the definition of e^x.

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* There are in fact cases where Taylor's expansion is convergent, whilst the sum is *not* equal to $\phi(a+x)$.

3°. Let $$f(x) = \cos x \quad\text{........................}(6).$$

It was shewn in Art. 63 that this makes

$$f^{(n)}(x) = \cos\left(x + \tfrac{1}{2}n\pi\right),$$

so that $$f(0) = 1, \quad f^{(n)}(0) = \cos \tfrac{1}{2}n\pi \quad\text{............}(7).$$

Hence $f^{(n)}(0)$ vanishes when n is odd, and is equal to ± 1 when n is even, according as $\frac{1}{2}n$ is even or odd. Substituting in Maclaurin's formula, we get

$$\cos x = 1 - \frac{x^2}{2\,!} + \frac{x^4}{4\,!} - \ldots + (-)^n \frac{x^{2n}}{2n\,!} + \ldots \quad\text{......}(8).$$

4°. Let $$f(x) = \sin x \quad\text{.....................}(9).$$

This makes $$f^{(n)}(x) = \sin\left(x + \tfrac{1}{2}n\pi\right),$$

so that

$$f(0) = 0,\ f^{(n)}(0) = \sin \tfrac{1}{2}n\pi = \sin\left\{\tfrac{1}{2}(n-1)\pi + \tfrac{1}{2}\pi\right\} \ \ldots(10).$$

Hence $f^{(n)}(0)$ vanishes when n is even, and is equal to ± 1 when n is odd, according as $\frac{1}{2}(n-1)$ is even or odd. Maclaurin's formula then gives

$$\sin x = x - \frac{x^3}{3\,!} + \frac{x^5}{5\,!} - \ldots + (-)^n \frac{x^{2n+1}}{(2n+1)\,!} + \ldots \ \ldots(11).$$

The results (8) and (11) have been established rigorously in Art. 199, 2°.

5°. Let $$f(x) = \log(1 + x) \quad\text{..................}(12).$$

This makes

$$f'(x) = \frac{1}{1+x}, \text{ and, for } n > 1,\ f^{(n)}(x) = \frac{(-)^{n-1}(n-1)\,!}{(1+x)^n}.$$

Hence $f(0) = 0$, $f'(0) = 1$, and, for $n > 1$,

$$f^{(n)}(0) = (-)^{n-1}(n-1)\,! \quad\text{............}(13).$$

Substituting in Maclaurin's formula, we get

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + (-)^{n-1}\frac{x^n}{n} + \ldots \ \ldots(14).$$

Cf. Art. 197, 1°.

When the general formula for the nth derivative of the given function is not known, the only plan is to calculate the derivatives in succession as far as may be considered necessary. The later stages of the work may sometimes be contracted by omitting terms which will contribute nothing to the final result, so far as it is proposed to carry it.

Ex. To expand $\tan x$ as far as x^7.

Putting $$f(x) = \tan x,$$
we find in succession

$$\begin{aligned}
f'\ (x) &= \sec^2 x = 1 + \tan^2 x,\\
f''\ (x) &= 2 \tan x \sec^2 x = 2 \tan x + 2 \tan^3 x,\\
f'''\ (x) &= (2 + 6 \tan^2 x) \sec^2 x = 2 + 8 \tan^2 x + 6 \tan^4 x,\\
f^{\text{iv}}\ (x) &= (16 \tan x + 24 \tan^3 x) \sec^2 x\\
&= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x,\\
f^{\text{v}}\ (x) &= (16 + 120 \tan^2 x + 120 \tan^4 x) \sec^2 x\\
&= 16 + 136 \tan^2 x + 240 \tan^4 x + 120 \tan^6 x,\\
f^{\text{vi}}\ (x) &= 272 \tan x \sec^2 x + \&c.,\\
f^{\text{vii}}\ (x) &= 272 \sec^4 x + \&c.,
\end{aligned}$$

where, in the last two lines, terms have been omitted which will contribute nothing to the value of $f^{\text{vii}}(0)$. Hence

$$\begin{aligned}
f\ (0) &= 0, & f'\ (0) &= 1,\\
f''\ (0) &= 0, & f'''\ (0) &= 2,\\
f^{\text{iv}}\ (0) &= 0, & f^{\text{v}}\ (0) &= 16,\\
f^{\text{vi}}\ (0) &= 0, & f^{\text{vii}}\ (0) &= 272,
\end{aligned}$$

and the expansion is

$$\begin{aligned}
\tan x &= x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \frac{272x^7}{7!} + \dots\\
&= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \dots \quad \dots\dots\dots (15).
\end{aligned}$$

That *odd* powers, only, of x would appear in this expansion might have been anticipated from the fact that $\tan x$ changes sign with x.

203. Proof of Maclaurin's and Taylor's Theorems.

Let $f(x)$ be a function of x which, together with its first $n - 1$ derivatives, is continuous for values of x ranging from 0 to h, inclusively; and let us write

$$f(x) = \Phi_n(x) + R_n(x) \dots\dots\dots\dots\dots\dots (1),$$

where

$$\Phi_n(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) \qquad \text{............(2)};$$

i.e. $\Phi_n(x)$ is the sum of the first n terms of Maclaurin's expansion, and $R_n(x)$ is at present merely a symbol for the difference, whatever it is, between $f(x)$ and $\Phi_n(x)$. The object aimed at, in any rigorous investigation of Maclaurin's Theorem, is to find (if possible) limits to the value of $R_n(x)$; in other words, to find limits to the error committed when $f(x)$ is replaced by the sum of the first n terms of Maclaurin's formula. If we can, in any particular case, shew that, by taking n great enough, a point can be reached after which the values of $R_n(x)$ will all be less than any assigned magnitude, however small, then Maclaurin's series is necessarily convergent, and its sum to infinity is $f(x)$. It is evident that the argument cannot be pushed to this conclusion if $f(x)$ or any of its derivatives be discontinuous for any value of x belonging to the range considered.

The notion of representing a function $f(x)$ approximately by a rational integral function of assigned degree, say

$$A_0 + A_1x + A_2x^2 + \ldots + A_{n-1}x^{n-1} \qquad \text{..............(3)},$$

has already been utilized in Art. 112. The plan there adopted was to determine the n coefficients $A_0, A_1, A_2, \ldots A_{n-1}$ so that the function (3) should be equal to $f(x)$ for n assigned values of x, which were distributed at equal intervals over a certain range. In the present case, the n values of x are taken to be ultimately coincident with 0; in other words, the coefficients are chosen so as to make the function (3) and its first $n-1$ derivatives coincide respectively with $f(x)$ and its first $n-1$ derivatives for the particular value $x=0$. The result of this determination is, by Art. 201, the function $\Phi_n(x)$.

In the graphical representation, the parabolic curve $y=\Phi_n(x)$ is determined so as to have contact of the $(n-1)$th order (see Art. 206) with a given curve $y=f(x)$ at the point $x=0$; and the problem is, to find limits to the possible deviation of one curve from the other, as measured by the difference of the ordinates, for values of x lying within a certain range. This is illustrated by Fig. 149, which shews the curve

$$y = \log(1+x) \qquad \text{.........................(4)},$$

and (by thinner lines) the curves

$$y = x, \; y = x - \tfrac{1}{2}x^2, \; y = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3, \; \ldots \quad \ldots\ldots\ldots(5),$$

obtained by taking 1, 2, 3, ... terms of the 'logarithmic series' (Art. 202 (14)). The dotted lines correspond to $x = \pm 1$, and so mark out the range of convergence of the latter series*.

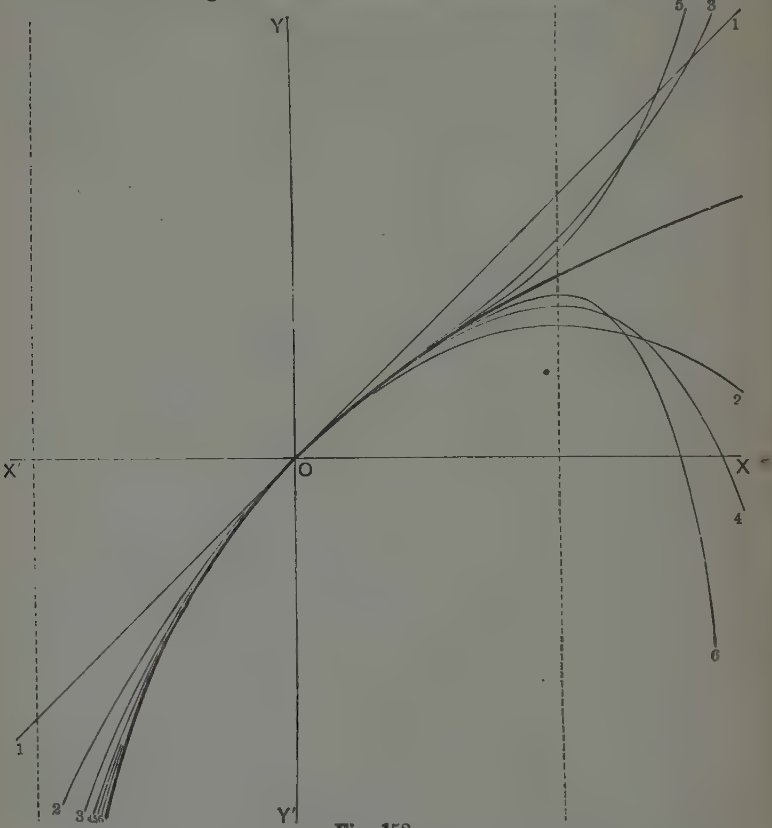

Fig. 152.

It appears, from the conditions which $\Phi_n(x)$ has been made to satisfy, that $R_n(x)$ and its first $n-1$ derivatives will be continuous from $x=0$ to $x=h$, and will all vanish for $x=0$. Now we can shew that any function which satisfies *these* conditions, and has a finite nth derivative, must lie between

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* This very instructive example is due to Prof. Felix Klein.

$$A\frac{x^n}{n!} \text{ and } B\frac{x^n}{n!},$$

where A and B are the lower and upper limits to the values which the nth derivative assumes in the interval from 0 to h.

For, let $F(x)$ be such a function. By hypothesis, we have

$$F(0)=0, \quad F'(0)=0, \quad F''(0)=0, \ldots F^{(n-1)}(0)=0 \ldots(6),$$

and
$$A < F^{(n)}(x) < B \ldots\ldots\ldots\ldots(7),$$

the latter condition holding from $x=0$ to $x=h$. It follows from (7), by Art. 89, 4°, that

$$\int_0^x A\,dx < \int_0^x F^{(n)}(x)\,dx < \int_0^x B\,dx^*,$$

or, since $F^{(n-1)}(0)=0$,

$$Ax < F^{(n-1)}(x) < Bx \ldots\ldots\ldots\ldots(8).$$

By a second application of the theorem referred to, we have

$$\int_0^x Ax\,dx < \int_0^x F^{(n-1)}(x)\,dx < \int_0^x Bx\,dx,$$

or, since $F^{(n-2)}(0)=0$,

$$A\frac{x^2}{2!} < F^{(n-2)}(x) < B\frac{x^2}{2!} \ldots\ldots\ldots\ldots(9).$$

A similar argument applies to shew that

$$A\frac{x^3}{3!} < F^{(n-3)}(x) < B\frac{x^3}{3!} \ldots\ldots\ldots\ldots(10),$$

and so on, until we arrive at the result

$$A\frac{x^n}{n!} < F(x) < B\frac{x^n}{n!} \ldots\ldots\ldots\ldots(11).$$

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* Provided x be positive. If x be negative the inequalities must be reversed, but the final conclusion is unaffected.

Hence we may write

$$F(x) = C\frac{x^n}{n!} \qquad \text{.................(12)},$$

where C is some quantity between A and B.

In the present application, since $\Phi_n(x)$ is a rational integral function of degree $n-1$, its nth derivative is zero (Art. 63); and the nth derivative of $R_n(x)$ is therefore, by (1), equal to $f^{(n)}(x)$, if this latter derivative exist. We infer, then, that

$$R_n(x) = C\frac{x^n}{n!} \qquad \text{.....................(13)},$$

where C is some quantity intermediate to the greatest and least values which $f^{(n)}(x)$ assumes in the interval from 0 to h. And if, as we will suppose, this latter derivative is continuous from $x=0$ to $x=h$, there will be some value of x, between 0 and h, for which $f^{(n)}(x)$ is equal to C. Denoting this value by θh, we have

$$R_n(x) = \frac{x^n}{n!}f^{(n)}(\theta h) \qquad \text{..................(14)},$$

where all we know as to the value of θ is that it lies between 0 and 1.

The formula (14) holds from $x = 0$ to $x = h$, inclusively Putting $x=h$, and substituting in (1), we obtain

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!}f''(0) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0)$$
$$+ \frac{h^n}{n!}f^{(n)}(\theta h) \text{...(15)}.$$

In this form Maclaurin's Theorem is exact, subject to the hypothesis that $f(x)$ and its derivatives up to the order n, inclusively, are continuous over the range from 0 to h. The conditions, however, that $f^{(n)}(x)$ is to exist, and to be continuous over the above range, include the rest.

If we write

$$f(x) = \phi(a+x) \qquad \text{..................(16)},$$

we deduce

$$\phi(a+h)=\phi(a)+h\phi'(a)+\frac{h^2}{2!}\phi''(a)+\ldots+\frac{h^{n-1}}{(n-1)!}\phi^{(n-1)}(a)$$
$$+\frac{h^n}{n!}\phi^{(n)}(a+\theta h) \;\ldots\ldots\; (17),$$

where as before, $1>\theta>0$.

This is an accurate form of Taylor's Theorem. It holds on the assumption that $\phi^{(n)}(x)$ exists and is continuous from $x=a$ to $x=a+h$, inclusively.

The last terms in (15) and (17) are known as Lagrange's forms of the 'remainder' in the respective theorems.

The formula (17) is a generalization of some results obtained in the course of this treatise. For example, putting $n=1$, we get

$$\phi(a+h)=\phi(a)+h\phi'(a+\theta h) \;\ldots\ldots\ldots\ldots\; (18);$$

and, putting $n=2$,

$$\phi(a+h)=\phi(a)+h\phi'(a)+\tfrac{1}{2}h^2\phi''(a+\theta h) \;\ldots\ldots (19).$$

These agree with Art. 56 (9) and Art. 66 (23), respectively.

Another form of remainder may be investigated as follows. If $F(x)$ be subject to the conditions stated in (6), we find by an integration by parts

$$\int_0^h\left(1-\frac{x}{h}\right)^{n-1}F^{(n)}(x)\,dx=\frac{n-1}{h}\int_0^h\left(1-\frac{x}{h}\right)^{n-2}F^{(n-1)}(x)\,dx$$
$$\ldots\ldots(20),$$

since the integrated term vanishes at both limits. Performing this process $n-1$ times, we obtain

$$\int_0^h\left(1-\frac{x}{h}\right)^{n-1}F^{(n)}(x)\,dx=\frac{(n-1)!}{h^{n-1}}\int_0^h F'(x)\,dx=\frac{(n-1)!}{h^{n-1}}F(h),$$

or

$$F(h)=\frac{h^{n-1}}{(n-1)!}\int_0^h\left(1-\frac{x}{h}\right)^{n-1}F^{(n)}(x)\,dx\ldots\ldots(21),$$

Since the function under the $\int$ sign is continuous, we infer, by Art. 89, 3°, that

$$F(h)=\frac{h^n}{(n-1)!}(1-\theta)^{n-1}F^{(n)}(\theta h). \;\ldots\ldots\ldots\ldots(22),$$

where $1>\theta>0$.

It appears, then, that the last term in (15) may be replaced by

$$\frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta h) \qquad (23),$$

and the last term in (17) by

$$\frac{h^n}{(n-1)!}(1-\theta)^{n-1}\phi^{(n)}(a+\theta h) \qquad (24).$$

These forms of remainder are due to Cauchy.

204. Another Proof.

The proof of Taylor's (or Maclaurin's) theorem which is most frequently given follows the lines of Art. 66, 2°.

Considering any given curve

$$y=f(x) \qquad (1),$$

we compare with it the curve

$$y=A_0+A_1x+A_2x^2+\ldots+A_{n-1}x^{n-1}+A_nx^n \quad \ldots(2),$$

in which the $n+1$ coefficients are determined so as to make the two curves intersect at $x=0$ and $x=h$, and, further, so as to make the values of

$$\frac{dy}{dx}, \ \frac{d^2y}{dx^2}, \ \ldots \ \frac{d^{n-1}y}{dx^{n-1}}$$

respectively the same in the two curves at the point $x=0$. These conditions give

$$A_0=f(0), \quad A_1=f'(0), \quad A_2=\frac{1}{2!}f''(0), \ \ldots$$

$$A_{n-1}=\frac{1}{(n-1)!}f^{(n-1)}(0) \quad \ldots(3),$$

as before, and

$$f(h)=A_0+A_1h+A_2h^2+\ldots+A_{n-1}h^{n-1}+A_nh^n \quad \ldots(4),$$

this latter equation determining A_n.

Denoting by $F(x)$ the difference of the ordinates of the two curves, it appears that

$$F(0)=0, \ F'(0)=0, \ F''(0)=0, \ \ldots \ F^{(n-1)}(0)=0 \quad \ldots(5),$$

and

$$F(h)=0 \qquad (6).$$

Since $F(x)$ vanishes for $x=0$ and $x=h$, it follows, under the usual conditions, that $F'(x)$ vanishes for some value of x

between 0 and h, say for $x = \theta_1 h$, where $1 > \theta_1 > 0$. Again, since $F'(x)$ vanishes for $x = 0$ and $x = \theta_1 h$, $F''(x)$ will vanish for some value of x between 0 and $\theta_1 h$, say for $x = \theta_2 h$, where $\theta_1 > \theta_2 > 0$. Proceeding in this way we find that $F^{(n-1)}(x)$ vanishes for $x = 0$ and $x = \theta_{n-1} h$, where $1 > \theta_{n-1} > 0$, and hence that

$$F^{(n)}(\theta h) = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7),$$

where $1 > \theta > 0$. Now, on reference to (1) and (2), we see that

$$F^{(n)}(x) = f^{(n)}(x) - n!\,A_n \quad\ldots\ldots\ldots\ldots\ldots(8).$$

It follows from (7) that

$$A_n = \frac{1}{n!} f^{(n)}(\theta h) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

Hence, substituting from (3) and (9) in (4) we obtain

$$f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f''(0) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n}{n!} f^{(n)}(\theta h) \quad\ldots\ldots(10),$$

as before. The conditions of validity are as stated in Art. 203, after equation (15)*.

205. Derivation of Certain Expansions.

We proceed to consider the value of the remainder for various forms of $f(x)$, or $\phi(x)$; and in particular to examine under what circumstances it tends, with increasing n, to the limit 0. In this way we are enabled to demonstrate several very important expansions; but it is right to warn the student that the method has a somewhat restricted application, since the general form of the nth derivative of a given function can be ascertained in only a few cases. Moreover, even when the method is successful, it is often far from being the most instructive way of arriving at the final result.

1°. If

$$f(x) = e^x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

we have

$$\frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} e^{\theta x} \quad\ldots\ldots\ldots\ldots\ldots(2).$$

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* The foregoing proof is substantially that given by Homersham Cox, *Camb. and Dub. Math. Journ.*, 1851.

Now, whatever the value of x, we have

$$\lim_{n=\infty} \frac{x^n}{n!} = 0 \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

since the successive fractions

$$\frac{x}{1}, \frac{x}{2}, \frac{x}{3}, \ldots \frac{x}{n}, \ldots$$

diminish indefinitely in absolute value. Hence the expansion (5) of Art. 202 holds for all values of x. As already pointed out, this merely amounts to a verification.

2°. If $$f(x) = \cos x \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4),$$

we have $$\frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!} \cos(\theta x + \tfrac{1}{2} n\pi) \quad\ldots\ldots\ldots\ldots(5).$$

The limiting value of the fraction $x^n/n!$ is zero, and the cosine lies always between ± 1. Hence the expansion (8) of Art. 202 holds for all values of x.

The same reasoning applies in the case of $\sin x$.

3°. If $$f(x) = (1 + x)^m \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

we find

$$\frac{x^n}{n!} f^{(n)}(\theta x) = \frac{m(m-1)\ldots(m-n+1)}{1 \,.\, 2 \ldots n} \; \frac{x^n}{(1+\theta x)^{n-m}} \;\ldots(7).$$

This may be regarded as the product of $(1 + \theta x)^m$ into n factors of the type

$$\frac{m - r + 1}{r} \cdot \frac{x}{1 + \theta x} \quad \text{or} \quad \left(-1 + \frac{m+1}{r}\right) \frac{x}{1 + \theta x} \;\ldots(8).$$

If $1 > x > 0$, the fraction $x/(1 + \theta x)$ lies between 0 and x, and since the first factor in (8) tends with increasing r to the limiting value -1, it appears that by taking n great enough the value of the expression (7) can be made less than any assignable magnitude. Hence, for $1 > x > 0$, we may write

$$(1 + x)^m = 1 + mx + \frac{m(m-1)}{1 \,.\, 2} x^2 + \ldots \quad \ldots\ldots(9)$$

ad infinitum.

We cannot make the same inference when x is negative, even if $|x| < 1$. For if we put $x = -x_1$, the fraction $x_1/(1-\theta x_1)$ is less than 1 only if $\theta < (1-x_1)/x_1$. And if $x_1 > \frac{1}{2}$, we have no warrant for assuming that θ lies below this value.

Cauchy's form of remainder (Art. 203 (23)) is now of service. We have, in place of (7),

$$\frac{m(m-1)\dots(m-n+1)}{1\,.\,2\dots(n-1)}\,\frac{(1-\theta)^{n-1}x^n}{(1+\theta x)^{n-m}} \;\dots\dots\dots\; (10).$$

This is equal to $mx(1+\theta x)^{m-1}$ multiplied into $n-1$ factors of the type

$$\left(-1+\frac{m}{r}\right)\frac{x-\theta x}{1+\theta x} \;\dots\dots\dots\dots\dots\dots\dots (11);$$

and, if $|x| < 1$, the fraction $(x-\theta x)/(1+\theta x)$ is easily seen to be less in absolute value than x, whether x be positive or negative. Hence the limiting value of the remainder (10) is 0, so long as $1 > x > -1$.

4°. If $$f(x) = \log(1+x) \;\dots\dots\dots\dots\dots\; (12),$$

we find $$\frac{x^n}{n!}f^{(n)}(\theta x) = \frac{(-1)^{n-1}}{n}\left(\frac{x}{1+\theta x}\right)^n \;\dots\dots\dots\; (13).$$

The limiting value of the first factor is 0, and, if x be positive and $\not> 1$, $x/(1+\theta x) \not> 1$. Hence the limiting value of (13), for $n = \infty$, is zero, and the expansion (14) of Art. 202 is valid from $x = 0$ to $x = 1$, inclusively. Cf. Fig 149.

The above form of remainder does not enable us to determine the case of x negative, even when $|x| < 1$.

In Cauchy's form, we have, in place of (13)

$$(-1)^{n-1}\frac{x}{1+\theta x}\left(\frac{x-\theta x}{1+\theta x}\right)^{n-1} \;\dots\dots\dots\dots\; (14).$$

If $|x| < 1$, the limiting value of this is 0, whether x be positive or negative.

206. Applications of Taylor's Theorem. Order of Contact of Curves.

If two curves intersect in *two* points, and if, by continuous modification of one curve, these two points be made to coalesce into a single point P, then the two curves are said ultimately to have contact 'of the first order' at P. An instance is the contact of a curve with its tangent line. And whenever two curves have contact of the first order, they have a common tangent line.

Again, if two curves intersect in *three* points, and if by continuous modification of one curve these three points are made to coalesce into a single point P, then the two curves are said ultimately to have contact 'of the second order' at P. An instance is the contact of a curve with its osculating circle (Art. 154).

Let us suppose that the two curves

$$y=\phi(x), \quad y=\psi(x) \;\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

intersect at the points for which $x=x_0, x_1, x_2$, respectively. The function

$$F(x)\equiv\phi(x)-\psi(x) \;\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

which represents the difference of the ordinates of the two curves, will vanish for $x=x_0, x_1, x_2$. Hence, on the usual assumptions as to the continuity of $F(x)$ and $F'(x)$, the derived function $F'(x)$ will, by the theorem of Art. 48, vanish for some value of x between x_0 and x_1, say for $x=x_0'$, and again for some value of x between x_1 and x_2, say for $x=x_1'$. Hence, by another application of the theorem referred to, if $F''(x)$ be continuous in the interval from x_0' to x_1', it will vanish for some value of x between x_0' and x_1', say for $x=x_0''$. Hence if, by continuous modification of one of the curves (1), the three points $x=x_0, x_1, x_2$ be made to coalesce into the one point $x=x_0$, the values of $F(x_0)$, $F'(x_0)$, $F''(x_0)$ will all be zero; *i.e.* we shall have

$$\phi(x)=\psi(x),\; \phi'(x)=\psi'(x),\; \phi''(x)=\psi''(x) \ldots(3),$$

simultaneously, for $x=x_0$.

In other words, if two curves have contact of the second order at any point, the values of

$$y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$$

will at that point be respectively identical for the two curves.

Ex. To determine the circle having contact of the second order with the curve

$$y = \phi(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

at a given point.

The equation of a circle with centre (ξ, η) and radius ρ is

$$(x-\xi)^2 + (y-\eta)^2 = \rho^2 \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

If we differentiate this twice with respect to x, we find

$$x - \xi + (y-\eta)\frac{dy}{dx} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (6),$$

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-\eta)\frac{d^2y}{dx^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots (7).$$

In these results, y is regarded as a function of x determined by the equation (5). But if the circle have contact of the second order with the curve (4) at the point (x, y), the values of y, dy/dx, and d^2y/dx^2 will be the same for the circle as for the curve. We may therefore suppose that in (5), (6), (7) the values of x, y, dy/dx, d^2y/dx^2 refer to the curve (4). The equations then determine the circle uniquely, viz. we find that the coordinates of the centre are

$$\xi = x - \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}, \quad \eta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} \quad \ldots\ldots(8),$$

and that the radius is

$$\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(9);$$

cf. Art. 152.

The above considerations may be extended, and we may say that if two curves intersect in $n+1$ consecutive points, or have contact 'of the nth order,' the values of

$$y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}$$

must be respectively identical for the two curves at the point in question.

The investigations of Arts. 203, 204 give a measure of the degree of closeness of two curves in the neighbourhood of a contact of the nth order. By hypothesis we have at the point $x=a$ (say)

$$\phi(a)=\psi(a), \quad \phi'(a)=\psi'(a), \ldots \quad \phi^{(n)}(a)=\psi^{(n)}(a) \ldots (10),$$

and therefore, with $F(x)$ defined by (2),

$$F(a)=0, \quad F'(a)=0, \quad F''(a)=0, \ldots F^{(n)}(a)=0 \ldots (11).$$

It follows that, under the usual conditions,

$$F(a+h)=\frac{h^{n+1}}{(n+1)!}F^{(n+1)}(a+\theta h) \ldots\ldots\ldots (12),$$

where $1>\theta>0$. Hence, if h be infinitely small, the difference of the ordinates is in general a small quantity of the order $n+1$. Moreover, it will or will not change sign with h, according as n is even or odd.

For example, the deviation of a curve from a tangent line, in the neighbourhood of the point of contact, is in general a small quantity of the second order, and the curve does not in general cross the tangent at the point. Again, the deviation of a curve from the osculating circle is a small quantity of the third order, and the curve in general crosses the circle. See Fig. 123, p. 422. But if the contact with the circle be of the fourth order, as at the vertex of a conic, the curve does not cross the circle. The same thing is further illustrated in Fig. 149, p. 572, where the curves numbered 1, 3, 5 do not cross the curve $y=\log(1+x)$ at the origin, whilst the curves numbered 2, 4, 6 do cross it.

207. Maxima and Minima.

If $\phi(x)$ be a function of x which with its first and second derivatives is finite and continuous for all values of the variable considered, we have

$$\phi(a+h)-\phi(a)=h\phi'(a)+\frac{h^2}{2!}\phi''(a+\theta h) \quad \ldots (1),$$

where $1>\theta>0$. By taking h sufficiently small, the second term on the right-hand can in general be made smaller in absolute value than the first, and $\phi(a+h)-\phi(a)$ will then have the same sign as $h\phi'(a)$, and will therefore change sign with h.

Now if $\phi(a)$ is a maximum or a minimum value of $\phi(x)$, the difference

$$\phi(a+h)-\phi(a)$$

must have the same sign for sufficiently small values of h, whether h be positive or negative. Hence we cannot, under the present conditions, have a maximum or a minimum unless $\phi'(a)=0$.

Let us now suppose that $\phi'(a)=0$, so that (1) reduces to

$$\phi(a+h)-\phi(a)=\frac{h^2}{2\,!}\phi''(a+\theta h) \text{ .... } (2).$$

When h is sufficiently small, the sign of the right-hand will be that of $\phi''(a)$. Hence if this be *positive* we shall have $\phi(a+h)>\phi(a)$, whether h be positive or negative; *i.e.* $\phi(a)$ is a *minimum*. Similarly, if $\phi''(a)$ be *negative* we have a *maximum*.

If $\phi''(a)$ vanish simultaneously with $\phi'(a)$, it is necessary to continue the expansion in (1) further. To take at once the general case, if we have

$$\phi'(a)=0, \quad \phi''(a)=0, \ldots \phi^{(n-1)}(a)=0 \text{ ......} (3),$$

simultaneously, but $\phi^{(n)}(a)\neq 0$ ....(4),

then $\phi(a+h)-\phi(a)=\dfrac{h^n}{n\,!}\phi^{(n)}(a+\theta h)$ .... (5).

If h be small enough, the sign is that of $h^n\phi^{(n)}(a)$. If n be odd, this changes sign with h, and we have neither a maximum nor a minimum. But if n be even, we have a maximum or minimum, according as $\phi^{(n)}(a)$ is negative or positive.

In words, $\phi(x)$ is either a maximum or a minimum for a given value of x if the first derivative which does not vanish for this value of x be of even order, but not otherwise. And the function is a maximum or a minimum according as this derivative is negative or positive.

Ex. $\phi(x)=\cosh x+\cos x$ ....(6).

This makes

$$\phi'(x)=\sinh x-\sin x, \qquad \phi''(x)=\cosh x-\cos x,$$
$$\phi'''(x)=\sinh x+\sin x, \qquad \phi^{\text{iv}}(x)=\cosh x+\cos x.$$

The first derivative which does not vanish for $x=0$ is $\phi^{\text{iv}}(x)$. And since $\phi^{\text{iv}}(0)$ is positive, we infer that $\phi(0)$ is a minimum value of $\phi(x)$.

208. Infinitesimal Geometry of Plane Curves.

Let the tangent and normal at any point O of a plane curve be taken as axes of coordinates; it is required to express the coordinates of a neighbouring point P of the curve in terms of the arc OP, $= s$, say.

If, for brevity, we use accents to denote differentiations with respect to s, we have, as in Art. 109,

$$x' = \cos\psi, \quad y' = \sin\psi \quad \ldots\ldots\ldots\ldots\ldots (1),$$

and thence

$$\left.\begin{aligned} x'' &= -\sin\psi \,.\, \psi', & x''' &= -\cos\psi \,.\, \psi'^2 - \sin\psi \,.\, \psi'', \ldots \\ y'' &= \cos\psi \,.\, \psi', & y''' &= -\sin\psi \,.\, \psi'^2 + \cos\psi \,.\, \psi'', \ldots \end{aligned}\right\} \ldots (2),$$

and so on.

Now, by Maclaurin's Theorem,

$$\left.\begin{aligned} x &= x_0 + \frac{s}{1} x_0' + \frac{s^2}{1\,.\,2} x_0'' + \ldots, \\ y &= y_0 + \frac{s}{1} y_0' + \frac{s^2}{1\,.\,2} y_0'' + \ldots \end{aligned}\right\} \ldots\ldots\ldots\ldots (3),$$

where the suffix is used to mark the values which the respective quantities assume for $s=0$. But, putting $\psi = 0$ in (1) and (2), we have

$$\left.\begin{aligned} x_0' &= 1, & x_0'' &= 0, & x_0''' &= -\frac{1}{\rho^2}, \ldots \\ y_0' &= 0, & y_0'' &= \frac{1}{\rho}, & y_0''' &= -\frac{1}{\rho^2}\frac{d\rho}{ds}, \ldots \end{aligned}\right\} \ldots\ldots (4),$$

where $1/\rho$ has been written for $d\psi/ds$. Hence

$$x = s - \frac{s^3}{6\rho^2} + \ldots, \quad y = \frac{s^2}{2\rho} - \frac{s^3}{6\rho^2}\frac{d\rho}{ds} + \ldots \quad \ldots (5),$$

where ρ and $d\rho/ds$ refer to the origin.

These formulæ are useful in various questions of 'infinitesimal geometry.'

Ex. 1. Thus the second formula in (5) shews that the deviation of the curve from the osculating circle at O is ultimately

$$\tfrac{1}{6}\frac{s^3}{\rho^2}\frac{d\rho}{ds} \quad\text{..........................}(6),$$

since $d\rho/ds = 0$ for the circle.

And, generally, for all purposes where s^3 can be neglected, the curve may be replaced by its osculating circle.

Ex. 2. Again, the normal at P meets the normal at O in a point whose distance from O is $y + x \cot \psi$. If we neglect terms of the second order in s, this

$$= \frac{s}{\dfrac{s}{\rho} - \tfrac{1}{2}\dfrac{s^2}{\rho^2}\dfrac{d\rho}{ds}} = \rho\left(1 + \tfrac{1}{2}\frac{s}{\rho}\frac{d\rho}{ds}\right).$$

Hence the distance of the intersection from the centre of curvature at s is ultimately

$$\tfrac{1}{2}s\frac{d\rho}{ds} \quad\text{..........................}(7).$$

When ρ is a maximum or minimum we have (in general) $d\rho/ds = 0$, and the distance is of a higher order, the evolute having then a cusp at the point corresponding to O.

EXAMPLES. LIV.

1.
$$\cosh x \cos x = 1 - \frac{2^2x^4}{4\,!} + \frac{2^4x^8}{8\,!} - \ldots$$
$$\sinh x \sin x = x^2 - \frac{2^3x^6}{6\,!} + \frac{2^5x^{10}}{10\,!} - \ldots.$$

2.
$$\cosh x \sin x = x + \frac{2x^3}{3\,!} - \frac{2^2x^5}{5\,!} - \ldots,$$
$$\sinh x \cos x = x - \frac{2x^3}{3\,!} - \frac{2^2x^5}{5\,!} + \ldots.$$

3.
$$e^x \cos x = 1 + x - \frac{2x^3}{3\,!} - \frac{2^2x^4}{4\,!} - \frac{2^2x^5}{5\,!} + \frac{2^3x^7}{7\,!} + \ldots,$$
$$e^x \sin x = x + x^2 + \frac{2x^3}{3\,!} - \frac{2^2x^5}{5\,!} - \frac{2^3x^6}{6\,!} - \frac{2^3x^7}{7\,!} + \ldots.$$

4. $$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots.$$

5. $$\log \sec x = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \dots.$$

6. $$\tanh x = x - \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 - \tfrac{17}{315}x^7 + \dots.$$

7. $$\cos^2 x = 1 - x^2 + \frac{2^3x^4}{4!} - \frac{2^5x^6}{6!} + \dots.$$

8. $$\cos^n x = 1 - \frac{n}{2!}x^2 + \frac{n(3n-2)}{4!}x^4 - \dots.$$

9. $$\left(\frac{\sin x}{x}\right)^n = 1 - \frac{n}{3!}x^2 + \frac{n(5n-2)}{3.5!}x^4 - \dots.$$

10. $$\frac{x}{\sin x} = 1 + \frac{x^2}{3!} + \frac{14x^4}{6!} + \dots,$$
$$\frac{x}{\sinh x} = 1 - \frac{x^2}{3!} + \frac{14x^4}{6!} - \dots.$$

11. $$\frac{x}{e^x - 1} = 1 - \tfrac{1}{2}x + \tfrac{1}{6}\frac{x^2}{2!} - \tfrac{1}{30}\frac{x^4}{4!} + \dots.$$

12. $$\tan(\tfrac{1}{4}\pi + x) = 1 + 2x + 2x^2 + \tfrac{8}{3}x^3 + \tfrac{10}{3}x^4 + \dots.$$

13. $$\log\tan(\tfrac{1}{4}\pi + x) = 2x + \tfrac{4}{3}x^3 + \tfrac{4}{3}x^5 + \dots.$$

14. $$\log \sec^2 \tfrac{1}{2}\theta = \frac{1}{2}\frac{\sin^2\theta}{2} + \frac{1.3}{2.4}\frac{\sin^4\theta}{4} + \frac{1.3.5}{2.4.6}\frac{\sin^6\theta}{6} + \dots.$$

15. If $D = d/dx$, prove that
$$D^n e^{x\cos a}\cos(x\sin a) = e^{x\cos a}\cos(x\sin a + na).$$
Hence shew that
$$e^{x\cos a}\cos(x\sin a) = 1 + x\cos a + \frac{x^2}{2!}\cos 2a + \frac{x^3}{3!}\cos 3a + \dots.$$

16. Draw graphs of the functions
$$x,\quad x - \frac{x^3}{3!},\quad x - \frac{x^3}{3!} + \frac{x^5}{5!},$$
respectively, and compare them with the graph of $\sin x$,

17. Draw graphs of the functions

$$1-\frac{x^2}{2!},\quad 1-\frac{x^2}{2!}+\frac{x^4}{4!},$$

respectively, and compare them with the graph of $\cos x$.

18. If
$$\phi(a)=0,\quad \psi(a)=0,$$
prove that in general
$$\lim_{x=a}\frac{\phi(x)}{\psi(x)}=\frac{\phi'(a)}{\psi'(a)}.$$

19. Prove that, in the formula (17) of Art. 203, the limiting value of θ, when h is indefinitely diminished, is in general $1/(n+1)$.

20. Prove that when h is sufficiently small the error in Simpson's formula (Art. 112 (8)) of approximate integration is equal to

$$\tfrac{1}{90}h^5\frac{d^4y}{dx^4},$$

nearly.

21. Prove that the mean value of a function $\phi(x)$ over the range extending from $x=a-h$ to $x=a+h$ is

$$\phi(a)+\frac{h^2}{3!}\phi''(a)+\frac{h^4}{5!}\phi^{\text{iv}}(a)+\ldots.$$

Shew that this falls short of the arithmetic mean of the values at the extremities of the range by

$$\frac{h^2}{1.3}\phi''(a)+\frac{h^4}{3!5}\phi^{\text{iv}}(a)+\ldots.$$

22. Shew that if, from a given curve, another curve be constructed whose ordinate, for any value a of x, is the mean of the ordinates of the first curve over the range bounded by $x=a\pm h$, where h is a given small constant, the ordinate of the second curve exceeds that of the first by one-third the sagitta of the arc whose extremities are $x=a\pm h$.

23. Prove that if the expansions (4) of Art. 208 be carried to the order s^4, the results are

$$x=s-\tfrac{1}{6}\frac{s^3}{\rho^2}+\tfrac{1}{8}\frac{s^4}{\rho^3}\frac{d\rho}{ds}+\ldots,$$

$$y=\tfrac{1}{2}\frac{s^2}{\rho}-\tfrac{1}{6}\frac{s^3}{\rho^2}\frac{d\rho}{ds}-\tfrac{1}{24}\frac{s^4}{\rho^3}\left\{1-2\left(\frac{d\rho}{ds}\right)^2+\rho\frac{d^2\rho}{ds^2}\right\}+\ldots.$$

24. If the values of x, y, referred to the tangent and normal at a point O of a curve, be developed in terms of ψ, the inclination of the tangent to the axis of x, the results are

$$x = \rho\psi + \tfrac{1}{2}\frac{d\rho}{d\psi}\psi^2 - \tfrac{1}{6}\left(\rho - \frac{d^2\rho}{d\psi^2}\right)\psi^3 + \dots,$$

$$y = \tfrac{1}{2}\rho\psi^2 + \tfrac{1}{3}\frac{d\rho}{d\psi}\psi^3 + \dots.$$

25. Prove that, with the same axes, the coordinates of the centre of curvature are

$$\xi = -\tfrac{1}{2}\frac{d\rho}{d\psi}\psi^2 - \tfrac{1}{3}\frac{d^2\rho}{d\psi^2}\psi^3 + \dots,$$

$$\eta = \rho + \frac{d\rho}{d\psi}\psi + \tfrac{1}{2}\frac{d^2\rho}{d\psi^2}\psi^3 + \dots.$$

26. If O, P be two adjacent points on a curve, and PQ be drawn perpendicular to the chord OP, to meet the normal at O in Q, then ultimately $OQ = 2\rho$.

27. If equal small lengths OP, OT be measured along the arc of a curve, and along the tangent at the point O, and if R be the limiting position of the intersection of PT produced with the normal at O, then $OR = 3\rho$.

28. The perpendicular to a chord OP at its middle point meets the normal at O a distance from the centre of curvature ultimately equal to $\tfrac{1}{3}s\,d\rho/ds$, where $s = OP$.

29. The tangents at two adjacent points P, Q of a curve meet in T, and V is the middle point of the chord PQ. Prove that TV makes with the normal to the curve an angle $\tan^{-1}(\tfrac{1}{3}d\rho/ds)$.

30. Prove that if PQ be a small arc (s) of a curve, the arc exceeds the chord by $\tfrac{1}{24}s^3/\rho^2$, and the sum of the tangents at P and Q exceeds the arc by $\tfrac{1}{12}s^3/\rho^2$.

31. Prove that the form of a curve near a cusp is given by

$$ay^2 = x^3,$$

approximately, where $a = \tfrac{9}{8}d\rho/d\psi$.

32. Prove that the form of a curve near a point of inflexion is given by

$$y = \tfrac{1}{6}\frac{dc}{ds}x^3,$$

approximately where c is the curvature.

33. If P be any point on a curve, the form of the evolute of the part near P, referred to the centre of curvature at P as origin, is in general given by

$$ay = x^2,$$

where $a = 2d\rho/d\psi$.

34. Prove that if P be a point of maximum or minimum curvature, the form of the evolute is

$$ay^2 = x^3,$$

approximately, where $a = \frac{9}{8}d^2\rho/d\psi^2$.

209. Functions of several Independent Variables. Partial Derivatives of Higher Orders.

If u be a function of two or more independent variables $x, y, \ldots$, the partial derivatives

$$\frac{\partial u}{\partial x}, \ \frac{\partial u}{\partial y}, \ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

will themselves in general be functions of $x, y, \ldots$, and be susceptible of differentiation with respect to these several variables.

Thus, if
$$u = \phi(x, y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
we can form the second derivatives

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right), \ \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right),$$

or, as they are usually written,

$$\frac{\partial^2 u}{\partial x^2}, \ \frac{\partial^2 u}{\partial y \partial x}, \ \frac{\partial^2 u}{\partial x \partial y}, \ \frac{\partial^2 u}{\partial y^2} \qquad \ldots\ldots\ldots\ldots\ldots (3).$$

It will be noticed that there is (primarily) a distinction of meaning between the second and third of these symbols, the operations indicated being performed in inverse orders in the two cases. It will be shewn, however, in Art. 210 that under certain conditions, which are generally satisfied in practice, the results are identical.

The first derivatives of $\phi(x, y)$ are sometimes denoted by

$$\phi_x(x, y), \quad \phi_y(x, y) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (4),$$

and the second derivatives (3) by

$$\phi_{xx}(x, y), \quad \phi_{yx}(x, y), \quad \phi_{xy}(x, y), \quad \phi_{yy}(x, y) \ldots (5).$$

These are often abbreviated into

$$\phi_x, \quad \phi_y \ldots\ldots\ldots\ldots (6),$$

and

$$\phi_{xx}, \quad \phi_{yx}, \quad \phi_{xy}, \quad \phi_{yy} \ldots\ldots\ldots\ldots (7),$$

respectively.

Ex. 1. If

$$u = Ax^m y^n \ldots\ldots\ldots\ldots (8),$$

we have

$$\frac{\partial u}{\partial x} = mAx^{m-1}y^n, \quad \frac{\partial u}{\partial y} = nAx^m y^{n-1} \ldots\ldots\ldots\ldots (9),$$

$$\frac{\partial^2 u}{\partial x^2} = m(m-1)Ax^{m-2}y, \quad \frac{\partial^2 u}{\partial y^2} = n(n-1)Ax^m y^{n-2},$$

$$\frac{\partial^2 u}{\partial y \partial x} = mnAx^{m-1}y^{n-1} = \frac{\partial^2 u}{\partial x \partial y} \ldots\ldots\ldots\ldots (10).$$

Ex. 2. If

$$z = a \tan^{-1}\frac{y}{x} \ldots\ldots\ldots\ldots (11),$$

we find

$$\frac{\partial z}{\partial x} = -\frac{ay}{x^2 + y^2}, \quad \frac{\partial z}{\partial y} = \frac{ax}{x^2 + y^2} \ldots\ldots\ldots\ldots (12),$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2axy}{(x^2 + y^2)^2}, \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{a(y^2 - x^2)}{(x^2 + y^2)^2} = \frac{\partial^2 z}{\partial x \partial y}, \quad \frac{\partial^2 z}{\partial y^2} = \frac{-2axy}{(x^2 + y^2)^2} \ldots (13).$$

210. Proof of the Commutative Property.

Let

$$u = \phi(x, y) \ldots\ldots\ldots\ldots (1),$$

and let us suppose that the functions

$$u, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial u}{\partial y}, \quad \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial^2 u}{\partial x \partial y} \ldots\ldots\ldots\ldots (2)$$

are continuous over a finite range of the variables, including the values considered. We proceed to shew that, under these conditions,

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} \ldots\ldots\ldots\ldots (3).$$

To this end, we consider the fraction

$$\chi(h, k) = \frac{\phi(x + h, y + k) - \phi(x + h, y) - \phi(x, y + k) + \phi(x, y)}{hk} \ldots\ldots\ldots\ldots (4),$$

in which x, y are regarded as fixed, whilst h, k will (finally) be made infinitely small.

Let us write, for a moment,

$$F(x) = \phi(x, y+k) - \phi(x, y) \quad \text{............(5).}$$

By the mean-value theorem of Art. 56 (8), we have

$$F(x+h) - F(x) = hF'(x+\theta_1 h) \quad \text{............(6),}$$

or, in full,

$$\{\phi(x+h,\ y+k) - \phi(x+h,\ y)\} - \{\phi(x,\ y+k) - \phi(x,\ y)\}$$
$$= h\{\phi_x(x+\theta_1 h,\ y+k) - \phi_x(x+\theta_1 h,\ y)\} \quad \text{... (7),}$$

where $1 > \theta_1 > 0$, the value of y not being varied in this process. Hence

$$\chi(h,\ k) = \frac{\phi_x(x+\theta_1 h,\ y+k) - \phi_x(x+\theta_1 h,\ y)}{k} \quad \text{...(8).}$$

If we now write

$$f(y) = \phi_x(x+\theta_1 h,\ y) \quad \text{..................(9),}$$

we have, by a second application of the theorem referred to,

$$f(y+k) - f(y) = kf'(y+\theta_2 k) \quad \text{............(10),}$$

or

$$\phi_x(x+\theta_1 h,\ y+k) - \phi_x(x+\theta_1 h,\ y) = k\phi_{yx}(x+\theta_1 h,\ y+\theta_2 k) \quad \text{............(11).}$$

Hence $$\chi(h,\ k) = \phi_{yx}(x+\theta_1 h,\ y+\theta_2 k) \quad \text{............(12),}$$

where θ_1, θ_2 lie between 0 and 1.

By a similar process we could shew that

$$\chi(h,\ k) = \phi_{xy}(x+\theta_1' h,\ y+\theta_2' k) \quad \text{.........(13),}$$

where θ_1', θ_2' also lie between 0 and 1.

These results are exact, provided $x+h$, $y+k$ lie within the range of the variables for which the conditions above postulated hold. If we now diminish h and k indefinitely, it follows from the comparison of (12) and (13), and from the continuity of the derivatives, that

$$\phi_{yx}(x,\ y) = \phi_{xy}(x,\ y) \quad \text{..............(14),}$$

as was to be proved*.

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* This proof appears to be due to Ossian Bonnet. An alternative proof is indicated in Art. 211.

It appears from (4) that

$$\lim_{h=0}\chi(h, k) = \frac{1}{k}\lim_{h=0}\frac{\phi(x+h, y+k)-\phi(x, y+k)}{h} - \frac{1}{k}\lim_{h=0}\frac{\phi(x+h, y)-\phi(x, y)}{h}$$

$$= \frac{\phi_x(x, y+k)-\phi_x(x, y)}{k} \quad \ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Hence $$\lim_{k=0}\lim_{h=0}\chi(h, k) = \phi_{yx}(x, y) \quad \ldots\ldots\ldots\ldots (16).$$

Similarly, we find

$$\lim_{h=0}\lim_{k=0}\chi(h, k) = \phi_{xy}(x, y) \quad \ldots\ldots\ldots\ldots (17).$$

If, then, we could assume that the limiting value of the fraction (4), when h and k are indefinitely diminished, is unique, and independent of the *order* in which these quantities are made to vanish, the theorem (3) would follow at once. A simple example shews, however, that the assumption is not legitimate without further examination. If

$$f(h, k) = \frac{h^2-k^2}{h^2+k^2},$$

we have

$$\lim_{k=0}\lim_{h=0}f(h, k) = -1, \quad \lim_{h=0}\lim_{k=0}f(h, k) = +1.$$

Ex. The condition that

$$Mdx + Ndy \quad \ldots\ldots\ldots\ldots\ldots\ldots (18)$$

should be an exact differential (Art. 174) is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19).$$

For if the expression (18) be equal to du, we have

$$M = \frac{\partial u}{\partial x}, \quad N = \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots\ldots (20),$$

and therefore each of the partial derivatives in (19) is equal to $\partial^2 u/\partial y\partial x$ or $\partial^2 u/\partial x\partial y$.

Conversely, we can shew that, if the condition (19) hold, (18) will be an exact differential. Let v denote the function $\int Mdx$, obtained by integrating as if y were constant. We have, then,

$$\frac{\partial v}{\partial x} = M \quad \ldots\ldots\ldots\ldots\ldots\ldots (21),$$

and therefore
$$\frac{\partial N}{\partial x}=\frac{\partial M}{\partial y}=\frac{\partial^2 v}{\partial x\partial y},$$

or
$$\frac{\partial}{\partial x}\left(N-\frac{\partial v}{\partial y}\right)=0 \quad\text{.......................}(22).$$

This shews (Art. 56) that the function $N-\partial v/\partial y$ is constant so far as x is concerned. Denoting its value by $f'(y)$, we have

$$N=\frac{\partial v}{\partial y}+f'(y) \quad\text{.....................}\ (23).$$

Hence, if we write
$$u=v+f(y) \quad\text{.........................}(24),$$
we have, by (21) and (23),
$$\frac{\partial u}{\partial x}=M,\quad \frac{\partial u}{\partial y}=N\text{......................}\ (25),$$

and therefore
$$Mdx+Ndy=du \quad\text{....................}\ (26)^*.$$

It follows from the above theorem that in the case of a function of any number of independent variables $x, y, z, \ldots$ the operations
$$\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z},\ldots$$

or, as we may denote them for shortness,
$$D_x,\ D_y,\ D_z,\ldots$$

are in general commutative, *i.e.* the result of any number of them is independent of the order in which they are performed.

For example,
$$D_xD_yD_zu=D_x(D_yD_zu)=D_x(D_zD_y)\,u=D_xD_z(D_yu)$$
$$=D_zD_xD_yu=\text{etc.}$$

211. Extension of Taylor's Theorem.

Let $\phi(x, y)$ be a function of x and y which, with its derivatives up to a certain order, is continuous for all values of the variables considered. It may be required to find the expansion of
$$\phi(a+h,\ b+k) \quad\text{.....................}(1)$$

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* Cf. Murray, *Differential Equations* (1897), p. 197.

in ascending powers of h and k. We shall in the first place give a direct investigation of the expansion as far as the terms of the *second* degree in h and k.

First expanding in powers of h, we have, by Taylor's Theorem,

$$\phi(a+h, b+k) = \phi(a, b+k) + h\phi_x(a, b+k) + \tfrac{1}{2}h^2\phi_{xx}(a, b+k) + \dots \quad \dots(2).$$

Again, by the same theorem,

$$\left.\begin{aligned} \phi(a, b+k) &= \phi(a, b) + k\phi_y(a, b) + \tfrac{1}{2}k^2\phi_{yy}(a, b) + \dots \\ \phi_x(a, b+k) &= \phi_x(a, b) + k\phi_{yx}(a, b) + \dots \\ \phi_{xx}(a, b+k) &= \phi_{xx}(a, b) + \dots \end{aligned}\right\} \quad (3).$$

Substituting in (2), we find

$$\phi(a+h, b+k) = \phi(a, b) + \{h\phi_x(a, b) + k\phi_y(a, b)\} + \tfrac{1}{2}\{h^2\phi_{xx}(a, b) + 2hk\phi_{yx}(a, b) + k^2\phi_{yy}(a, b)\} + \dots \quad \dots(4).$$

If we regard the forms of the several 'remainders' (Art. 203) in the preliminary expansions, it appears that the remainder in (4) will be of the form

$$\frac{1}{3!}\{Rh^3 + 3Sh^2k + 3Thk^2 + Uk^3\} \quad \dots\dots\dots\dots(5),$$

where R, S, T, U are functions of a, b, h, k which remain finite when h, k are indefinitely diminished. The remainder is therefore of the *third* order in h, k.

The conditions for the validity of the foregoing result are that $\phi(x, y)$ and its derivatives up to the third order should be continuous for all values of the variable considered.

It may be remarked that in the proof of (4) it was not necessary to assume that

$$\phi_{yx}(a, b) = \phi_{xy}(a, b) \quad \dots\dots\dots\dots\dots\dots\dots(6).$$

If we had begun by expanding (1) in powers of k (instead of h) we should have arrived at a result similar to (4), but with $\phi_{xy}(a, b)$ in place of $\phi_{yx}(a, b)$. From a comparison of the two forms we can obtain an independent proof of the theorem of Art. 210.

With a slight change of notation we may write (4) in the form

$$\phi(x+h, y+k) = \phi(x, y) + \left(h\frac{\partial\phi}{\partial x} + k\frac{\partial\phi}{\partial y}\right)$$
$$+ \tfrac{1}{2}\left(h^2\frac{\partial^2\phi}{\partial x^2} + 2hk\frac{\partial^2\phi}{\partial x\partial y} + k^2\frac{\partial^2\phi}{\partial y^2}\right) + \dots \dots (7),$$

where, on the right-hand, ϕ stands for $\phi(x, y)$. A more compact form is

$$\phi(x+h, y+k) = \phi(x, y) + (h\phi_x + k\phi_y)$$
$$+ \tfrac{1}{2}(h^2\phi_{xx} + 2hk\phi_{xy} + k^2\phi_{yy}) + \dots \dots (8).$$

Again, if u be any function of the independent variables x, y, and if, as in Art. 60, δu denote the increment of u due to given increments δx, δy of these variables, the formula is equivalent to

$$\delta u = \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \tfrac{1}{2}\left\{\frac{\partial^2 u}{\partial x^2}(\delta x)^2 + 2\frac{\partial^2 u}{\partial x\partial y}\delta x\delta y + \frac{\partial^2 u}{\partial y^2}(\delta y)^2\right\} + \dots$$
$$\dots\dots\dots\dots(9)^*.$$

An independent investigation, giving the general term of the expansion (7), is as follows. We write $h = \alpha t$, $k = \beta t$,

and $$F(t) = \phi(x+h,\ y+k) = \phi(x+\alpha t,\ y+\beta t) \dots (10).$$

Regarded as a function of t, this can be expanded by Maclaurin's theorem, and the general term is

$$\frac{t^n}{n!}F^{(n)}(0) \quad \dots\dots\dots\dots\dots\dots(11).$$

Now if we put for a moment $x + \alpha t = u$, $y + \beta t = v$, we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial\phi}{\partial u}, \quad \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial\phi}{\partial v} \quad \dots\dots(12),$$

where ϕ is written for $\phi(u, v)$. Hence

$$F'(t) = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial t} = \alpha\frac{\partial\phi}{\partial x} + \beta\frac{\partial\phi}{\partial y}$$
$$= \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)\phi(u, v) \dots\dots\dots\dots(13).$$

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* The extension of the investigations of this Art. to cases where there are three or more independent variables will be obvious.

The result is evidently a function of u and v; hence, by a repetition of the argument,

$$F''(t) = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)\phi(u, v)$$

$$= \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)^2 \phi(u, v) \quad\ldots\ldots\ldots\ldots\ldots\ldots(14),$$

and, generally,

$$F^{(n)}(t) = \left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)^n \phi(u, v) \quad\ldots\ldots\ldots(15),$$

where the operator admits of expansion by the Binomial Theorem, in virtue of the commutative property of the operators $\partial/\partial x$ and $\partial/\partial y$. Since t only occurs in the combinations $x + \alpha t$, $y + \beta t$, it is evidently immaterial in (15) whether we put $t = 0$ before or after the differentiations indicated on the right-hand side. The general term of our expansion is therefore

$$\frac{t^n}{n!}F^{(n)}(0) = \frac{t^n}{n!}\left(\alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y}\right)^n \phi(x, y) = \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n \phi(x, y)$$

$$= \frac{1}{n!}\left(h^n\frac{\partial^n\phi}{\partial x^n} + nh^{n-1}k\frac{\partial^n\phi}{\partial x^{n-1}\partial y} + \frac{n(n-1)}{1.2}h^{n-2}k^2\frac{\partial^n\phi}{\partial x^{n-2}\partial y^2} + \ldots\right) \quad\ldots\ldots(16),$$

where ϕ is now written for $\phi(x, y)$.

Ex. To prove that if $\phi(x, y)$ be a homogeneous function of x, y, of degree m, we have

$$x\phi_x + y\phi_y = m\phi \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(17).$$

$$x^2\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy} = m(m-1)\phi \quad\ldots\ldots\ldots(18).$$

The general definition of a homogeneous function of degree m is that if x and y be altered in any ratio μ, the function is altered in the ratio μ^m, or

$$\phi(\mu x, \mu y) = \mu^m \phi(x, y) \quad\ldots\ldots\ldots\ldots\ldots\ldots(19).$$

In this equality, let us put $\mu = 1 + t$. Since

$$\phi(x + xt, y + yt) = \phi(x, y) + t(x\phi_x + y\phi_y)$$
$$+ \tfrac{1}{2}t^2(x^2\phi_{xx} + 2xy\phi_{xy} + y^2\phi_{yy}) + \ldots$$

by (8), and

$$(1 + t)^m \phi(x, y) = \left(1 + mt + \frac{m(m-1)}{1.2}t^2 + \ldots\right)\phi,$$

by the Binomial Theorem, the results (18) and (19) will follow, on equating coefficients of t and t^2. More generally, equating coefficients of t^n, and making use of (16), we find

$$x^n\frac{\partial^n\phi}{\partial x^n}+nx^{n-1}y\,\frac{\partial^n\phi}{\partial x^{n-1}\partial y}+\frac{n(n-1)}{1\,.\,2}\,x^{n-2}y^2\,\frac{\partial^n\phi}{\partial x^{n-2}\partial y^2}+\ldots$$
$$=m(m-1)(m-2)\ldots(m-n+1)\,\phi\ldots\ldots(20).$$

This is 'Euler's Theorem of Homogeneous Functions,' for the case of two independent variables. The extension to three or more independent variables will be obvious.

212. Maxima and Minima of a Function of Two Variables. Geometrical Interpretation.

We may utilize the generalized form of Taylor's Theorem to carry a step further the discussion (see Art. 59) of the maxima and minima of a function (u) of two independent variables (x, y).

It appears from Art. 211 (9) that when δx, δy are continually diminished in absolute value, preserving any given ratio to one another, the sign of δu is ultimately that of

$$\frac{\partial u}{\partial x}\,\delta x+\frac{\partial u}{\partial y}\,\delta y\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

Unless $\partial u/\partial x$ and $\partial u/\partial y$ both vanish, the sign of (1) is reversed by reversing the signs of δx and δy. Hence for some variations δu will be positive, and for others negative. In other words, u cannot be a maximum or minimum unless we have

$$\frac{\partial u}{\partial x}=0,\quad \frac{\partial u}{\partial y}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)^*$$

simultaneously.

Let us now suppose the conditions (2) to be fulfilled. We have, then,

$$\delta u=\tfrac{1}{2}\left\{\frac{\partial^2u}{\partial x^2}(\delta x)^2+2\,\frac{\partial^2u}{\partial x\partial y}\,\delta x\delta y+\frac{\partial^2u}{\partial y^2}(\delta y)^2\right\}+\ldots\ldots(3).$$

When δx and δy are sufficiently small, the sign of δu will be that of the terms written. Now it is known from Algebra that the sign of a homogeneous quadratic function

$$A\xi^2+2H\xi\eta+B\eta^2\ \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

is invariable, if (and only if)

$$AB>H^2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

---

* It is assumed in the investigation of Art. 211 that these derivatives are continuous and therefore finite. That is, we exclude *ab initio* the two-dimensional analogues of the cases considered in Art. 51.

and that the sign is then that of A (or B). We infer that when the conditions (2) are satisfied δu will have the same sign for all values of $|\delta x|$ and $|\delta y|$ not exceeding certain limits, provided

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} > \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \qquad\qquad (6),$$

and that the sign will then be that of $\partial^2 u/\partial x^2$ and $\partial^2 u/\partial y^2$. And u will be a maximum or minimum according as this sign is negative or positive.

If
$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} < \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \qquad\qquad (7),$$

then for some values of the ratio $\delta y/\delta x$ the increment of u will be positive, for others negative, and the value of u, though 'stationary' (cf. Art. 50) is neither a maximum nor a minimum.

If
$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 \qquad\qquad (8)$$

the terms which appear on the right-hand side of (3) are equal to ± the square of a linear function of δx and δy, and therefore vanish for a particular value of the ratio $\delta y/\delta x$. Since δu is then of the third order it appears that there is in general neither a maximum nor a minimum, but the question cannot be absolutely decided without continuing the expansion further. The same remark applies when the second derivatives $\partial^2 u/\partial x^2$, $\partial^2 u/\partial x \partial y$, $\partial^2 u/\partial y^2$ all vanish.

The preceding investigation has an interesting geometrical interpretation. If, as in Art. 45, z be the vertical ordinate of a surface, and x, y rectangular coordinates in a horizontal plane, the first condition for a point of maximum or minimum altitude is that

$$\frac{\partial z}{\partial x} = 0, \quad \frac{\partial z}{\partial y} = 0 \qquad\qquad (9)$$

simultaneously. Since these equations ensure that δz shall be of the second order in δx, δy, it follows that at the point (P, say,) in question the tangent line to every vertical section through P will be horizontal; in other words, we have a horizontal *tangent plane*.

We have next to examine whether the surface cuts the tangent plane at P. Along the line of intersection (if any), we shall have $\delta z = 0$, and therefore from (3), if we put $\delta y = m\delta x$, and finally make δx vanish, the directions of the tangent lines at P to the curve of intersection are determined by

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} m + \frac{\partial^2 z}{\partial y^2} m^2 = 0 \quad \text{..............(10).}$$

This quadratic in m will have imaginary roots if

$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2} > \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{.....................(11);}$$

the surface then, in the immediate neighbourhood of P, will lie wholly on one side of the tangent plane, and the contour-line at P reduces to a point. Hence P will be a point of maximum or minimum altitude according as $\partial^2 z/\partial x^2$ and $\partial^2 z/\partial y^2$ are negative or positive, *i.e.* (Art. 68) according as the vertical sections parallel to the planes zx and zy are convex or concave upwards. If we imagine the axes of x, y to be rotated in their own plane, we can infer that *every* vertical section through P is in this case convex upwards, or concave upwards, respectively.

But if
$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2} < \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{....................(12),}$$

the roots of (10) are real and distinct. The contour-line has a node at P, the two branches separating the parts of the surface which lie above the tangent plane from those which lie below.

If
$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 \quad \text{....................(13),}$$

the roots of (10) are real and coincident. The contour-line has in general a cusp at P, and the question as to whether the altitude at P is a maximum or minimum cannot be determined without further investigation.

Ex. 1. Let
$$z = x^3 - 3ax^2 - 4ay^2 + C \quad \text{...............(14).}$$

This makes
$$\frac{\partial z}{\partial x} = 3x(x-2a), \quad \frac{\partial z}{\partial y} = -8ay \quad \text{..............(15),}$$

$$\frac{\partial^2 z}{\partial x^2} = 6(x-a), \quad \frac{\partial^2 z}{\partial x \partial y} = 0, \quad \frac{\partial^2 z}{\partial y^2} = -8a \quad \text{........(16).}$$

The conditions (9) are satisfied by $x = 0$, $y = 0$, and also by $x = 2a$, $y = 0$. The former solution satisfies the inequality (11), and since $\partial^2 z/\partial y^2$ is negative, z is a *maximum*. The latter solution comes under (12); z is then neither a maximum nor a minimum. The contour-lines for this case are shewn in Fig. 74, p. 335.

Ex. 2. Let $z = (x^2 + y^2)^2 - 2a^2 (x^2 - y^2) + C$ ....(17).

We find $\dfrac{\partial z}{\partial x} = 4x (x^2 + y^2 - a^2)$, $\dfrac{\partial z}{\partial y} = 4y (x^2 + y^2 + a^2)$ ......(18),

$$\frac{\partial^2 z}{\partial x^2} = 4 (3x^2 + y^2 - a^2),\ \frac{\partial^2 z}{\partial x \partial y} = 8xy,\ \frac{\partial^2 z}{\partial y^2} = 4 (x^2 + 3y^2 + a^2) \ \dots (19).$$

The real solutions of (9) are, in this case, $x = 0$, $y = 0$, and $x = \pm a$, $y = 0$. The former values fulfil the relation (12), so that z is neither a maximum nor a minimum. The solutions $x = \pm a$, $y = 0$ satisfy (11), and, since they make $\partial^2 z/\partial x^2$ positive, z is then a *minimum*. The contour-lines of the surface (17) are shewn in Fig. 113, p. 389. The surface has two symmetrical hollows with the 'bar' between them. If we reverse the sign of the right-hand side of (17), we get two peaks, with the 'pass' between them.

213. Applications of Partial Differentiation.

Numerous problems of partial differentiation present themselves in geometrical and physical questions. As a rule, they are best dealt with as they arise; but we give one or two simple cases which may serve to elucidate the chief points to be attended to.

1°. Let $u = \phi (v)$ ....(1),

where v is a function of the independent variables x, y; and let it be required to form the successive partial derivatives of u with respect to these variables.

By Art. 39 we have

$$\frac{\partial u}{\partial x} = \phi' (v) \frac{\partial v}{\partial x},\quad \frac{\partial u}{\partial y} = \phi' (v) \frac{\partial v}{\partial y} \ \dots\dots\dots\dots(2).$$

Again,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \phi'(v) \cdot \frac{\partial v}{\partial x} + \phi' (v) \frac{\partial^2 v}{\partial x^2} = \phi'' (v) \left(\frac{\partial v}{\partial x}\right)^2 + \phi' (v) \frac{\partial^2 v}{\partial x^2} \dots (3),$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y} \phi' (v) \cdot \frac{\partial v}{\partial x} + \phi' (v) \frac{\partial^2 v}{\partial x \partial y} = \phi'' (v) \frac{\partial v}{\partial x} \frac{\partial v}{\partial x} + \phi' (v) \frac{\partial^2 v}{\partial x \partial y} \ \dots\dots\dots\dots(4),$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\phi'(v)\,.\,\frac{\partial v}{\partial y} + \phi'(v)\frac{\partial^2 v}{\partial y^2} = \phi''(v)\left(\frac{\partial v}{\partial y}\right)^2 + \phi''(v)\frac{\partial^2 v}{\partial y^2} \ldots (5),$$

and so on.

Ex. 1. Let $z = \phi(x - ct) + \chi(x + ct)$ ....(6),
where the variables x and t are independent.

Putting $x - ct = u, \quad x + ct = v,$
for shortness, we find

$$\frac{\partial z}{\partial x} = \phi'(u) + \chi'(u), \quad \frac{\partial z}{\partial t} = -c\phi'(u) + c\chi'(u) \;\ldots\ldots.(7),$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(u) + \chi''(u), \quad \frac{\partial^2 z}{\partial t^2} = c^2\phi''(u) + c^2\chi''(u) \;\ldots(8).$$

Hence

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9).$$

2°. Let $u = \phi(x, y)$ ....(10),

where x, y are given functions of the independent variable t; and let it be required to calculate the derivatives of u with respect to t. We have, by Art. 62, 1°,

$$\frac{du}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} \;\ldots\ldots\ldots\ldots\ldots (11).$$

Differentiating again, we find

$$\frac{d^2u}{dt^2} = \frac{\partial\phi}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial\phi}{\partial y}\frac{d^2y}{dt^2} + \frac{d}{dt}\left(\frac{\partial\phi}{\partial x}\right)\frac{dx}{dt} + \frac{d}{dt}\left(\frac{\partial\phi}{\partial y}\right)\frac{dy}{dt} \ldots (12).$$

Now, by the theorem referred to,

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\frac{dy}{dt},$$

and

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)\frac{dy}{dt}.$$

Substituting in (12), and recalling the commutative property established in Art. 210, we have

$$\frac{d^2u}{dt^2} = \frac{\partial\phi}{\partial x}\frac{d^2x}{dt^2} + \frac{\partial\phi}{\partial y}\frac{d^2y}{dt^2} + \frac{\partial^2\phi}{\partial x^2}\left(\frac{dx}{dt}\right)^2 + 2\frac{\partial^2\phi}{\partial x\partial y}\frac{dx}{dt}\frac{dy}{dt} + \frac{\partial^2\phi}{\partial y^2}\left(\frac{dy}{dt}\right)^2$$
$$\ldots\ldots\ldots\ldots(13).$$

The process might be continued, but it is seldom necessary to proceed beyond this stage.

This process is sometimes required when we transform the coordinates in a dynamical problem. Thus, to change from rectangular to polar coordinates in two dimensions, we have

$$x = r\cos\theta, \quad y = r\sin\theta,$$

and the above method enables us to express d^2x/dt^2 and d^2y/dt^2 in terms of the differential coefficients of r and θ with respect to t.

3°. Let y be a function of x, defined 'implicitly' by the equation

$$\phi(x, y) = 0 \text{ ....}(14);$$

it is required to calculate the successive derivatives of y with respect to x.

We have, as in Art. 62,

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \text{ .... } (15).$$

If we differentiate this with respect to x, we find

$$\frac{d}{dx}\left(\frac{\partial\phi}{\partial x}\right) + \frac{d}{dx}\left(\frac{\partial\phi}{\partial y}\right)\frac{dy}{dx} + \frac{\partial\phi}{\partial y}\frac{d^2y}{dx^2} = 0 \text{ ... } (16).$$

Now, by Art. 62,

$$\frac{d}{dx}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\frac{dy}{dx},$$

$$\frac{d}{dx}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{dy}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)\frac{dy}{dx}.$$

Hence (16) becomes

$$\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial x\partial y}\frac{dy}{dx} + \frac{\partial^2\phi}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{\partial\phi}{\partial y}\frac{d^2y}{dx^2} = 0 \text{ ... } (17).$$

If we substitute in this the value of dy/dx from (15), we can find d^2y/dx^2, Again, differentiating (17) we could find d^3y/dx^3 and so on.

The above process is sometimes useful in the geometrical applications of the subject. It may be noticed that (17) is included in (13) by putting $t = x$, $u = 0$.

4°. Let $$u = \phi(\xi, \eta) \quad \text{.....................} (18),$$
where ξ, η are given functions of x and y, and let it be required to calculate the second partial derivatives of u with respect to x and y.

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \text{......} (19)$$

Hence
$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right)\frac{\partial \xi}{\partial x} \\ &\quad + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right)\frac{\partial \eta}{\partial x} \\ &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial x}\right)^2 \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} \text{ .... } (20).\end{aligned}$$

In like manner we should find

$$\begin{aligned}\frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right) + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x \partial y} \text{............} (21),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 \\ &\quad + \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial y^2} \text{ .... } (22).\end{aligned}$$

Ex. 2. To change from rectangular to polar coordinates in the expression

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \text{ .... } (23).$$

Putting $$x = r\cos\theta, \quad y = r\sin\theta \text{.................} (24),$$
we find

$$\left.\begin{aligned}\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}, \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = r\left(-\sin\theta\frac{\partial u}{\partial x} + \cos\theta\frac{\partial u}{\partial y}\right)\end{aligned}\right\} \text{...} (25),$$

whence

$$\frac{\partial u}{\partial x} = \cos\theta\,\frac{\partial u}{\partial r} - \sin\theta\,\frac{\partial u}{r\partial\theta}, \quad \frac{\partial u}{\partial y} = \sin\theta\,\frac{\partial u}{\partial r} + \cos\theta\,\frac{\partial u}{r\partial\theta} \quad \text{...(26)}.$$

Hence

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \left(\cos\theta\,\frac{\partial}{\partial r} - \sin\theta\,\frac{\partial}{r\partial\theta}\right)\left(\cos\theta\,\frac{\partial u}{\partial r} - \sin\theta\,\frac{\partial u}{r\partial\theta}\right), \\ \frac{\partial^2 u}{\partial y^2} &= \left(\sin\theta\,\frac{\partial}{\partial r} + \cos\theta\,\frac{\partial}{r\partial\theta}\right)\left(\sin\theta\,\frac{\partial u}{\partial r} + \cos\theta\,\frac{\partial u}{r\partial\theta}\right) \end{aligned}\right\} \text{...(27)}.$$

It is not necessary to perform all the operations indicated, as several of the terms will obviously cancel when we form the sum (23). The remaining terms give

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \quad \text{............(28)}.$$

EXAMPLES. LV.

1. If $$u = \frac{xy}{x+y},$$

verify the relations $$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u,$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

2. If $$z = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y},$$

prove that $$\frac{\partial^2 z}{\partial x\partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

3. If $$z = F(x) + f(y),$$

prove that $$\frac{\partial^2 z}{\partial x\partial y} = 0.$$

Conversely shew that, if $\partial^2 z/\partial x\partial y = 0$, z must have the above form.

4. Prove that the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

is satisfied by $$\phi = \left(Ar^n + \frac{B}{r^n}\right)\cos n(\theta - \alpha).$$

5. Prove that any differential equation of the type

$$F(x^2+y^2)(xdx+ydy)+f\left(\frac{y}{x}\right)(xdy-ydx)=0$$

becomes exact on division by x^2+y^2.

6. Prove that

$$\frac{\sinh y\,dx-\sin x\,dy}{\cosh y-\cos x}$$

is an exact differential of a function u; and find u.

7. If $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0,$$

prove that a function ψ exists such that

$$\frac{\partial\phi}{\partial x}=-\frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x},$$

and that $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0.$$

8. If $$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}=0,$$

prove that a function ψ exists such that

$$\frac{\partial\phi}{\partial r}=-\frac{\partial\psi}{rd\theta},\quad \frac{\partial\phi}{r\partial\theta}=\frac{\partial\psi}{\partial r},$$

and that ψ satisfies the same partial differential equation as ϕ.

9. If $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{1}{y}\frac{\partial\phi}{\partial y}=0,$$

prove that a function ψ exists such that

$$\frac{\partial\phi}{\partial x}=-\frac{1}{y}\frac{\partial\psi}{\partial y},\quad \frac{\partial\phi}{\partial y}=\frac{1}{y}\frac{\partial\psi}{\partial x},$$

and that $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}-\frac{1}{y}\frac{\partial\psi}{\partial y}=0.$$

10. Prove that in the surface

$$az=x^2-y^2,$$

the ordinate (z) is stationary, but not a maximum or minimum, when $x=0$, $y=0$. Sketch the contour-lines of the surface.

11. Prove by means of the rule of Art. 212 that the parallelepiped of least surface for a given volume is a cube.

12. If A, B, C be the vertices of a triangle, and P a variable point, the sum

$$PA^2 + PB^2 + PC^2$$

is a minimum when P coincides with the mass-centre of three equal particles at A, B, C.

13. With the same notation, the sum

$$m_1 . PA^2 + m_2 . PB^2 + m_3 . PC^2$$

is a minimum when P coincides with the mass-centre of three particles m_1, m_2, m_3 situate at A, B, C.

14. Find for what values of x, y the ordinate of the surface

$$z = x^3 + y^3 - 3axy$$

is stationary.

[The values are a, a, and 0, 0. The latter do not make z a maximum or minimum.]

15. Prove that the ordinate of the surface

$$c^2 z = ay^2 - x^3$$

is stationary, but not a maximum or minimum, when $x = 0$, $y = 0$. Sketch the contour lines.

16. Find for what values of x, y the function

$$x^4 + y^4 - 2(x - y)^2$$

is stationary.

[There is a stationary value when $x = 0$, $y = 0$, and two minima when $x = \pm\sqrt{2}$, $y = \mp\sqrt{2}$.]

17. Prove that the function

$$(x^2 + y^2)\, e^{x^2 - y^2}$$

has a minimum value when $x = 0$, $y = 0$, and a stationary value which is neither a maximum nor a minimum when $x = 0$, $y = \pm 1$.

18. Prove that the ordinate of the surface

$$z = f(ax^2 + 2hxy + by^2)$$

is in general stationary when $x = 0$, $y = 0$, and examine whether it is a maximum or minimum. Sketch the contour-lines in the several cases.

19. Find the stationary points of the function

$$(x^2-a^2)^2+(x^2-a^2)(y^2-b^2)+(y^2-b^2)^2,$$

and examine their nature.

Sketch the contour-lines of the function.

20. If

$$u=f\left(\frac{y}{x}\right),$$

prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0,$$

and

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{1}{x^4}\left\{(x^2+y^2)f''\left(\frac{y}{x}\right)+2xyf'\left(\frac{y}{x}\right)\right\}.$$

21. If

$$z=f(x^2+y^2),$$

prove that

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=4(x^2+y^2)f''(x^2+y^2)+4f'(x^2+y^2).$$

22. If

$$u=f(r),$$

where $r=\sqrt{(x^2+y^2)}$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

23. If ∇_1^2 stand for the operator $\partial^2/\partial x^2+\partial^2/\partial y^2$, prove that

$$\nabla_1^2\log r=0,$$

where

$$r=\sqrt{\{(x-\alpha)^2+(y-\beta)^2\}}.$$

24. If

$$u=f(x^2+y^2+z^2),$$

prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4(x^2+y^2+z^2)f''(x^2+y^2+z^2)+6f'(x^2+y^2+z^2).$$

25. If

$$u=f(r),$$

where $r=\sqrt{(x^2+y^2+z^2)}$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=f''(r)+\frac{2}{r}f'(r).$$

26. If ∇^2 stand for the operator $\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$, prove that

$$\nabla^2 r=\frac{2}{r},\qquad \nabla^2\frac{1}{r}=0,$$

where

$$r=\sqrt{\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\}}.$$

27. With the same meaning of ∇^2, prove that if

$$\nabla^2 u = 0, \qquad \nabla^2 v = 0, \qquad \nabla^2 w = 0,$$

and
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

then
$$\nabla^2 (xu + yv + zw) = 0.$$

28. If u, v be two functions of x, y, z satisfying the equations $\nabla^2 u = 0$, $\nabla^2 v = 0$, and if v be a function of u, then v must be of the form $Au + B$.

29. If
$$x = r \cos\theta, \qquad y = r \sin\theta,$$

where r, θ are functions of t, prove that

$$\frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt^2}\sin\theta = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2,$$

$$-\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta = \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right).$$

30. If
$$u = \frac{1}{r}\phi(ct - r) + \frac{1}{r}\chi(ct + r),$$

prove that
$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right).$$

31. If
$$u = t^{-\frac{1}{2}} e^{-x^2/4\kappa t},$$

prove that
$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}.$$

32. If
$$u = t^{-\frac{3}{2}} e^{-r^2/4\kappa t},$$

prove that
$$\frac{\partial u}{\partial t} = \kappa\left(\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r}\right).$$

33.
$$u = x^n \phi\left(\frac{y}{x}\right),$$

verify that
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$$

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

34. If
$$y - nz = f(x - mz),$$

prove that
$$m\frac{\partial z}{\partial x} + n\frac{\partial z}{\partial y} = 1.$$

35. $$z-\gamma=(x-a)f\left(\frac{y-\beta}{x-a}\right),$$

prove that $$(x-a)\frac{\partial z}{\partial x}+(y-\beta)\frac{\partial z}{\partial y}=z-\gamma.$$

36. If $x=c\cosh\xi\cos\eta$, $y=c\sinh\xi\sin\eta$,
prove that
$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}=\tfrac{1}{2}c^2(\cosh 2\xi-\cos 2\eta)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right).$$

37. If ξ, η be current coordinates, the equations of the tangent and normal at any point (x, y) on the curve
$$\phi(x, y)=0$$
are $$(\xi-x)\phi_x+(\eta-y)\phi_y=0,$$
and $$(\xi-x)/\phi_x=(\eta-y)/\phi_y,$$
respectively.

38. If y be a function of x determined by the equation
$$\phi(x, y)=0,$$
prove that $$\frac{d^2y}{dx^2}=-\frac{\phi_y{}^2\phi_{xx}-2\phi_x\phi_y\phi_{xy}+\phi_x{}^2\phi_{yy}}{\phi_y{}^3}.$$

39. Prove that the curvature at any point of the curve
$$\phi(x, y)=0$$
is $$\pm\frac{\phi_y{}^2\phi_{xx}-2\phi_x\phi_y\phi_{xy}+\phi_x{}^2\phi_{yy}}{(\phi_x{}^2+\phi_y{}^2)^{\frac{3}{2}}}.$$

40. Prove that if at a point on the curve
$$\phi(x, y)=0$$
we have $$\phi_x=0, \qquad \phi_y=0,$$
simultaneously, then two (real or imaginary) branches of the curve pass through that point, whose directions are given by the quadratic
$$\phi_{xx}+2\phi_{xy}\frac{dy}{dx}+\phi_{yy}\left(\frac{dy}{dx}\right)^2=0.$$

Hence shew that the point is a node, a cusp, or an isolated point, according as
$$(\phi_{xy})^2 \gtreqless \phi_{xx}\phi_{yy}.$$

APPENDIX.

NUMERICAL TABLES.

A. Table of Squares of Numbers from 10 to 100.

	0	1	2	3	4	5	6	7	8	9
1	100	121	144	169	196	225	256	289	324	361
2	400	441	484	529	576	625	676	729	784	841
3	900	961	1024	1089	1156	1225	1296	1369	1444	1521
4	1600	1681	1764	1849	1936	2025	2116	2209	2304	2401
5	2500	2601	2704	2809	2916	3025	3136	3249	3364	3481
6	3600	3721	3844	3969	4096	4225	4356	4489	4624	4761
7	4900	5041	5184	5329	5476	5625	5776	5929	6084	6241
8	6400	6561	6724	6889	7056	7225	7396	7569	7744	7921
9	8100	8281	8464	8649	8836	9025	9216	9409	9604	9801

B. 1. Table of Square-Roots of Numbers from 0 to 10, at Intervals of ·1.

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
0	0	·316	·447	·548	·632	·707	·775	·837	·894	·949
1	1·000	1·049	1·095	1·140	1·183	1·225	1·265	1·304	1·342	1·378
2	1·414	1·449	1·483	1·517	1·549	1·581	1·612	1·643	1·673	1·703
3	1·732	1·761	1·789	1·817	1·844	1·871	1·897	1·924	1·949	1·975
4	2·000	2·025	2·049	2·074	2·098	2·121	2·145	2·168	2·191	2·214
5	2·236	2·258	2·280	2·302	2·324	2·345	2·366	2·387	2·408	2·429
6	2·449	2·470	2·490	2·510	2·530	2·550	2·569	2·588	2·608	2·627
7	2·646	2·665	2·683	2·702	2·720	2·739	2·757	2·775	2·793	2·811
8	2·828	2·846	2·864	2·881	2·898	2·915	2·933	2·950	2·966	2·983
9	3·000	3·017	3·033	3·050	3·066	3·082	3·098	3·114	3·130	3·146
10	3·162									

B. 2. Table of Square-Roots of Numbers from 10 to 100, at Intervals of 1.

	0	1	2	3	4	5	6	7	8	9
1	3·162	3·317	3·464	3·606	3·742	3·873	4·000	4·123	4·243	4·359
2	4·472	4·583	4·690	4·796	4·899	5·000	5·099	5·196	5·292	5·385
3	5·477	5·568	5·657	5·745	5·831	5·916	6·000	6·083	6·164	6·245
4	6·325	6·403	6·481	6·557	6·633	6·708	6·782	6·856	6·928	7·000
5	7·071	7·141	7·211	7·280	7·348	7·416	7·483	7·550	7·616	7·681
6	7·746	7·810	7·874	7·937	8·000	8·062	8·124	8·185	8·246	8·307
7	8·367	8·426	8·485	8·544	8·602	8·660	8·718	8·775	8·832	8·888
8	8·944	9·000	9·055	9·110	9·165	9·220	9·274	9·327	9·381	9·434
9	9·487	9·539	9·592	9·644	9·695	9·747	9·798	9·849	9·899	9·950

C. Table of Reciprocals of Numbers from 1 to 10, at Intervals of ·1.

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1	1·000	·909	·833	·769	·714	·667	·625	·588	·556	·526
2	·500	·476	·455	·435	·417	·400	·385	·370	·357	·345
3	·333	·323	·313	·303	·294	·286	·278	·270	·263	·256
4	·250	·244	·238	·233	·227	·222	·217	·213	·208	·204
5	·200	·196	·192	·189	·185	·182	·179	·175	·172	·169
6	·167	·164	·161	·159	·156	·154	·152	·149	·147	·145
7	·143	·141	·139	·137	·135	·133	·132	·130	·128	·127
8	·125	·123	·122	·120	·119	·118	·116	·115	·114	·112
9	·111	·110	·109	·108	·106	·105	·104	·103	·102	·101

D. Table of the Circular Functions at Intervals of One-Twentieth of the Quadrant.

$\theta/\frac{1}{2}\pi$	sin θ	cosec θ	tan θ	cot θ	sec θ	cos θ	
0	0	∞	0	∞	1·000	1·000	1·00
·05	·078	12·745	·079	12·706	1·003	·997	·95
·10	·156	6·392	·158	6·314	1·012	·988	·90
·15	·233	4·284	·240	4·165	1·028	·972	·85
·20	·309	3·236	·325	3·078	1·051	·951	·80
·25	·383	2·613	·414	2·414	1·082	·924	·75
·30	·454	2·203	·510	1·963	1·122	·891	·70
·35	·522	1·914	·613	1·632	1·173	·853	·65
·40	·588	1·701	·727	1·376	1·236	·809	·60
·45	·649	1·540	·854	1·171	1·315	·760	·55
·50	·707	1·414	1·000	1·000	1·414	·707	·50
	cos θ	sec θ	cot θ	tan θ	cosec θ	sin θ	$\theta/\frac{1}{2}\pi$

E. Table of the Exponential and Hyperbolic Functions of Numbers from 0 to 2·5, at Intervals of ·1.

x	e^x	e^{-x}	cosh x	sinh x	tanh x
0	1·000	1·000	1·000	0	0
·1	1·105	·905	1·005	·100	·100
·2	1·221	·819	1·020	·201	·197
·3	1·350	·741	1·045	·305	·291
·4	1·492	·670	1·081	·411	·380
·5	1·649	·607	1·128	·521	·462
·6	1·822	·549	1·185	·637	·537
·7	2·014	·497	1·255	·759	·604
·8	2·226	·449	1·337	·888	·664
·9	2·460	·407	1·433	1·027	·716
1·0	2·718	·368	1·543	1·175	·762
1·1	3·004	·333	1·669	1·336	·801
1·2	3·320	·301	1·811	1·509	·834
1·3	3·669	·273	1·971	1·698	·862
1·4	4·055	·247	2·151	1·904	·885
1·5	4·482	·223	2·352	2·129	·905
1·6	4·953	·202	2·577	2·376	·922
1·7	5·474	·183	2·828	2·646	·935
1·8	6·050	·165	3·107	2·942	·947
1·9	6·686	·150	3·418	3·268	·956
2·0	7·389	·135	3·762	3·627	·964
2·1	8·166	·122	4·144	4·022	·970
2·2	9·025	·111	4·568	4·457	·976
2·3	9·974	·100	5·037	4·937	·980
2·4	11·023	·091	5·557	5·466	·984
2·5	12·182	·082	6·132	6·050	·987

F. Table of Logarithms to Base *e*.

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1	0	·095	·182	·262	·336	·405	·470	·531	·588	·642
2	·693	·742	·788	·833	·875	·916	·956	·993	1·030	1·065
3	1·099	1·131	1·163	1·194	1·224	1·253	1·281	1·308	1·335	1·361
4	1·386	1·411	1·435	1·459	1·482	1·504	1·526	1·548	1·569	1·589
5	1·609	1·629	1·649	1·668	1·686	1·705	1·723	1·740	1·758	1·775
6	1·792	1·808	1·825	1·841	1·856	1·872	1·887	1·902	1·917	1·932
7	1·946	1·960	1·974	1·988	2·001	2·015	2·028	2·041	2·054	2·067
8	2·079	2·092	2·104	2·116	2·128	2·140	2·152	2·163	2·175	2·186
9	2·197	2·208	2·219	2·230	2·241	2·251	2·262	2·272	2·282	2·293

log 10 = 2·303, log 10^2 = 4·605, log 10^3 = 6·908.

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